Reaching for the prize: investment with uncertain costs*

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Abstract

Key words: Investment, uncertainty. *JEL classifications*:

1 Introduction and motivation

In many investment scenarios the return is (relatively) certain, but the investor is uncertain about the cost (or the effort) required to obtain the return. For example, the United Kingdom has committed itself to purchase up to 300 million doses of a malaria vaccine, if any company can invent it (Surowiecki 2004). Given that there is a limited market for these types of vaccines, it is then the case that the return to a firm from developing the vaccine is more or less known - but it is the cost of developing the new drug that is uncertain.[1] Similar situation exist in construction. The size of an oil reserve may be known to a mining company, but the cost of extracting it may not be; likewise, the benefits of a new bridge or dam may be obvious, but the funds required to design and build such a project could be difficult to accurately predict. An individual,

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[1]This practice, advocated by Kremer (2001), involves a government agency, or an international body such as the World Bank, committing to provide a return for the development of drugs of major benefit in developing countries.

also, may know the benefits of achieving some person accomplishment (education, being able to play a musical instrument, etc) but there could be uncertainty about the effort cost required.

Inspired by the examples above, we investigate the following problem. If a monopolist makes a new discovery (for example a vaccine) there is a certain benefit. However, the costs of making the discovery are not known with certainty; only the possible distribution is known. In the first period a monopolist must decide how much to invest. If the amount equals or exceeds the required amount the discovery is realized. If this amount of resources is less than required amount the invention is not realized and the game proceeds to the next period. In this case the firm has learned something about the costs required, and can decide how much to invest in the next period, and so on. Also, costs expended by the firm accumulate over the periods (so that once the total costs expended exceed the cut-off, the discovery is realized). Finally, the benefits of the project only accumulate once the discovery is made (in other words when the project is finished). The essential tradeoff facing the firm is: incurring unnecessarily costs of investment by overshooting the amount required to complete the project; and incurring costs of delay by investing too little and undershooting the amount required to complete the project. Given we have a closed-form solution to this problem, we can analyze how investment and the probability of success changes with various aspects to the problem. For example, we examine how investment is altered by changes in the return and the distribution. **This is a natural framework to describe staged financing. For example, construction of the Opera House in Sydney was done in 4 stages.**

Grossman and Shapiro (1986) study a similar scenario in their investigation of R&D programs. In a project with a given benefit but where the progress required for its completion is unknown, they find that there is a positive relationship between the rate of investment and the expected value of the project.

Given our different approach, our model provides a complement to the work of Grossman and Shapiro (1986). In their model, time is continuous. Here, time is discrete. This reflects that many investment decisions are lumpy and cannot be adjusted continuously. For example, a board of directors may need to decide on the funding of a research laboratory's funding for the coming year of investment. Similarly, funding of a public project is typically provided in tranches. Second, Grossman and Shapiro (1986) consider projects with convex costs, while we focus on a uniform distribution of costs. Finally, note, while there are several similarities (investment costs are sunk and irreversible, for example) the approach we take here is distinct from the framework of Dixit and Pindyck (1995) who investigated the scenario in which there is an incentive to wait before investing as some uncertainty is resolved by waiting. A crucial distinction is that in our model it is only by the act of investing that the monopolist can update her information regarding the costs of the project.

For an excellent review of literature under uncertainty see Pindyck (1991).

Roberts and Weitzman (1981) get properties of a more general model but they did not derive a closed form solution.

2 Model specification

Let us consider the following investment model in discrete time. An agent has an option to finance a project with a known return $R > 0$, but unknown costs c. The costs are assumed to be distributed uniformly on $[0, x]$[2], where x is given. It means that if $0 \leq I \leq x$ is invested then the project is either finished immediately (with probability I/x) or with probability $1 - I/x$ the investment will continue next period. The costs in the next period will be distributed uniformly on $[0, y]$, where $y := x - I$. We assume that the agent is risk-neutral: he maximizes an expected discounted sum of net returns, where δ is a discount factor.

