Political Campaigning with Momentum Effects

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Abstract

This paper considers a game between two players (e.g., politicians) engaged in an advertising race which is subject to momentum effects. Momentum is modeled as a complementarity between current and past advertising spending in a way that is reminiscent of models of addiction and habit formation: the more effective a player's past spending has been, the more effective her current and future spending will be. We analyze a two-agent zero-sum differential game. We show that for symmetric games with a Cobb-Douglas payoff function the closed-loop equilibrium leads to an outcome that is equivalent to that of an open-loop equilibrium of the same game, but in which the momentum effect is more pronounced. In the polar case where the advertising stock does not decay over time, the advertising path is a decreasing power function. We also look at non-symmetric games, for which we compute the open-loop equilibria numerically by solving the discrete-time analog of the continuous-time game. Our numerical results suggest that in games where one politician has more resources than her opponent, the gap between the politicians' advertising intensity increases during the race.

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1 Introduction

This paper provides a framework for analyzing advertising races between two players (e.g., politicians) where their competition is subject to *momentum effects*. To fix ideas, consider a race between two politicians running for the same office. Suppose that one of them has succeeded in getting ahead in the polls, thus having gained momentum over her opponent. Then, one might expect that it will be easier, loosely speaking, for the politician leading the polls to maintain her lead and, perhaps, reinforce it even further. For example, if from this point onwards until election time both politicians match each other's advertising spending, we might expect the leading politician to not only maintain her lead to the end, but strengthen it even further.

For clarity, we will maintain a running example of a political contest throughout the paper, although one should keep in mind that the general framework introduced in this paper lends itself to other applications as well. For example, an alternative natural application of our framework is in the marketing of competing products, such as the looming launch of two rival game consoles or the release of two potential blockbuster movies in peak season. Another application might be in sports where an early lead by one team may provide a psychological boost to the leading team.

Depending on the application at hand, momentum effects may arise due to different underlying forces and our formulation does not take a position on the precise source of momentum effects. For example, in a political campaign momentum may arise due to the electorate's imperfect memory (i.e., bounded rationality). More specifically, Mullainathan (2002) points to a large body of psychological research suggesting that memory operates on the principles of *cue-dependence* and *rehearsal*. According to cue-dependence, events tend to trigger memories of similar past memories. As these past events are triggered, they get rehearsed and thus their memories are strengthened. In turn, this makes it even more likely that they will be further rehearsed by similar events that may follow in the future. Returning to our motivating example, one can thus interpret a lead by one politician in the polls as that politician having created a superior stock of good memories in the electorate's mind, relative to her opponent. As a result, due to the principles of cue-dependence and rehearsal, her future advertising spending will be, all else equal, more successful than her opponent's, as it will be rehearsing a superior stock of existing memories, i.e., the leading politician has gained momentum.

In other applications, there may be other forces that give rise to momentum. For instance, in the context of our console example, momentum may arise due to the fact that consumers whose interest on a product has been piqued by prior advertising may pay more attention to subsequent advertising. Network effects may be another source of momentum as consumers may pay more attention to a console that is showing early signs of popularity.

Formally, we incorporate momentum in our model by imposing the following assumption: the more effective a politician's past spending has been, the more effective her current and future spending will be. To use the economics jargon, there exists a complementarity between current and past advertising spending. In this sense, we model the notion of momentum in a way that is reminiscent of addiction and habit formation models, such as Becker and Murphy (1988) and Constantinides (1991), respectively. In fact, as we shall see, the single-agent analog of the game we consider is isomorphic to a standard utility-maximization model with addiction or habit formation. We show that extending these ideas into a game-theoretic setting gives rise to a game that is both tractable and has several convenient and intuitive properties. Thus, in our opinion, the paper's main methodological contribution is to show that a modeling device previously used in single-agent decision problems of addiction and habit formation can also be fruitfully applied in dynamic games to model situations where a race between the players is subject to momentum effects.

We concede upfront that our modeling of momentum as a complementarity between current and past advertising spending is by no means the only way to proceed. We have made this particular modeling choice out of concern for tractability. In Section 5, we briefly outline two alternative modeling choices (left for future research) through which one could capture the notion of momentum.

Below, we provide a sketch of our model and foreshadow the main results.

1.1 The set up and main results

To set the stage, consider a single politician with a fixed advertising budget. Following the advertising literature in the vein of Nerlove and Arrow (1962), we allow advertising to generate a stock of goodwill, which decays at a fixed rate, δ. Our politician wishes to maximise her stock of goodwill at a critical time, T (e.g. at election time). We allow the marginal effect of her advertising spending in each period to depend on her existing stock of goodwill. That is, it is possible for her to create momentum during the campaign. How should she then sequence her scarce advertising budget to maximise her stock of goodwill at election time?

We show that this problem is isomorphic to a standard inter-temporal utility maximization model with addiction or habit formation and there is nothing here we claim to be novel. We thus view the single-agent problem as a stepping stone towards the two-agent game. As we will see, it also provides a useful benchmark. We show that under a payoff function of the Cobb-Douglas form (with the two arguments being the advertising flow and the existing stock of goodwill) the politician's optimal strategy is to advertise at an increasing exponential rate (as in the model with no momentum effects), with this rate being decreasing in the parameter that measures the magnitude of the momentum effect. In other words, relative to the model with no momentum effects, the optimal path has the same qualitative features, but is *flatter*, as the politician strives to build early on a healthy stock of goodwill that will enhance the impact of future advertising.

We then analyze a differential game between two such politicians. Each politician now nurtures a separate stock of goodwill. Not only must each politician consider the effect of her stock of goodwill on her own advertising effectiveness, but she should also contemplate the impact her advertising has on her rival's spending pattern and stock of goodwill, as well as the manner in which this feeds back into her own stock of goodwill. Guided by a concern for tractability, we make the following five modeling choices.

1. We assume that each politician's objective is to maximize the lead over her opponent at election time. Thus, we consider a zero-sum game. An alternative formulation

that might be of some interest would be to consider a binary objective function. To pursue our political analogy, this would be the case if the politician simply wanted to win the election by being ahead on election day. We think our formulation is of interest for at least two reasons. First, our politician will care, at least to some extent, about the margin of her victory as this will influence her mandate for action (and potentially stoke her ego). In non-political settings, we may expect our decision maker to care about both stocks of goodwill. For example, if two related products are to be launched at a similar time, the success of a launch will hinge on the ability to not only win over consumers but to wrest those consumers from the clutches of competitors. Second, even if our politician is primarily concerned about winning per se, in a stochastic setting she will be more secure in her likelihood of winning the greater her margin over her opponent.

2. We permit the effectiveness of each politician's advertising spending to depend not only on her own stock of goodwill, but also on her rival's stock. In fact, we show that in our framework this dependence is the source of the strategic interaction. In other words, if the effectiveness of one's advertising spending depended only on one's own stock of goodwill, the game would collapse to two separate (single-agent) decision problems.[1]

3. We consider a Cobb-Douglas specification of preferences, according to which the marginal impact of advertising for one politician depends on the ratio between her existing stock of goodwill and that of her opponent. Throughout the discussion, we present our results in two forms. We consider a general formulation of our problem and draw some conclusions based on Euler conditions. We also maintain a running example based on the Cobb-Douglas payoff function. This allows us to draw out intuition not evident in the general formulation and present more detailed results. The special properties of the Cobb-Douglas preferences play a central role in particular results, and we highlight this when this is the case. We note that in the habit-formation literature, which our modeling framework closely resembles, the use of Cobb-Douglas preferences is prevalent. See, for example, Abel (1990).

4. Our benchmark model restricts attention to symmetric games, that is games where politicians have the same budget, same initial stock of goodwill and same preferences. We also investigate extensions of the benchmark model that relax the symmetry assumption, by looking at discrete-time analogs of the game which we solve numerically.

5. We consider both open- and closed-loop equilibria. In open-loop equilibria agents choose their advertsing path by taking as given the advertising path of their rivals. In closed-loop equilibria agents choose a state-contingent strategy taking as given the state-contingent strategy of their rivals. Thus, we see open-loop equilibria as better suited to environments in which agents must plan their advertising strategy

[1] We note that the alternative case in which the politician's advertising effectiveness does not depend on her rival's goodwill stock might be of intrinsic interest to some readers. This case can be understood through our single agent problem of Section 2.

in advance. On the contrary, we see closed-loop equilibria as more relevant when candidates have the flexibility to tailor their advertising spending to developments on the campaign as they arise.