This investment game can be described by the following Bellman equation

$$V(x) = \max_{0 \leq I \leq x} \{-I + RI/x + \delta(1 - I/x)V(y)\}. \tag{1}$$

[2] We focus our attention on uniform distribution because this represents a situation when the maximal value of the costs is known and all positive values of the costs below the maximal one are equally possible.

The first term in this equation is the costs of investment incurred by the agent. The second term is the expected value of finishing the project after the first period. The last term is the expected value of project continuation.

In terms of $y = x - I$ the problem is

$$V(x) = \max_{0 \le y \le x} \{(R - x)(x - y)/x + \delta y V(y)/x\}. \tag{2}$$

3 Solving the model

Note, x, y, V, R are of the same unit measure. For convenience, we make the following substitution

$$x := x/R, \;\; y := y/R, \;\; V := V/R, \;\; R := 1; \tag{3}$$

to work with unit free variables. Equation 2 transforms to

$$V(x) = \max_{0 \le y \le 1} \{(1 - y/x)(1 - x) + \delta y V(y)/x\} =: BV(x). \tag{4}$$

It makes sense to expect the value of the project to be non-negative and not greater than one. In addition, when the costs are zero the value of the project is equal to one.

Assumption 1. *V(x) satisfies the following conditions* $0 \le V(x) \le 1 \;\forall\; x$ *and* $V(0) = 1$.

Therefore, we will look for a solution of equation 4 in the class of continuous functions defined on R_+ with upperbound 1. As usual, the natural ordering and uniform metrics are assumed.

Result 1. *When* $\delta = 1$*, function* $\hat{V}(x) := (1 - x/2)^+$ *is the unique solution of problem 4. The optimal strategy is* $\quad Y(x) = x \quad \forall \quad x \ge 0$.

Proof: Substitute $\hat{V}(x)$ into equation 4 and confirm that it is satisfied. Note that Assumption 1 is satisfied, which makes the solution unique. □

From economic context the solution of equation 4 increases in δ and consequently, $V \leq \hat{V} \quad \forall \quad \delta < 1$. Further, it is convenient to introduce function $\Psi(x) := xV(x)$, and then equation 4 becomes

$$\Psi(x) = \max_{0 \leq y \leq x} \{(1-x)(x-y) + \delta\Psi(y)\} =: B\Psi(x). \tag{5}$$

The following Lemma follows from the contraction mapping theorem.

Lemma 1. *When $\delta < 1$ the operator on the right side of equation 5 is a contraction operator. Therefore, equation 5 has a unique solution, that can be obtained as a limit of the following increasing with k sequence*

$$\Psi_0 \equiv 0 \quad , \quad \Psi_k := B\Psi_{k-1} \quad k = 1, 2, ... \quad \square \tag{6}$$

Let us specify the necessary condition for optimal value $y = Y(x)$ to be an internal point in $[0, x]$. Denote the expression under maximization in equation 5 as $A(y)$, then the necessary condition is

$$A'(y) = -(1-x) + \delta\Psi'(y) = 0. \tag{7}$$

3.1 Construction of sequence Ψ_k

Note that construction of sequence $\{\Psi^k\}$ with the help of Lemma 1 is equivalent to using the backward induction argument. Let us start from the end of the investment process. Specifically, what will be the value of the game, if the agent is allowed to invest only once. In that case the agent's objective function is linear in investment and he either invests whole x in one period or does not invest at all.

$$\Psi_1(x) = \max_{0 \leq y \leq x} \{(1-x)(x-y)\} = \begin{cases} (1-x)x & \text{if} \quad x < u_1, \\ 0 & \text{if} \quad x \geq u_1, \end{cases}$$