For symmetric games with Cobb-Douglas preferences, we establish two key results. First, we show that the closed-loop equilibrium is closely related to the open-loop equilibrium in the following sense. The closed-loop equilibrium leads to an equilibrium path that coincides with the open-loop equilibrium of the same game, but in which the parameter measuring the extent of the momentum effect in the Cobb-Douglas payoff function is higher. An agent that builds up momentum can discourage their rival from advertising. The feedback inherent in a closed-loop equilibrium thus accentuates the effect of momentum. This result eliminates the need for discussing closed- and open-loop equilibria separately (at least for symmetric games), because once parameters are appropriately re-interpreted, the properties of the open-loop equilibria also apply to the equilibrium paths associated with closed-loop equilibria.

Second, we show that the open-loop equilibrium (and, in turn, the equilibrium path associated with the closed-loop equilibrium) can be described by a relatively simple differential equation with an analytic solution. In the polar case where the stock of goodwill does not decay, the equilibrium advertising path is a decreasing power function: our politicians spend a substantial fraction of their budget at the early stages of the campaign, decreasing the intensity of their advertising until election time.

The single-agent case serves as a useful natural benchmark for comparing and interpreting these results. Recall that in the single-agent problem the optimal path is always increasing; a more pronounced momentum effect makes the optimal path flatter, but does not change the optimal path's qualitative feature. Put differently, in the single-agent case, the model with momentum is observationally equivalent to the model without momentum. This is, however, not the case in the game. As foreshadowed above, in the absence of momentum, the game reduces to two separate single-agent problems and, hence, in equilibrium the two politicians advertise at an increasing exponential rate. In the presence of momentum, however, the equilibrium advertising path may be decreasing or U-shaped over time. In fact, if there is no decay the equilibrium advertising path will surely be decreasing over time.

Finally, we provide numerical results for non-symmetric games while maintaining the Cobb-Douglas specification of preferences. We find that sample paths for asymmetric games can be qualitatively different relative to those that arise for symmetric games. Of particular interest are asymmetries in the available budget allocations between agents. More specifically, our simulations suggest that in equilibrium the gap between the advertising intensity of the two politicians increases during the race. In other words, in early periods the politician with the lower budget tries to match the advertising spending of her opponent, to the extent that her lower budget allows her to do so. In this way, she can minimize the momentum that her opponent will inevitably accumulate. Eventually, she runs out of money and her richer opponent bombards the electorate with her advertising messages. The notion that the gap between the advertising intensity of two politicians with different resources should widen during the race is a testable prediction that emerges from our model.

1.2 Related literature

This paper pulls together ideas from several distinct strands of literature. First, our model's main building block, the premise that agents build a stock of goodwill over time, dates back to the seminal work of Nerlove and Arrow (1962). The Nerlove-Arrow framework has been the workhorse behind a large and well-established literature in marketing. Sethi (1977) analyzes a variant of the Nerlove-Arrow framework under a fixed budget constraint, a feature that is also a key part of our model. There is also a large body of work within this literature analyzing advertising decisions through the lenses of differential games. This line of work is surveyed in Erickson (2003) and Jorgensen and Zaccour (2003).

Second, the way in which we introduce momentum effects in the Nerlove-Arrow framework borrows from models of rational addiction, such as Becker and Murphy (1988), and habit formation, such as Abel (1990) and Constantinides (1990). An earlier predecessor of this line of work is Ryder and Heal (1973) who analyze a growth model with intertemporally-dependent preferences. The common thread across these papers is the idea that there is a complementarity between current and past consumption. Sarafidis (2007) shows that a similar type of complementarity between current and past information releases (e.g., advertising spending) may arise when agents process information under bounded memory. Specifically, building on Mullainathan (2002), he shows that when an agent (e.g., electorate) forgets with a memory technology that operates on the principles of cue-dependence (similarity) and rehearsal (repetition), the impact of current information releases will depend on the impact of past information releases. In this sense, current and past information releases are complements. Bounded memory thus provides a micro-founded explanation for how the particular type of momentum that we model in this paper may arise.

Third, there is an existing body of work that studies momentum effects in games, primarily in the context of political contests. Ali and Kartik (2006) provide an informational theory for how momentum can arise endogenously in sequential elections. Specifically, they consider a model in which some voters receive a noisy signal regarding who the better candidate is. In a sequential election where past votes are observable to all future voters, initial voters will thus be able to convey their private information to future voters. The authors show that there exists an equilibrium where voters take into account (when voting) the information revealed in the prior voting history. Moreover, in such an equilibrium there is momentum in the sense that herding can occur on a candidate with positive probability, which approaches one in large populations. We view our work as complementary to Ali and Kartik (2006) in the following sense. A natural question that arises in their framework is: if candidates can affect the signals that voters receive through increased campaign spending, how should they time their spending during the election? Our work can thus be seen as an attempt to address this issue.[2]

Furthermore, Klumpp and Polborn (2006) provide an alternative theory of momentum in the context of a political contest that is decided in a series of sequential elections, such

[2]So far, our model has been cast in the context of two politicians facing an election at a critical time T. Notice, however, that the results of Ali and Kartik (2006) provide an alternative interpretation of our model as one in which politicians face a series of smaller sequential elections ending at time T.

as the US primaries. They employ a model with similar features to an all-pay auction in which politicians can increase the probability of winning a primary by increasing their campaign spending. Unlike our framework, in theirs politicians do not face any budget constraints. They show that the winner of the first primary has an incentive to spend more than his opponent in the second primary, which in turn increases the probability that the winner of the first primary will also win the second one, and so on. Thus, in their model early success breeds future success which amounts to momentum. A key difference between our framework and theirs is that in the former momentum is built directly into the payoff function, whereas in the latter momentum arises endogenously.

Finally, our work has some, though more distant connection, to a prior literature on races, such as Harris and Vickers (1987) and Horner (2004), a common theme in which is how players' effort levels vary with the intensity of their rivalry.

The rest of this paper is organized as follows. Section 2 introduces the model in the context of the single-agent case. Section 3 discusses the two-agent symmetric game and derives the open- and closed-loop equilibria. Section 4 relaxes the assumption of symmetry and provides numerical results for asymmetric games. Section 5 concludes and outlines two alternative formulations for modeling momentum.

2 A single-agent optimal advertising problem with momentum effects

We first look at a single-agent problem with momentum effects. As we shall see, this problem reduces to a standard consumption-smoothing model with habit formation. We thus ask the reader to treat the single-agent case only as a stepping stone towards the two-agent game. Throughout, we provide some limited results for the general case before turning to a parametric example based on Cobb-Douglas preferences for more detailed results.

A politician is facing an election at a future date T. For her campaign, our politician has an exogenous, fixed advertising budget equal to B. Advertising creates a stock of goodwill through a payoff function $u(\cdot,\cdot)$, whose two arguments are the amount spent on advertising at time t, denoted by c_t, and the existing stock of goodwill at time t, denoted by S_t. We allow the stock of goodwill to decay at a rate δ. Hence, the stock of goodwill evolves according to:

$$\frac{dS}{dt} = u(c_t, S_t) - \delta S_t \tag{1}$$

The politician's objective is to maximize the stock of goodwill at election time, S_T. Integrating forward equation (1) shows that the stock of goodwill at election time, S_T, is equal to:

$$S_T = e^{-\delta(T-t_0)} S_{t_0} + \int_{t_0}^{T} e^{-\delta(T-t)} u(c_t, S_t) dt \tag{2}$$

We can thus cast the politician's optimal control problem as:

$$\max_{c_t} \quad \int_{t_0}^{T} e^{\delta t} u(c_t, S_t) dt \tag{3}$$

$$\text{s.t.} \quad \frac{dS}{dt} = u(c_t, S_t) - \delta S_t$$

$$\int_{t_0}^{T} c_t dt = B$$

Two features of our formulation are worth noting. First, even though the politician cares only about the final stock of goodwill, S_T, we have reformulated her problem so that it is as if she cares about the discounted flow of payoffs, u_t. Second, the marginal effect of current spending depends on the *effectiveness* of past spending, not on past spending per se.[3]

It can be shown that the optimal advertising path follows the following Euler equation (see Appendix 6.1):

$$\left[u_{cc}\frac{dc}{dt} + u_{cS}\frac{dS}{dt}\right]\frac{1}{u_c} - u_S + \delta = 0 \tag{4}$$

2.1 Cobb-Douglas preferences

Next, we assume that the payoff function is of the Cobb-Douglas family,

$$u(c, S) = c^{\alpha} S^{\beta}, \qquad 0 < \alpha < 1, \qquad 0 < \beta < 1, \qquad \alpha + \beta < 1. \tag{5}$$

The concavity parameter α permits decreasing returns to advertising at time t and could be interpreted as a coefficient of risk aversion. The strength of the momentum effect is calibrated through β, with $\beta = 0$ indicating the absence of momentum. The optimal advertising path can then be shown to be equal to (see Appendix 6.1.1):

$$c^*(t) = Ke^{\rho t}, \qquad \rho = \frac{\delta(1-\beta)}{1-\alpha} \tag{6}$$

The constant of integration K can be evaluated from the isoperimetric budget constraint, which yields $K = B\frac{\rho}{e^{\rho T} - e^{\rho t_0}}$.