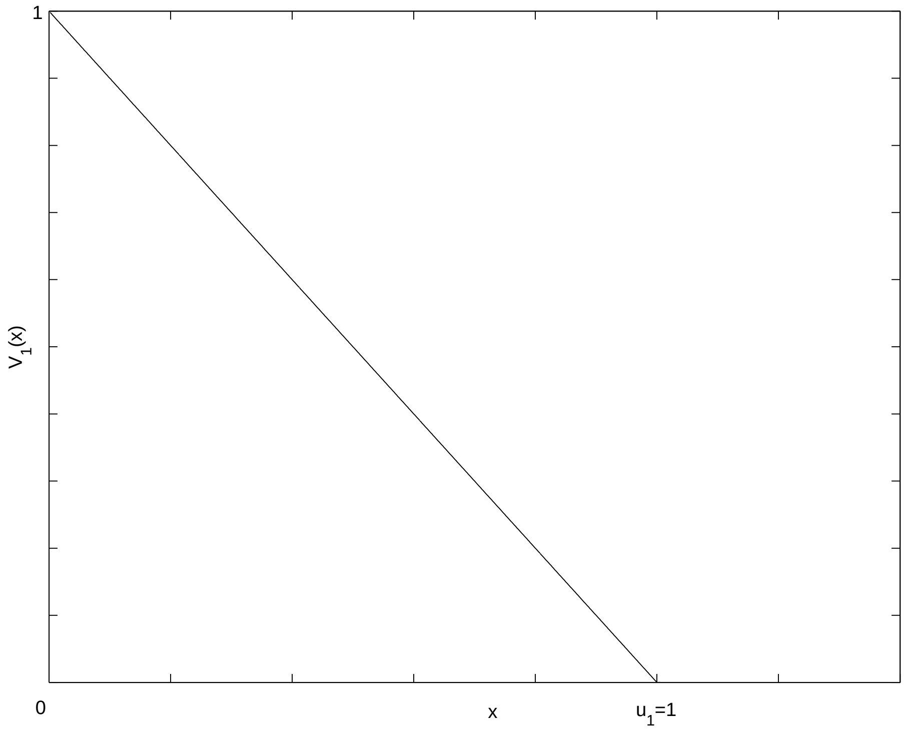

Figure 1: Value function $V_1(x)$

where $u_1 = 1$ is the positive root of polynomial $(1 - x)x$. Note that the project is started only if it has a positive value. The optimal strategy is

$$Y_1(x) = \begin{cases} 0 & \text{if} \quad x < u_1, \\ x & \text{if} \quad x \geq u_1. \end{cases}$$

The value function is

$$V_1(x) = \Psi_1(x)/x = \begin{cases} 1 - x & \text{if} \quad x < u_1, \\ 0 & \text{if} \quad x \geq u_1. \end{cases}$$

The value function in the case when the agent can only invest once is plotted in Figure 1. Note that it does not depend on δ because the project is finished without any delay.

Now let us find the value of the project if the agent is allowed to invest at most twice. This means that for some values of x the agent's optimal strategy is to invest twice, for some other values is to invest once, and finally for the remainder of values he will not invest at all.

$$\Psi_2(x) = \max_{0 \le y \le x} \{(1-x)(x-y) + \delta\Psi_1(y)\}. \tag{8}$$

Let us show that if $\Psi_2(x) \ne \Psi_1(x)$ then $\Psi_1(y) = (1-y)y$, i.e the value of the two-period game is different from the value of one-period game only if the agent finds it optimal to invest twice.

If the optimal value $y = Y_2(x) > u_1$ (which is possible only when $x > u_1$) function $\Psi_2(x)$ has the same right side as function $\Psi_1(x)$, which means $\Psi_2(x) = \Psi_1(x) = 0$. Consequently, in the case when functions are different the optimal value $y = Y_2(x) \le u_1$, i.e.

$$\Psi_2(x) > \Psi_1(x) \quad \Rightarrow \quad 0 < Y_2(x) \le u_1. \tag{9}$$

Thus in the case when the value of the two-period game is different from the value of one-period game, $\Psi_1(y) = (1-y)y$ and equation 8 becomes

$$\Psi_2(x) = \max_{0 \le y \le x} \{(1-x)(x-y) + \delta y(1-y)\}. \tag{10}$$

From condition 7 it follows

$$A_1'(y) = -(1-x) + \delta(1-2y) = 0.$$

Consequently,

$$y = \frac{x-(1-\delta)}{2\delta}. \tag{11}$$

Substitute y into equation 10 to get

$$\Psi_2(x) = \begin{cases} -\left(1-\frac{1}{4\delta}\right)(1-x)^2 + \frac{1}{2}(1-x) + \frac{\delta}{4} & \text{if} \quad x < u_2, \\ 0 & \text{if} \quad x \ge u_2; \end{cases} \tag{12}$$

where $u_2 = \frac{(1+\sqrt{\delta})^2}{1+2\sqrt{\delta}}$ is the positive root of polynomial in (12). The solution of two-period game is described by (12) only when in (11) $y \geq 0$, otherwise the project is financed in one period and $\Psi_2(x) = \Psi_1(x)$. Let us find x_1, the point where there is a change in regime. Equating in (11) $y = 0$ results in $x_1 = 1 - \delta$. Consequently,

$$\Psi_2(x) = \begin{cases} \Psi_1(x) & \text{if} \quad x < x_1, \\ -\left(1 - \frac{1}{4\delta}\right)(1-x)^2 + \frac{1}{2}(1-x) + \frac{\delta}{4} & \text{if} \quad x_1 \leq x < u_2, \\ 0 & \text{if} \quad x \geq u_2. \end{cases} \tag{13}$$

The optimal strategy is

$$Y_2(x) = \begin{cases} 0 & if \quad x < x_1, \\ \frac{x-(1-\delta)}{2\delta} & if \quad x_1 \leq x < u_2, \\ x & if \quad x \geq u_2. \end{cases} \tag{14}$$

The value function is

$$V_2(x) = \Psi_2(x)/x = \begin{cases} V_1(x) & \text{if} \quad x < x_1, \\ \frac{1}{x}\left(-\left(1 - \frac{1}{4\delta}\right)(1-x)^2 + \frac{1}{2}(1-x) + \frac{\delta}{4}\right) & \text{if} \quad x_1 \leq x < u_2, \\ 0 & \text{if} \quad x \geq u_2. \end{cases} \tag{15}$$

The value function in the case when the agent can invest not more than twice is plotted in Figure 2.

Note that $\forall\ 0 < \delta < 1$ $V_2(x_1) = \delta > 0$, which means that the value function always has at least 2 steps. [3]

Now let us find the value of the game if the agent is allowed to invest not more than k times. This means that for some values of x the agent's optimal strategy is to invest k times, for some other values is to invest $k-1$ times, etc.

Note that both $\Psi_2(x)$ and $\Psi_1(x)$ are functions in the following form:

[3] Result 2 describes how many steps the value function has in a general case.