Hence, in the single-agent case and under a Cobb-Douglas payoff function, in the optimal advertising path the politician advertises at an increasing exponential rate, ρ. The rate ρ is decreasing in the parameter that measures the magnitude of the momentum effect β, increasing in the rate of decay δ, and increasing in the concavity parameter α. We provide intuition for these comparative statics results below.

[3] This is in contrast with the formulation of Becker and Murphy (1988), for example, in which the marginal utility of current consumption depends on past consumption, rather than the utility of past consumption. In other words, the state equation in their model is cast as: $\frac{dS}{dt} = c_t - \delta S_t$.

2.2 Relation to the textbook consumption-smoothing model

Readers familiar with inter-temporal utility maximization models will have recognized the similarities between these models and our set up. In fact, when there are no momentum effects, i.e., $\beta = 0$, our set up is identical to the textbook model.[4] In this case, the optimal advertising path is dictated by the concavity of the payoff function and the rate of decay of the stock of goodwill. These are the analogs of the coefficient of risk aversion and the discount rate, respectively, in the jargon of the inter-temporal utility maximization models. When the rate of decay is greater than zero, the politician would like to concentrate her spending as close as possible to election time to prevent her advertising from decaying. This is equivalent to the desire of a utility maximizer to consume as much as possible early on due to her impatience. The only superficial difference between the two models is that the politician discounts past flows, whereas the utility maximizer discounts future ones. But, this is purely semantics. The concavity of the payoff function induces the politician to smooth her spending over time, in the same way that a utility maximizer would smooth her consumption over her lifetime. For example, in the extreme case where there is no decay, i.e., $\delta = 0$, the politician perfectly smooths her spending and advertises at a constant level.

Introducing momentum effects to our politician's problem as a complementarity between past and present spending in the payoff function is similar to introducing habit formation or addiction to the utility function in a model of inter-temporal utility maximization. As shown, when the payoff function is of the Cobb-Douglas form, in the optimal path our politician advertises at an increasing exponential rate with the property that as the magnitude of the momentum effect rises, the optimal path becomes *flatter*. This is intuitive because by shifting advertising to earlier periods, our politician can increase the effectiveness of her future advertising.

For future reference, we would like to note two additional features of the optimal path. As we shall see below, neither of these relatively unattractive features arise in the equilibrium of the game between two politicians. First, the advertising behavior of our politician in the presence of momentum effects is observationally equivalent to that of a politician who does not face any momentum effects, but instead faces a lower decay parameter δ or a higher concavity parameter α. Second, in the optimal path decay and momentum are intertwined in the sense that a higher decay rate makes the impact of momentum more pronounced, but, under Cobb-Douglas preferences, this effect takes an extreme form: when decay is not present ($\delta = 0$), momentum has no impact whatsoever on the advertising path. [5]

[4]See, for example, exercises 5 and 6 of section 5 in Kamien and Schwartz (1981).

[5]To see this, consider first the case where there is no decay ($\delta = 0$) and momentum is absent ($\beta = 0$). As explained above, in this case the politician's optimal advertising path would be flat. Suppose now that the momentum parameter β increased and consider the politician's incentive to shift some advertising to earlier periods. On the one hand, this incremental advertising in earlier periods would have a smaller impact at the time of spending (because the stock of goodwill is small). On the other hand, it would increase the impact of future advertising as it would increase momentum. In the special case of Cobb-Douglas preferences and no decay, these two factors exactly balance out and the optimal path is flat, regardless of the magnitude of the momentum effect. For more general preferences, this extreme result does not hold, but the same general intuition applies: momentum effects are accentuated if decay is more severe. In the presence of rapid decay, the stock of goodwill deteriorates quickly. If advertising is left

3 A zero-sum advertising game with momentum effects

We now extend the single-agent model of the previous section to analyze a zero-sum game between two politicians competing for goodwill.

There are two politicians, i and j, who accumulate two separate stocks of goodwill, S_i and S_j, respectively. The effectiveness of advertising by politician i at time t depends on both her and her opponent's stock of goodwill according to a payoff function $u^i(c_i, S_i, S_j)$, with the property that u^i_c is increasing in the politician's own stock S_i and decreasing in her opponent's stock, S_j. And similarly for politician j. We assume that each politician's objective is to maximize the difference between her and her opponent's stock of goodwill at the critical time T.

Hence, politician i now solves (suppressing the time subscript):

$$\max_{c_i} \quad \int_{t_0}^{T} e^{\delta t} u^i(c_i, S_i, S_j) dt - \int_{t_0}^{T} e^{\delta t} u^j(c_j, S_j, S_i) dt \tag{7}$$

$$\text{s.t.} \quad \frac{dS_i}{dt} = u^i(c_i, S_i, S_j) - \delta S_i \tag{8}$$

$$\frac{dS_j}{dt} = u^j(c_j, S_j, S_i) - \delta S_j \tag{9}$$

$$\int_{t_0}^{T} c_i dt = B_i \tag{10}$$

Notice that our assumption that each politician's objective is to maximize the size of the lead over her opponent gives rise to a zero-sum game. In an alternative formulation one might consider a binary objective, i.e., each politician simply wants to win the election by being ahead on election day. As we noted in the Introduction, we view that our formulation has merit on two grounds: the environment may be stochastic, with our politician seeking to maximise her chances of winning; or the margin of victory may be of intrinsic value (either as a mandate in a political setting, or an indicator of profitability in a market setting).

3.1 Open-loop equilibrium in symmetric games

We begin our equilibrium analysis by describing the open-loop equilibrium in which agents commit to a path of spending, taking as given the path of their rival. Later, we will describe the closed-loop equilibrium in which agents choose a state-contingent strategy, taking as given the state-contingent strategy of their rival. In fact, we will show that in our game (and under certain conditions) there is a close connection between these two solution concepts.

until late, it will be less effective because it acts on a diminished stock of goodwill. The presence of momentum thus ameliorates the impact of decay.

Following similar steps as in the single-agent case, it can be shown (see Appendix 6.2.1) that under the open-loop equilibrium the Euler equation for politician i (i.e., the analog of equation (4) in the single-agent problem) is given by:

$$\frac{1}{u^i_{c_i}}\left(u^i_{c_ic_i}\frac{dc_i}{dt}+u^i_{c_iS_i}\frac{dS_i}{dt}+u^i_{c_iS_j}\frac{dS_j}{dt}\right)-u^i_{S_i}+u^j_{S_i}\frac{1-\chi_i}{1+\psi_i}=-\delta \tag{11}$$

The variables ψ_i and χ_i are related to the lagrange multipliers associated with the state equations (8) and (9), as shown in Appendix 6.2.1.

3.1.1 The source of strategic interaction

Let us pause to compare this Euler equation to the single agent analog (equation (4)), and suppose for a moment that momentum effects are absent. That is, suppose that we can write the payoff function for agent i as $u^i(c_i)$. It is immediately evident that the two Euler equations then coincide. In other words, when there are no momentum effects, the two-agent problem reduces to two separate decision problems. Intuitively, the only reason I consider my rival's stock of goodwill when choosing how much to advertise is because of the payoff function $u^i(\cdot)$. I care about my rival's terminal stock because it enters my objective function, but this does not influence my decision making. Hence, strategic interaction derives directly from momentum. In fact, we can narrow the source of strategic interaction a little further. If my rival's stock of goodwill does not enter my payoff function *and* my stock of goodwill does not enter my rival's payoff (that is, *both* payoff functions are of the form $u^i(c_i, S_i)$), then the two-agent problem once more reduces to the single agent problem (again, we can see this by comparing Euler equations). Thus, it is only if momentum takes on a particular character that strategic interaction plays a role: at least one player's stock must influence the effectiveness of their rival's advertising spending.