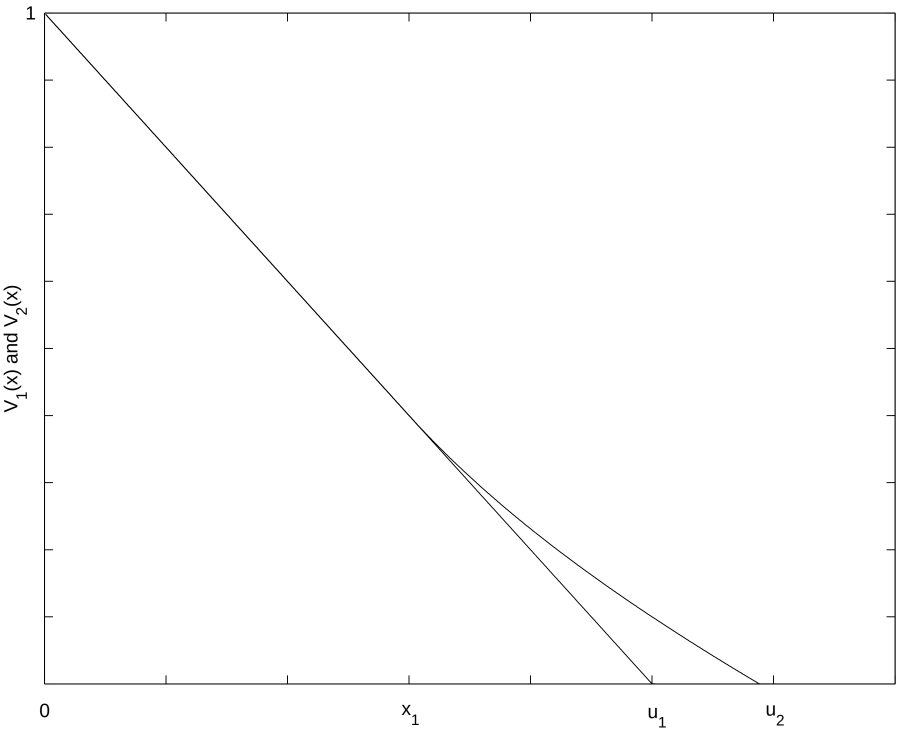

Figure 2: $V_2(x)$ and $V_1(x)$, $\delta = 0.4$

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x) & \text{if} \quad x < x_{k-1}, \\ -\frac{a_k}{2}(1-x)^2 + b_k(1-x) + c_k, & \text{if} \quad x_{k-1} \le x < u_k, \\ 0 & \text{if} \quad x \ge u_k. \end{cases} \tag{16}$$

From above we have

$$a_1 = 2, \quad b_1 = 1, \quad c_1 = 0; \quad a_2 = 2 - \frac{1}{2\delta}, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4}.$$

The transition in general can be described by the following equation

$$\Psi_k(x) = \max_{0 \le y \le x} \{(1-x)(x-y) + \delta \Psi_{k-1}(y)\} =: B\Psi_{k-1}(y), \tag{17}$$

where function Ψ_{k-1} in (17) is given by (16). The next function Ψ_k is equal to $B\Psi_{k-1}$.

Let us find a_k, b_k, and c_k for any k. From condition (7) it follows [4]

$$1 - x = \delta \Psi'_{k-1}(y) = \delta[a_{k-1}(1-y) - b_{k-1}]. \tag{18}$$

Therefore,

$$y(x) = \frac{(a_{k-1} - b_{k-1})\delta + x - 1}{a_{k-1}\delta}. \tag{19}$$

Substitute (16) into (17) to get

$$\Psi_k(x) = (1-x)(x-y) + \delta\left(-\frac{a_{k-1}}{2}(1-y)^2 + b_{k-1}(1-y) + c_{k-1}\right). \tag{20}$$

Substitute (19) into (20) and use (16) to derive

$$a_k = 2 - \frac{1}{\delta a_{k-1}}, \quad b_k = \frac{b_{k-1}}{a_{k-1}}, \quad c_k = \delta\left[c_{k-1} + \frac{b_{k-1}^2}{2a_{k-1}}\right]. \tag{21}$$

This system of difference equations has the following solution derived in the Appendix.

$$a_k = \frac{\sin{(k+1)\varphi}}{v \sin k\varphi}, \;\; b_k = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \;\; c_k = \frac{v^{2k-1}\sin{(k-1)\varphi}}{2\sin k\varphi}, \;\; k \geq 2, \tag{22}$$

where $v = \sqrt{\delta}$ and $\varphi = \arccos v$.

To find the point where regime changes from Ψ_{k-1} to Ψ_k, we iterate equation (19) where $y = x_{k-1}$ and $x = x_k$. It gives the following difference equation

$$x_k - 1 = a_{k-1}\delta(x_{k-1} - 1) + b_{k-1}\delta, \tag{23}$$

with the initial condition $x_1 = 1 - \delta$. This difference equations has the following solution derived in the Appendix

$$x_k = 1 - v^k \cos k\varphi. \tag{24}$$

[4] We apply a similar argument to above that $\Psi_k > \Psi_{k-1}$ only when $x_{k-2} < y < u_{k-1}$. When $y \leq x_{k-2} \quad \Psi_k = \Psi_{k-1} = \Psi_{k-2}$, while when $y \geq u_{k-1} \quad \Psi_k = \Psi_{k-1} = 0$.