3.1.2 Cobb-Douglas preferences

Now, let us further assume that the payoff function $u^i(\cdot,\cdot,\cdot)$ is of the form:

$$u^i(c_i, S_i, S_j) = c_i^{\alpha_i}\left(\frac{S_i}{S_j}\right)^{\beta_i} \tag{12}$$

The dependence on rival goodwill induces strategic interaction between agents. Exploiting the properties of the Cobb-Douglas payoff function, as shown in Appendix 6.2.2, the euler equation in (11) reduces to:

$$(\alpha_i-1)\frac{\frac{dc_i}{dt}}{c_i}-u^j_{S_j}\frac{\beta_i}{\beta_j}+u^j_{S_i}\frac{1-\chi_i}{1+\psi_i}+\delta=0 \tag{13}$$

For now, we also restrict attention to games where our politicians are symmetric with regard to their budgets, their initial stocks and the parameters of their payoff functions. Specifically, we assume: $B_i = B_j = B$, $S_i(t_0) = S_j(t_0) = S(t_0)$, $\alpha_i = \alpha_j = \alpha$ and $\beta_i = \beta_j = \beta$. We look for symmetric equilibria where $c_i(t) = c_j(t) = c(t)$ and $S_i(t) = S_j(t) = S(t)$.

Under these symmetry conditions, we have $u^i_{S_i} = u^j_{S_j}$, $u^i_{S_j} = u^j_{S_i}$ and $u^i_{S_i} = -u^j_{S_i}$. Moreover, as discussed in Appendix 6.2.3, we have $\psi_i = -\chi_i$ and $\psi_j = -\chi_j$. Substituting these into the Euler equation in equation (13) and invoking the state equation (8), as shown in Appendix 6.2.3, yields a differential equation that must be satisfied by the stock of goodwill in a symmetric open-loop equilibrium:

$$\frac{\frac{d^2S}{dt^2} + \delta\frac{dS}{dt}}{\frac{dS}{dt} + \delta S} = -\frac{2\alpha\beta}{1-\alpha}\frac{\frac{dS}{dt}}{S} + \frac{\alpha\delta}{1-\alpha}(1-2\beta) \tag{14}$$

Letting $S^*(t)$ denote the solution to this differential equation, and invoking equation (8), we can write the open-loop equilibrium advertising path as:

$$c^*(t) = \left(\frac{dS^*}{dt} + \delta S^*\right)^{\frac{1}{\alpha}} \tag{15}$$

The polar cases with $\beta = 0$ and $\delta = 0$

To motivate intuition, we first consider two polar cases. First, if there are no momentum effects (i.e. $\beta = 0$), the solution to the differential equation in (14) is of the form:

$$S^*(t) = K_1 e^{\frac{\alpha\delta}{1-\alpha}t} + K_2 e^{-\delta t} \tag{16}$$

In turn, the corresponding equilibrium advertising path coincides with that for the single agent case shown in equation (6). This confirms our result for the more general case: absent momentum, there is no strategic interaction between agents. The equilibrium advertising path is increasing.

Consider next the case where there are momentum effects, $\beta > 0$, but suppose that the rate of decay of the stock of goodwill, δ, is zero. In this case, the solution to the differential equation (14), $S^*(t)$, and the corresponding advertising path, $c^*(t)$, are power functions, as shown below:

$$S^*(t) = K_1(t + K_2)^{\gamma} \tag{17}$$

$$c^*(t) = (K_1\gamma)^{\frac{1}{\alpha}}(t + K_2)^{\frac{\gamma-1}{\alpha}} \tag{18}$$

$$\text{where } \gamma = \frac{1-\alpha}{1-\alpha+2\alpha\beta}$$

where the constants of integration K_1 and K_2 can be evaluated from the initial condition $S(t_0)$ and the isoperimetric budget constraint. Notice that advertising now decreases over time. Further, the power $\frac{\gamma-1}{\alpha}$ in the expression for the equilibrium advertising path, $c^*(t)$, is decreasing in β, the magnitude of the momentum effect. In other words, the higher the magnitude of the momentum effect, the more intensely the politicians advertise early on.

Notice also that this contrasts with our results for the single-agent case. There, in the absence of decay, the momentum parameter had no effect on the optimal spending path.

Beyond the polar cases

For the general case in which $\beta > 0$ and $\delta > 0$, the solutions $S^*(t)$ and $c^*(t)$, derived in Appendix 6.2.4, are given by:

$$S^*(t) = \left(K_1 e^{\alpha\eta t} + K_2 e^{-\frac{\delta}{\gamma}t}\right)^{\gamma}, \tag{19}$$

$$c^*(t) = \left(\frac{\delta\gamma}{1-\alpha}\right)^{\frac{1}{\alpha}} K_1^{\frac{1}{\alpha}} e^{\eta t} \left(K_1 e^{\eta\alpha t} + K_2 e^{-\frac{\delta}{\gamma}t}\right)^{\frac{\gamma-1}{\alpha}}, \tag{20}$$

where $\eta = \dfrac{\delta(1-2\beta)}{1-\alpha}$

Figures 1 to 3 illustrate some of the properties of the symmetric open-loop equilibrium for several parameter permutations with regard to α, β and δ.

The effect of the concavity parameter α is illustrated in Figures 1.a (without decay) and 1.b (with decay). A decrease in α increases the concavity of the payoff function and thus encourages the agent to smooth her spending over time, leading to a *flatter* advertising profile. Notice, however, that this effect may materialize in two different ways. With low decay and/or high momentum (e.g., panel a), each agent engages in an initially flurry of advertising early in the game in order to build up momentum (and prevent her opponent from doing the same). Additional concavity (lower α) tempers with the agents' incentive to advertise with an initial burst and thus agents shift advertising to *later* periods. On contrary, with high decay and/or low momentum (e.g., panel b), each agent has an increased incentive to engage in a flurry of advertising late in the game. Additional concavity (lower α) now tempers with the agents' incentive to advertise intensively at the end and thus agents shift advertising to earlier periods.

The effect of momentum is illustrated in Figures 2.a (without decay) and Figure 2.b (with decay). An increase in the magnitude of the momentum effect (i.e., a higher β), induces the agent to concentrate more in earlier periods in an effort to build up momentum (and prevent her opponent from doing the same).

Figure 3 further illustrates the role of discounting. With greater discounting, the agent concentrates more of her spending nearer the terminal date.

Returning to Figures 2.a and 2.b, another key take-away of this section is the fact that, unlike in the single-agent case, the presence of momentum effects can change the qualitative features of the equilibrium path. Recall that in the single-agent case, the optimal path is always (weakly) increasing. In the game, however, the presence of momentum effects can affect the monotonicity of the equilibrium path. To see this, recall that when there are no momentum effects, the game reduces to two separate single-agent problems and the equilibrium path will be (weakly) increasing. In contrast, in the presence of momentum effects, as long as the decay rate is not too pronounced, the equilibrium advertising path with momentum effects can be decreasing or U-shaped.

Figure 1. Comparative Statics of Advertising Rates with Respect to α

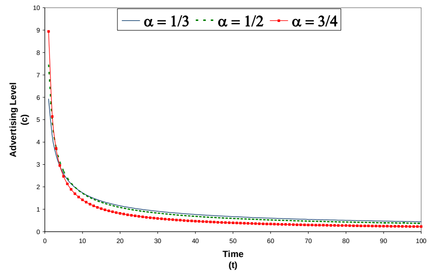


Figure 1.a. Case δ = 0

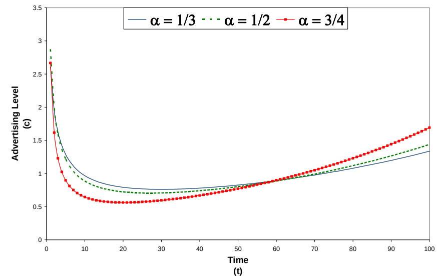


Figure 1.b. Case δ = 0.02

Notes
Both figures are obtained for initial period *t*=0, terminal period *T=100,* initial condition *S(0)=1*, budget *B=100*, and momentum effect parameter of *β=1/4*
Figures do not plot values of advertising level for *t<1*