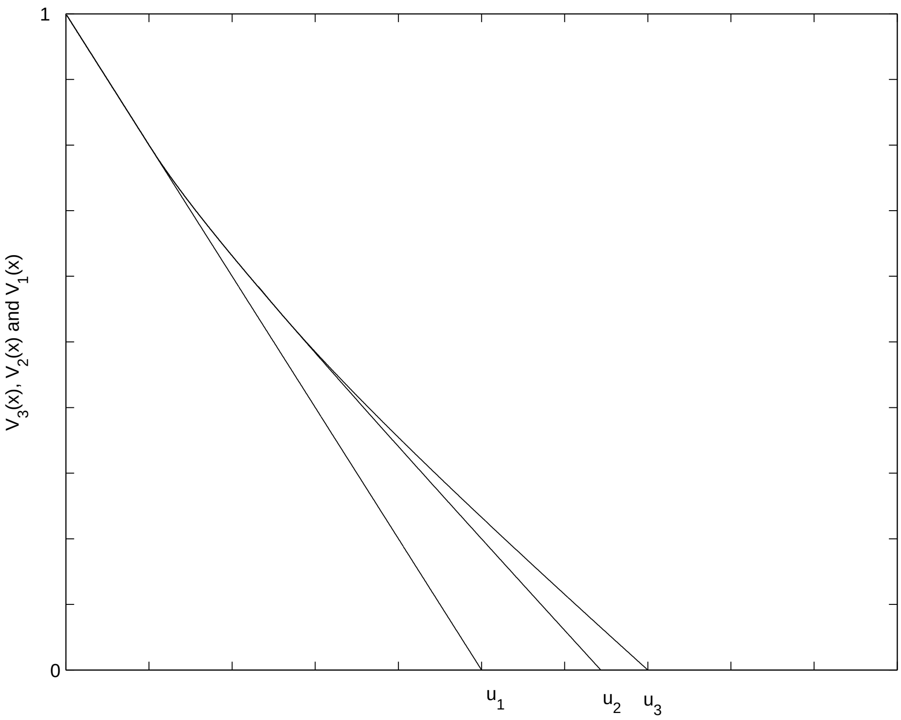

Figure 3: $V_3(x)$, $V_2(x)$ and $V_1(x)$, $\delta = 0.8$

To find u_k, the minimal value where $\Psi_k = 0$ we equate

$$-\frac{a_k}{2}(1 - u_k)^2 + b_k(1 - u_k) + c_k = 0 \tag{25}$$

and derive [5]

$$u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}. \tag{26}$$

Thus $\Psi_k(x)$ is described by (16) with coefficients given by (22) and points of regime change given by (24) and (26). The value function of the game when the agent is allowed to invest not more than k times is $V_k(x) = \Psi_k(x)/x$. The value function in the case when the agent can invest not more than 3 times is plotted in Figure 3.

[5] See the Appendix.