Figure 2. Comparative Statics of Advertising Rates with Respect to β

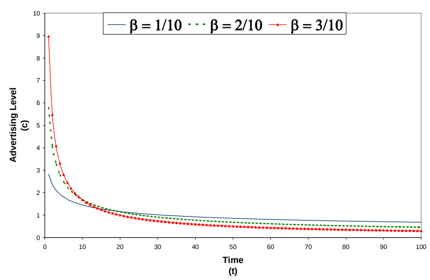


Figure 2.a. Case δ = 0

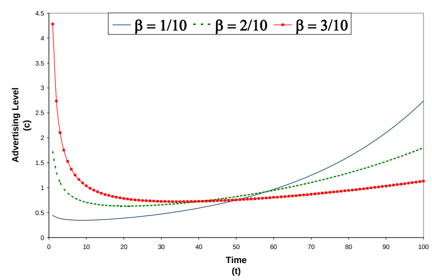
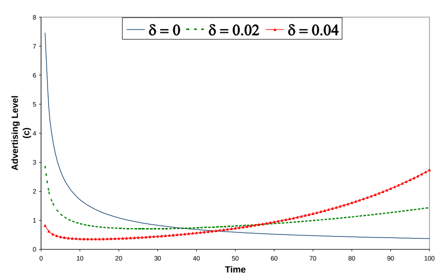


Figure 2.b. Case δ = 0.02

Notes
Both figures are obtained for initial period *t*=0, terminal period *T=100,* initial condition *S(0)=1*, budget *B=100*, and coefficient of risk aversion parameter of *α=1/2*
Figures do not plot values of advertising level for *t<1*

Figure 3. Comparative Statics of Advertising Rates with Respect to δ



Notes
Figure is obtained for initial period *t*=0, terminal period *T=100,* initial condition of *S(0)=1*, budget of *B=100*, coefficient of risk aversion parameter of *α=1/2*, and momentum effect parameter of *β=1/4*
Figure does not plot values of advertising level for *t<1*

3.2 Closed-loop equilibrium in symmetric games

In a closed-loop equilibrium politician i takes as given not the opponent's advertising path $c_j(t)$, but her advertising strategy $c_j(S_j, S_i)$. Therefore, when contemplating a deviation politician i needs to take into account how this deviation will alter the stocks of goodwill, S_i and S_j, and how, in turn, these changes will affect the opponent's advertising path.

While we cannot describe the closed-loop equilibrium *strategies* in closed form, we can describe the associated *equilibrium path*. Exploiting again the properties of the Cobb-Douglas payoff function and invoking symmetry, it can be shown (see Appendix 6.3) that the closed-loop equilibrium gives rise to an equilibrium path that is qualitatively very similar to the open-loop equilibrium. To be more precise, on the equilibrium path of a closed-loop equilibrium the stock of goodwill must satisfy the following differential equation:

$$\frac{\frac{d^2S}{dt^2} + \delta\frac{dS}{dt}}{\frac{dS}{dt} + \delta S} = -\alpha\beta\frac{2-\alpha}{(1-\alpha)^2}\frac{\frac{dS}{dt}}{S} + \frac{\alpha\delta}{1-\alpha}\left(1 - \beta\frac{2-\alpha}{1-\alpha}\right) \tag{21}$$

Comparing (21) with the corresponding differential equation (14) that arises under the open-loop equilibrium, ones sees that the two differential equations are identical, but for the fact that the parameter β in (14) is replaced with the term $\beta\frac{2-\alpha}{2-2\alpha}$ in (21).[6]

In other words, the closed-loop equilibrium is equivalent to the open-loop equilibrium of a slightly modified game where the extent of the momentum effect is parametrized differently. In fact, because $\frac{2-\alpha}{2-2\alpha} > 1$, the closed-loop equilibrium is equivalent to the open-loop equilibrium of the game when solved under a *higher* degree of momentum. We could think of the open-loop problem as one in which strategic effects are mitigated. In this context, the presence of momentum effects encourages agents to concentrate more advertising spending earlier in the campaign. The introduction of strategic incentives to advertise through the closed-loop problem provides an additional incentive to spend more early in the campaign. An increase in agent i's goodwill stock discourages agent j from advertising. Because of the momentum effect, this has a lasting effect on agent j's accumulation of goodwill. Effectively, the momentum effect is amplified. Notice also that, in the absence of the momentum effect, the open loop and closed loop equilibria are identical; it is through the momentum effect that strategic incentives enter.

4 Asymmetric games

In this section, we look at asymmetric games that arise when agents have different budgets, different initial stocks of goodwill, or payoff functions with different risk aversion and momentum parameters. We rely on Cobb-Douglas preferences throughout.

We first note that if the only source of asymmetry is the momentum effect parameter (i.e., $\beta_i \neq \beta_j$), it turns out that there still exists a symmetric equilibrium. Mimicking the steps in Appendix 6.2.3, it can be shown that in a symmetric equilibrium the stock of

[6]To see this in a more transparent way, re-write (21) as: $\frac{\frac{d^2S}{dt^2}+\delta\frac{dS}{dt}}{\frac{dS}{dt}+\delta S} = -\frac{2\alpha\beta\frac{2-\alpha}{2-2\alpha}}{1-\alpha}\frac{\frac{dS}{dt}}{S} + \frac{\alpha\delta}{1-\alpha}(1-2\beta\frac{2-\alpha}{2-2\alpha})$.

goodwill satisfies the same differential equation (14), but replacing the term β by the term $\frac{\beta_i+\beta_j}{2}$. This result is somewhat intuitive because a higher β_i makes it more worthwhile for agent i to advertise earlier on in an effort to gain momentum, while a higher β_j makes it more worthwhile for the same agent i to do the same, but in an effort to prevent her opponent from gaining momentum. Two aspects of our formulation make these two concerns entirely symmetric, making only the "average" β matter. First, in our Cobb-Douglas specification, the payoff function $u^i(c_i, S_i, S_j)$ depends on the ratio of the goodwill stocks. Second, each agent has a symmetric concern for their rival's payoff stream as the objective function depends on the difference between payoffs. Hence, agent i's incentive to advertise to build up S_i is exactly matched by agent j's incentive to advertise to prevent this.

When there are asymmetries in other parameters, in order to solve the resulting asymmetric game we resort to numerical methods by looking at the discrete-time analog of the original continuous-time game. Specifically, we look at the game where each politician i solves:

$$\max_{c_{i,1},c_{i,2},\ldots,c_{i,T}} \quad \sum_{t=1}^{T}(1-\delta)^{T-t}[u^i(c_{i,t}, S_{i,t-1}, S_{j,t-1}) - u^j(c_{j,t}, S_{j,t-1}, S_{i,t-1})] \tag{22}$$

$$\text{s.t.} \quad S_{i,t} = u^i(c_{i,t}, S_{i,t-1}, S_{j,t-1}) + (1-\delta)S_{i,t-1} \tag{23}$$

$$S_{j,t} = u^j(c_{j,t}, S_{j,t-1}, S_{i,t-1}) + (1-\delta)S_{j,t-1} \tag{24}$$

$$\sum_{t=1}^{T} c_{i,t} = B_i \tag{25}$$

where the payoff function is of the Cobb-Douglas family as shown in (12).

Figures 4-6 illustrate the effect of asymmetries in the budget, the parameter of risk aversion, α, and the initial stock of goodwill, respectively. As a benchmark, each figure plots the open-loop equilibrium strategy for a 100-period symmetric game (solid path). Then, each figure introduces an asymmetry in one dimension (budget, coefficient of risk aversion, and initial stock) and shows the resulting open-loop equilibrium strategy for each agent.

Our simulations suggest that, in games with positive decay, asymmetries due to different budgets can lead to equilibrium advertising paths that are qualitatively different from the equilibrium paths of symmetric games. As illustrated by Figure 4.b, the agent with the lower budget tries to match, to the extent that her lower budget allows her to do so, the advertising level of her opponent, in order to minimize the momentum that her opponent will inevitably accumulate. Eventually, she runs out of money and the agent with the higher budget can finish the race with an advertising blitz.

Asymmetries due to different coefficients of risk aversion also seem to lead to qualitatively different paths, relative to the symmetric case, in games with positive decay. For example, when decay is high and/or momentum effects are not very pronounced (as in Figure 5.b), all else equal the agents concentrate their advertising towards the end. The agent with the lower coefficient of risk aversion, however, is more eager to smooth her spending profile, so she finds it worthwhile to shift some of her advertising towards the beginning of the race. Notice, that in this asymmetric equilibrium the agent with the higher coefficient of risk aversion concedes the momentum early in the race to her opponent.

Finally, as illustrated in Figure 6 asymmetries in the initial stock of goodwill seem to have a negligible effect on the equilibrium, both qualitatively as well as quantitatively. Taken together, however, these results demonstrate that asymmetries can generate qualitative differences in advertising dynamics. It is worth emphasising that this stems directly from the strategic interaction induced by momentum. In the absence of momentum, the qualitative character of the advertising path is unaffected by asymmetries in any dimension (recall equation (6)).