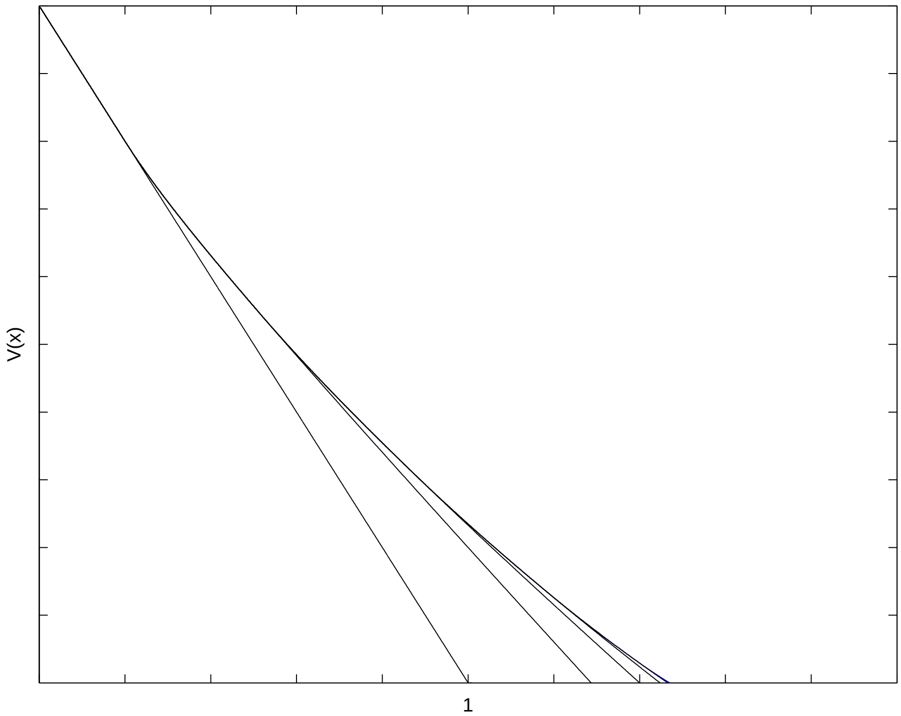

Figure 4: $V(x)$, $\delta = 0.8$

3.2 Construction of value function $V(x)$

Now let us address the question of what is the minimum k that $V(x) \equiv V_k(x)$; denote this value by $\bar{K}$. To answer this question observe that [6]

$$\Psi_k(x_k) = \frac{v^{2k-1}\sin{(k+1)\varphi}\sin{k\varphi}}{2} = \frac{v^{2k}a_k\sin^2{k\varphi}}{2}, \tag{27}$$

which suggests that $\Psi_k(x_k)$ is non-negative if and only if a_k is non-negative. Consequently,

$$\bar{K} = \max\{k| \quad a_k(\delta) > 0\} + 1 \qquad \delta \in (0,1). \tag{28}$$

Let us define point δ_k to be the root of the following equation [7]

$$\delta_k \qquad : \qquad a_{k-1}(\delta) = 0 \quad k = 3, 4, ... \tag{29}$$

[6]For details see the Appendix.
[7]Note that indexes are designed in such a way that for $\delta \geq \delta_k$ number of zones is at least k.

Result 2. *For any natural number k, equation 29 has the following unique solution*

$$\delta_k = \cos^2 \frac{\pi}{k} \qquad k = 3, 4, \dots \tag{30}$$

Proof: The proof is given in the Appendix. □

Thus for any δ we can first find $\bar{K}$ using result 2. Then we can construct the value function $V(x) = V_{\bar{K}}(x)$ and the optimal strategy $Y(x) = Y_{\bar{K}}(x)$.

Let us illustrate the algorithm of finding the solution in the case when $\delta = 0.8$. Using result 2 we can find that $\delta_6 < 0.8 < \delta_7$, which means $\bar{K} = 6$ and $V(x) = V_6(x)$. The value function when $\delta = 0.8$ is plotted in Figure 4.

4 Appendix

4.1 Derivation of a_k, b_k and c_k

Using

$$a_k = 2 - \frac{1}{\delta a_{k-1}}, \quad b_k = \frac{b_{k-1}}{a_{k-1}}, \quad c_k = \delta \left[c_{k-1} + \frac{b_{k-1}^2}{2a_{k-1}} \right] \tag{21}$$

let us show that

$$a_k = \frac{\sin{(k+1)\varphi}}{v \sin k\varphi}, \;\; b_k = \frac{v^{k-1} \sin \varphi}{\sin k\varphi}, \;\; c_k = \frac{v^{2k-1} \sin{(k-1)\varphi}}{2 \sin k\varphi}, \;\; k \geq 2, \tag{22}$$

where $v = \sqrt{\delta}$ and $\varphi = \arccos v$.

Define

$$P_k := v^k \cdot \prod_{j=1}^{k} a_j \qquad k = 1, 2, \dots \tag{31}$$

Using (21) we get the following second-order difference equation

$$P_{k+1} = vP_k \cdot \left(2 - \frac{1}{\delta a_k} \right) = 2vP_k - P_{k-1} \quad k \geq 2. \tag{32}$$

The initial conditions are $P_0 = 1$ and $P_1 := 2v$. The characteristic equation