With a view towards applications, we would like to look more closely into asymmetries that arise due to different budgets. This is because, unlike asymmetries in the parameter α, asymmetries in budgets are both likely to be encountered in practice and observable. A natural question that arises is how the gap between the advertising intensity of the rich and the poor politician will evolve during the race. For example, one may ask if it can ever be the case that, in equilibrium, the poor politician outspends her opponent early in the race in an effort to build some momentum and thus compensate for her lower budget. Figure 7 attempts to address this question by plotting the percentage difference between the advertising intensity of the two politicians for several permutations of the parameter values. In all cases shown in the figure, as well as all others we privately considered, the percentage gap (in absolute value) increases over time. Thus, a testable prediction of the model is that, all else equal, in races where one politician has more resources than her opponent we expect to see the gap in the advertising intensity of the two politicians widening over the duration of the race.

5 Discussion

This paper provided a framework for analyzing advertising games in the presence of momentum effects. We showed that the proposed framework can be fruitfully applied to a two-agent game that exhibits qualitatively different properties to an analogous single-agent decision problem. For symmetric games and Cobb-Douglas preferences, in particular, the game is tractable allowing us to establish several sharp results, such as (1) describing the open-loop equilibrium and closed-loop equilibrium path in closed form and (2) establishing that the closed-loop game can be recast as an open-loop one with a larger momentum parameter.

A key ingredient of our framework has been that momentum takes the form of a complementarity between current and past advertising spending in the players' payoff function. We believe that this modeling approach, which builds on the well-established habit-formation and addiction literature, is an intuitive way of capturing the notion of momentum in an advertising race. Nevertheless, we cannot deny that it also served our own selfish interest of working with a tractable model. Below, we conclude by suggesting

Figure 4. Asymmetries in the Budget (B_1 = 200; B_2 = 100)

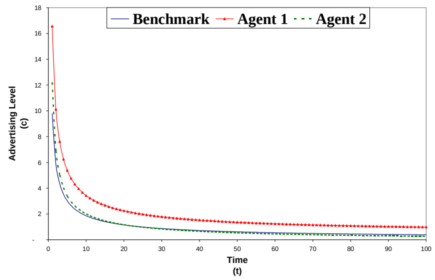


Figure 4.a. Case δ = 0

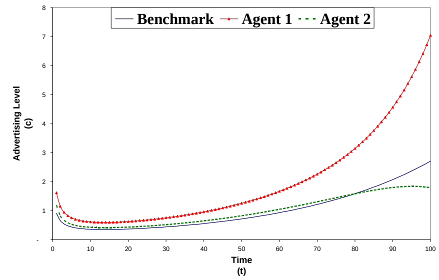


Figure 4.b. Case δ = 0.04

Notes
The "benchmark" scenario is obtained for a symmetric 100-period game with budget B = *100*, parameter of risk aversion α = *1/2*, momentum effect parameter β = *1/4* and initial condition S(0) = 1

Figure 5. Asymmetries in the Coefficient of Risk Aversion (α_1 = 0.9 ; α_2 = 0.5)

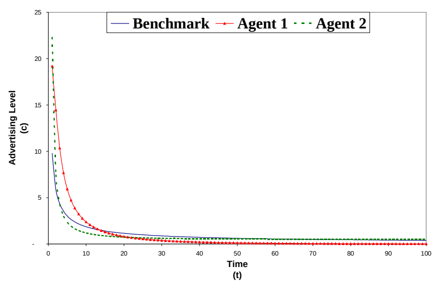


Figure 5.a. Case δ = 0

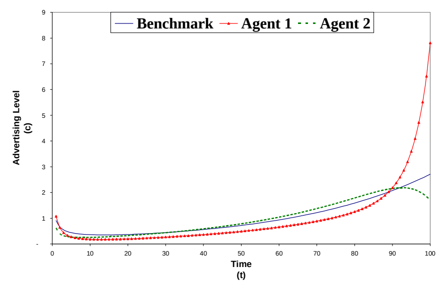


Figure 5.b. Case δ = 0.04

Notes
The "benchmark" scenario is obtained for a symmetric 100-period game with budget B = *100*, parameter of risk aversion α = *1/2*, momentum effect parameter β = *1/4* and initial condition S(0) = 1

Figure 6. Asymmetries in the Initial Stock ($S_1(0)$ = 50 ; $S_2(0)$ = 1)

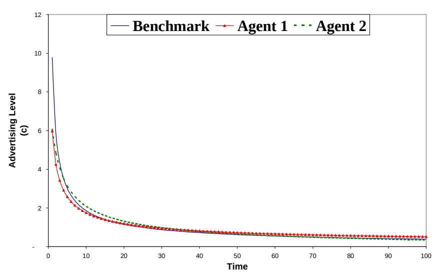


Figure 6.a. Case δ = 0

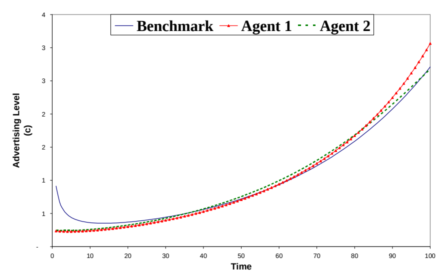
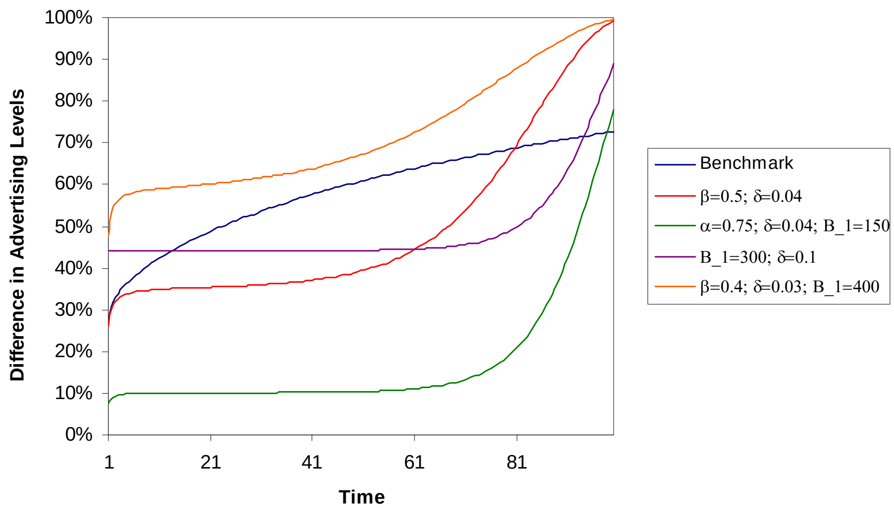


Figure 6.b. Case δ = 0.04

Notes
The "benchmark" scenario is obtained for a symmetric 100-period game with budget B = *100*, parameter of risk aversion α = *1/2*, momentum effect parameter β = *1/4* and initial condition S(0) = 1

Figure 7. Percentage Difference between the Advertising Levels of the "Rich" and the "Poor" Politicians



Notes

The "benchmark" scenario is obtained for a 100-period game with budget $B\_1 = 200$, $B\_2 = 100$, parameter of risk aversion $\alpha = 1/2$, momentum effect parameter $\beta = ¼$, zero decay rate and initial conditions S(0) = 1 for both players.

two alternative ways of modeling momentum as promising extensions for future research.

First, one could alternatively capture the notion of momentum by treating the agent's budget as an endogenous variable which depends on the stock of goodwill. This alternative model may be particularly relevant for applications where politicians rely on campaign contributions by strategic donors who decide how to allocate their contributions between the two politicians by factoring in which politician is more likely to win the race. Consider, for example, a variant of our model in which the relationship between spending and goodwill is stochastic. Even if our politicians started on an equal footing, we may anticipate asymmetric outcomes. One would expect that initially our politicians will receive equal contributions from donors and thus will advertise with equal intensity. Once one of two politicians gets a lead, due to stochastic events, then she will receive more money from donors, which will enable her to spend more and thus become more likely to maintain her lead. In turn, this will lead to even more contributions flowing into her coffers, and so on.

Second, instead of incorporating our momentum term β, one could consider endogenizing the decay rate δ by treating it as a decreasing function of goodwill. The idea here is that as a politician gains momentum, her advertising messages make a more lasting impression on the electorate, thus decaying more slowly. A model along these lines would be reminiscent of growth models with endogenous discount rates, pioneered by Uzawa (1968).

6 Appendix

6.1 Derivation of the optimal path for the single-agent case

This section of the appendix provides a a detailed derivation of the optimal path for the single-agent case discussed in section 2.

The first order conditions are given by (supressing the time subscript):

$$\partial c : e^{\delta t} u_c + \lambda u_c = \mu \tag{26}$$

$$\partial S : \frac{d\lambda}{dt} + e^{\delta t} u_S + \lambda u_S - \lambda \delta = 0 \tag{27}$$

where λ and μ denote the lagrange multipliers corresponding to the two constraints, respectively. Notice that the lagrange multiplier associated with the isoperimetric budget constraint, μ, is a constant that does not depend on time, t.

Defining $\psi = \lambda e^{-\delta t}$ and $\phi = \mu e^{-\delta t}$, the first order conditions reduce to:

$$u_c[1 + \psi] = \phi \tag{28}$$

$$\frac{d\psi}{dt} + u_S[1 + \psi] = 0 \tag{29}$$

Differentiating (28) with respect to time yields:

$$[u_{cc} \frac{dc}{dt} + u_{cS} \frac{dS}{dt}][1 + \psi] + u_c \frac{d\psi}{dt} = \frac{d\phi}{dt} \tag{30}$$

Dividing by $u_c[1 + \psi] = \phi$, substituting in equation (29), and using the fact that $\frac{\frac{d\phi}{dt}}{\phi} = -\delta$ we obtain the euler equation shown in equation (4) and re-produced below for completeness:

$$[u_{cc} \frac{dc}{dt} + u_{cS} \frac{dS}{dt}] \frac{1}{u_c} - u_S + \delta = 0 \tag{31}$$

6.1.1 The optimal path under a Cobb-Douglas payoff function

When the payoff function is of the Cobb-Douglas family, as shown in equation (5), we also have:

$$\frac{u_{cc}}{u_c} = (\alpha - 1)\frac{1}{c}; \;\; \frac{u_{cS}}{u_c} = \beta \frac{1}{S}; \text{ and } u_S = \beta \frac{\frac{dS}{dt}}{S} + \delta \beta$$

where the third expression follows from dividing both sides of the state equation (1) by the state S and observing that $\beta \frac{u(c,S)}{S} = u_S$.

Substituting these expressions into the euler equation in (31) and re-arranging, we obtain the following differential equation:

$$(1 - \alpha) \frac{\frac{dc}{dt}}{c} = \delta(1 - \beta) \tag{32}$$

whose solution yields the optimal advertising path:

$$c^*(t) = Ke^{\rho t}, \text{ where } \rho = \frac{\delta(1-\beta)}{1-\alpha} \tag{33}$$

6.2 Derivation of the open-loop equilibrium

This section of the appendix provides a detailed derivation of the open-loop equilibrium discussed in section (3.1).

6.2.1 Euler equation for general games

The first-order conditions for politician i are given by:

$$\partial c_i \quad : \quad e^{\delta t} u^i_{c_i} - \mu_i + \lambda_i u^i_{c_i} = 0 \tag{34}$$

$$\partial S_i \quad : \quad \frac{d\lambda_i}{dt} + \lambda_i u^i_{S_i} - \delta\lambda_i + \nu_i u^j_{S_i} + e^{\delta t}(u^i_{S_i} - u^j_{S_i}) = 0 \tag{35}$$

$$\partial S_j \quad : \quad \frac{d\nu_i}{dt} + \nu_i u^j_{S_j} - \delta\nu_i + \lambda_i u^i_{S_j} + e^{\delta t}(u^i_{S_j} - u^j_{S_j}) = 0 \tag{36}$$

where λ_i, ν_i and μ_i denote the lagrange multipliers associated with equations (8)-(10), respectively.

Defining $\psi_i = \lambda_i e^{-\delta t}$, $\chi_i = \nu_i e^{-\delta t}$ and $\phi_i = \mu_i e^{-\delta t}$, the first-order conditions reduce to:

$$\partial c_i \quad : \quad u^i_{c_i}(1+\psi_i) = \phi_i \tag{37}$$

$$\partial S_i \quad : \quad \frac{d\psi_i}{dt} + u^i_{S_i}(1+\psi_i) - u^j_{S_i}(1-\chi_i) = 0 \tag{38}$$

$$\partial S_j \quad : \quad \frac{d\chi_i}{dt} - u^j_{S_j}(1-\chi_i) + u^i_{S_j}(1+\psi_i) = 0 \tag{39}$$

Differentiating both sides of the first-order condition (37) with respect to t, yields:

$$(u^i_{c_i,c_i}\frac{dc_i}{dt} + u^i_{c_i,S_i}\frac{dS_i}{dt} + u^i_{c_i,S_j}\frac{dS_j}{dt})(1+\psi_i) + u^i_{c_i}\frac{d\psi_i}{dt} = \frac{d\phi_i}{dt} \tag{40}$$

Dividing by $u^i_{c_i}(1+\psi_i) = \phi_i$, substituting (38) and noting that $\frac{\frac{d\phi_i}{dt}}{\phi_i} = -\delta$, yields the euler equation shown in (11) and repeated below for completeness:

$$\frac{1}{u^i_{c_i}}(u^i_{c_i,c_i}\frac{dc_i}{dt} + u^i_{c_i,S_i}\frac{dS_i}{dt} + u^i_{c_i,S_j}\frac{dS_j}{dt}) - u^i_{S_i} + u^j_{S_i}\frac{1-\chi_i}{1+\psi_i} = -\delta \tag{41}$$

6.2.2 Exploiting the Cobb-Douglas payoff function

When the payoff function $u^i(c_i, S_i, S_j)$ is of the Cobb-Douglas form as shown in equation (12), we have:

$$\frac{u^i_{c_ic_i}}{u^i_{c_i}} = (\alpha_i - 1)c_i^{-1};\ \frac{u^i_{c_iS_i}}{u^i_{c_i}} = \beta_i\frac{1}{S_i};\ \ \frac{u^i_{c_iS_j}}{u^i_{c_i}} = -\beta_i\frac{1}{S_j};\ u^i_{S_i} = \beta_i\frac{\frac{dS_i}{dt}}{S_i} + \delta\beta_i;\ \text{and } u^j_{S_j} = \beta_j\frac{\frac{dS_j}{dt}}{S_j} + \delta\beta_j$$

where the last two expressions follow from equations (8) and (9), respectively. Substituting into the general euler equation, we obtain the euler equation shown in (13) and repeated below for completeness:

$$(\alpha_i - 1)\frac{\frac{dc_i}{dt}}{c_i} - u^j_{S_j}\frac{\beta_i}{\beta_j} + u^j_{S_i}\frac{1-\chi_i}{1+\psi_i} + \delta = 0 \tag{42}$$

6.2.3 Exploiting symmetry

When the two politicians have the same budget, the same initial stock of goodwill and the same payoff function parameters, i.e., $B_i = B_j = B$, $S_i(t_0) = S_j(t_0) = S(t_0)$, $\alpha_i = \alpha_j = \alpha$ and $\beta_i = \beta_j = \beta$ we look for symmetric equilibria where $c_i(t) = c_j(t) = c(t)$ and $S_i(t) = S_j(t) = S(t)$.

Under these symmetry conditions, we have $u^i_{S_i} = u^j_{S_j}$ and $u^i_{S_j} = u^j_{S_i}$ and $u^i_{S_i} = -u^j_{S_i}$. Substituting these into the first-order conditions (38) and (39), we deduce that $\frac{d\psi_i}{dt} = -\frac{d\chi_i}{dt}$. Combining this with the free-endpoint transversality conditions $\psi(T) = \chi(T) = 0$, we obtain that $\psi_i(t) = -\chi_i(t)$. Hence, the euler equation reduces to:

$$(\alpha - 1)\frac{\frac{dc}{dt}}{c} - 2\beta\frac{c^\alpha}{S} + \delta = 0 \tag{43}$$

We now invoke equation (8). Re-arranging terms, differentiating both sides with respect to t and imposing symmetry yields:

$$c^\alpha = \frac{dS}{dt} + \delta S \Rightarrow \tag{44}$$

$$\frac{\frac{dc}{dt}}{c} = \frac{1}{\alpha c^\alpha}[\frac{d^2S}{dt^2} + \delta\frac{dS}{dt}] \tag{45}$$

Substituting equations (44) and (45) into the euler equation in (43) yields the differential equation in (14).