$z^2 - 2vz + 1 = 0$ has two complex roots

$$z_1 = v + is, \quad z_2 = v - is, \qquad s := \sqrt{1 - v^2} > 0. \tag{33}$$

Denote $\varphi := \{\arg z_1 \in [0, \pi/2]\} = \arccos v$, then $z_{1,2} = e^{\pm i\varphi}$. Further, write solutions to equation (32) in form $P_k = Az_1^{k+1} - Bz_2^{k+1}$ and use initial conditions to get $A = B = -\frac{i}{2\sin\varphi}$. Consequently

$$P_k = -\frac{i}{2\sin\varphi}(e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}) = \frac{\sin[(k+1)\varphi]}{\sin\varphi}. \tag{34}$$

Apply (31) and (21) to get

$$a_k = \frac{P_k}{vP_{k-1}} = \frac{\sin(k+1)\varphi}{v\sin k\varphi}, \tag{35}$$

$$b_k = \frac{b_{k-1}}{a_{k-1}} = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \tag{36}$$

$$c_k = \delta\left[c_{k-1} + \frac{b_{k-1}^2}{2a_{k-1}}\right] == \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi}. \tag{37}$$

4.2 Derivation of x_k

Using

$$x_k - 1 = a_{k-1}\delta(x_{k-1} - 1) + b_{k-1}\delta, \tag{23}$$

and the initial condition $x_1 = 1 - \delta$ *let us show that*

$$x_k = 1 - v^k \cos k\varphi. \tag{24}$$

Substitute (22) into (23) to get

$$(x_k - 1)\sin(k-1)\varphi = v(x_{k-1} - 1)\sin k\varphi + v^k \sin\varphi. \tag{38}$$

Substitute the following equality

$$\sin\varphi = \sin k\varphi \cos(k-1)\varphi - \cos k\varphi \sin(k-1)\varphi \tag{39}$$

into (38) to get

$$(x_k - 1 + v^k \cos k\varphi)\sin(k-1)\varphi = (x_{k-1} - 1 + v^{k-1}\cos(k-1)\varphi)v \sin k\varphi. \tag{40}$$

It is easy to see that (24) is the solution to (40) that satisfies the initial condition $x_1 = 1 - \delta$.

4.3 Derivation of u_k

Using

$$-\frac{a_k}{2}(1-u_k)^2 + b_k(1-u_k) + c_k = 0 \tag{25}$$

let us show that

$$u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}. \tag{26}$$

Substitute (22) into (25) to get

$$-(1-u_k)^2 \sin(k+1)\varphi + 2(1-u_k)v^k \sin\varphi + v^{2k}\sin(k-1)\varphi. \tag{41}$$

Solving this quadratic equation results in (26).

4.4 Derivation of formula (27)

Substitute (24) into (16) to get

$$\Psi_k(x_k) = \frac{v^{2k-1}}{2\sin k\varphi}(-\sin(k+1)\varphi\cos^2 k\varphi + 2\sin\varphi\cos k\varphi + \sin(k-1)\varphi). \tag{42}$$

Substitute the following equality

$$\sin(k+1)\varphi = \sin(k-1)\varphi + 2\sin\varphi\cos k\varphi \tag{43}$$

into (42) to get formula (27).

4.5 Proof of Proposition 4

Equation (29) (which defines δ_k) is equivalent to condition $P(k-1) = 0$ (P_k is defined in (31)). Apply (34) to get

$$\varphi_k \quad : \qquad k\varphi = n\pi, \tag{44}$$

where $n \geq 1$ is some natural number which in general can depend on k, i.e. $n = n_k$.

Let us prove by induction that $n_k = 1 \ \forall \ k$. It is easy to see that for $k = 1$ the statement is correct. Substitute $k' := k + 1$ in (44) to get (using assumption that $n_k = 1$ and the fact that $\varphi_k = \arccos\sqrt{\delta_k}$ is monotonically decreasing in k)

$$n_{k+1}\pi = (k+1)\varphi_{k+1} \leq (k+1)\varphi_k = \frac{k+1}{k}\pi < 2\pi.$$

Given that n_{k+1} is a natural number, it has to be the case that $n_{k+1} = 1$. Substitute $n = 1$ in (44) to get $\varphi_k = \pi/k$, which means (30) is proved. □

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