6.2.4 The general differential equation

We can rewrite the differential equation (14) as

$$\frac{\frac{d^2S}{dt^2} + \delta\frac{dS}{dt}}{\frac{dS}{dt} + \delta S} = A\frac{\frac{dS}{dt}}{S} + B, \qquad A = -\frac{2\alpha\beta}{1-\alpha}, \qquad B = \frac{\alpha\delta}{1-\delta}(1-2\beta). \tag{46}$$

Rewriting this in log form and integrating over t, we obtain

$$\ln\left(\frac{dS}{dt} + \delta S\right) = A\ln S + Bt + K_1(\theta),$$

where we make explicit the dependence of the integration constant, K_1, on the parameter vector $\theta \equiv \{\alpha, \beta, \delta\}$. Taking exponents, we have a first order differential equation for the stock of goodwill, S:

$$\frac{dS}{dt} + \delta S = e^{K_1(\theta)} e^{Bt} S^A. \tag{47}$$

Multiplying through by S^{-A} and introducing the change of variables $v = S^{1-A}$, we have

$$\frac{\frac{dv}{dt}}{1-A} + \delta v = e^{K_1(\theta)} e^{Bt}. \tag{48}$$

Multiplying through by $(1-A)e^{\delta(1-A)t}$

$$\frac{dv}{dt} e^{\delta(1-A)t} + \delta(1-A) v e^{\delta(1-A)t} = (1-A) e^{K_1(\theta)} e^{(B+\delta(1-A))t}, \tag{49}$$

and integrating, we obtain

$$v e^{\delta(1-A)t} = K_1^*(\theta) e^{(B+\delta(1-A))t} + K_2(\theta), \qquad K_1^*(\theta) = \frac{(1-A)e^{K_1(\theta)}}{B+\delta(1-A)}, \tag{50}$$

where we note the dependence of the second integration constant, K_2, on the parameter vector θ; and $K_1^*(\theta)$ is a transformation of the original integration constant. In this step, we implicitly assume that $\delta \neq 0$.[7] We can then solve for v:

$$v = K_1^*(\theta) e^{Bt} + K_2(\theta) e^{-\delta(1-A)t}, \tag{51}$$

Solving for S, we have:

$$S = (K_1^*(\theta) e^{Bt} + K_2(\theta) e^{-\delta(1-A)t})^{\gamma}, \tag{52}$$

corresponding to equation (19) in the text.

6.3 Derivation of the closed-loop equilibrium

This section of the appendix provides a detailed derivation of the closed-loop equilibrium for symmetric games.

6.3.1 Euler equation for general games

The first-order conditions for politician i are given by:

$$\partial c_i \; : \; e^{\delta t} u^i_{c_i} - \mu_i + \lambda_i u^i_{c_i} = 0 \tag{53}$$

$$\partial S_i \; : \; \frac{d\lambda_i}{dt} + \lambda_i u^i_{S_i} - \delta \lambda_i + \nu_i u^j_{S_i} + e^{\delta t}(u^i_{S_i} - u^j_{S_i}) - e^{\delta t} u^j_{c_j} \frac{\partial c_j}{\partial S_i} + \nu_i u^j_{c_j} \frac{\partial c_j}{\partial S_i} = 0 \tag{54}$$

$$\partial S_j \; : \; \frac{d\nu_i}{dt} + \nu_i u^j_{S_j} - \delta \nu_i + \lambda_i u^i_{S_j} + e^{\delta t}(u^i_{S_j} - u^j_{S_j}) - e^{\delta t} u^j_{c_j} \frac{\partial c_j}{\partial S_j} + \nu_i u^j_{c_j} \frac{\partial c_j}{\partial S_j} = 0 \tag{55}$$

[7] In the case where $\delta = 0$, the right hand side of equation (49) simplifies to a constant. The resulting paths of consumption and the stock of goodwill are discussed in the text.

where λ_i, ν_i and μ_i denote the lagrange multipliers associated with equations (8)-(10), respectively.

Defining $\psi_i = \lambda_i e^{-\delta t}$, $\chi_i = \nu_i e^{-\delta t}$ and $\phi_i = \mu_i e^{-\delta t}$, the first-order conditions reduce to:

$$\partial c_i \;:\; u^i_{c_i}(1+\psi_i) = \phi_i \tag{56}$$

$$\partial S_i \;:\; \frac{d\psi_i}{dt} + u^i_{S_i}(1+\psi_i) - (u^j_{S_i} + u^j_{c_j}\frac{\partial c_j}{\partial S_i})(1-\chi_i) = 0 \tag{57}$$

$$\partial S_j \;:\; \frac{d\chi_i}{dt} - (u^j_{S_j} + u^j_{c_j}\frac{\partial c_j}{\partial S_j})(1-\chi_i) + u^i_{S_j}(1+\psi_i) = 0 \tag{58}$$

Differentiating both sides of the first-order condition (56) with respect to t, yields:

$$(u^i_{c_i,c_i}\frac{dc_i}{dt} + u^i_{c_i,S_i}\frac{dS_i}{dt} + u^i_{c_i,S_j}\frac{dS_j}{dt})(1+\psi_i) + u^i_{c_i}\frac{d\psi_i}{dt} = \frac{d\phi_i}{dt} \tag{59}$$

Dividing by $u^i_{c_i}(1+\psi_i) = \phi_i$, substituting (57) and noting that $\frac{\frac{d\phi_i}{dt}}{\phi_i} = -\delta$, yields the following euler equation for the closed-loop equilibrium:

$$\frac{1}{u^i_{c_i}}(u^i_{c_i,c_i}\frac{dc_i}{dt} + u^i_{c_i,S_i}\frac{dS_i}{dt} + u^i_{c_i,S_j}\frac{dS_j}{dt}) - u^i_{S_i} + (u^j_{S_i} + u^j_{c_j}\frac{\partial c_j}{\partial S_i})\frac{1-\chi_i}{1+\psi_i} = -\delta \tag{60}$$

6.3.2 Exploiting the Cobb-Douglas payoff function

When the payoff function $u^i(c_i, S_i, S_j)$ is of the Cobb-Douglas form, we can re-arrange (56) to obtain:

$$c_i = (\frac{\phi_i}{\alpha(1+\psi_i)})^{\frac{1}{\alpha-1}}(\frac{S_i}{S_j})^{\frac{\beta}{1-\alpha}} \tag{61}$$

where we assumed that the coefficient of risk aversion, α, and the momentum effects parameter, β, are common across the two agents.

In turn, this implies:

$$u^j_{c_j}\frac{\partial c_j}{\partial S_i} = -\frac{\alpha\beta}{1-\alpha}\frac{c_j^\alpha}{S_i}(\frac{S_j}{S_i})^\beta = \frac{\alpha}{1-\alpha}u^j_{S_i} \tag{62}$$

Substituting (62) and the expressions derived above in Appendix 6.2.2 into the euler equation yields:

$$(\alpha-1)\frac{\frac{dc_i}{dt}}{c_i} - u^j_{S_j} + u^j_{S_i}(\frac{1}{1-\alpha})\frac{1-\chi_i}{1+\psi_i} + \delta = 0 \tag{63}$$

6.3.3 Exploiting symmetry

When the two politicians have the same budget and initial stock of goodwill, i.e., $B_i = B_j = B$ and $S_i(t_0) = S_j(t_0) = S(t_0)$, we look for symmetric equilibria where $c_i(t) = c_j(t) = c(t)$ and $S_i(t) = S_j(t) = S(t)$.

Under these symmetry conditions, we have $u^i_{S_i} = u^j_{S_j}$ and $u^i_{S_j} = u^j_{S_i}$ and $u^i_{S_i} = -u^j_{S_i}$. Substituting these into the first-order conditions (57) and (58), we deduce that $\frac{d\psi_i}{dt} = -\frac{d\chi_i}{dt}$. Combining this with the transversality conditions $\psi(T) = \chi(T) = 0$, we obtain that $\psi_i(t) = -\chi_i(t)$. Hence, the euler equation reduces to:

$$(\alpha - 1)\frac{\frac{dc}{dt}}{c} - 2\beta\frac{2-\alpha}{2-2\alpha}\frac{c^{\alpha}}{S} + \delta = 0 \tag{64}$$

This is identical to equation (43) derived in the context of the open-loop equilibrium, but for the fact the term β in equation (43) is replaced by the term $\beta\frac{2-\alpha}{2-2\alpha}$ in equation (64). Hence, on the equilibrium path of the closed-loop equilibrium the stock of goodwill must satisfy the differential equation shown in (21), which replaces the term β in (14) with the term $\beta\frac{2-\alpha}{2-2\alpha}$.

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