“Theoretical Decompositions of the Cross-Sectional Dispersion of Stock Returns”

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Theoretical Decompositions of the Cross-Sectional Dispersion of Stock Returns

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Abstract
We present theoretical decompositions of cross-sectional return dispersion, assuming either a one-factor model, or a constant parameter model. This allows us to calculate expected return dispersion, based on dispersions in alpha and beta, and their cross-sectional correlation. We find that expected dispersion matches up reasonably well with actual realised dispersion - periods of high expected dispersion correspond to periods of high realised dispersion. Using U.S. equity portfolio data, we find that approximately 80\% of expected dispersion is determined by extreme returns in the market.

Keywords: Cross-Sectional Volatility, Decomposition, Predictability

JEL Classification: G11, G17

1. Introduction
The cross-sectional standard deviation of stock returns, or dispersion as it is frequently called, has attracted an increasing amount of attention in recent years. In the funds management industry, dispersion is of interest because a stock-selector may only reasonably add value when stock returns are disparate - that is, when one set of stocks outperform others. Ankrim and Ding (2002) and De Silva, Sapra and Thorley (2001) find that the dispersion

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in the alphas of managed funds is higher when cross-sectional volatility is higher, but that this dispersion in fund alpha is not related to managerial talent. Connor and Li (2009) find a similar result for hedge fund performance. Gorman, Sapra and Weigand (2010) show that dispersion in stock returns help to forecast the future dispersion in fund manager performance.

In this paper we seek to determine the drivers of cross-sectional volatility. It is of interest as to whether the cross-sectional variation in stock performance is due to variations in systematic risk (as measured by variation in betas), in idiosyncratic components of returns, or in fact in variations due to unpriced return factors (alpha), or some combination thereof. For example, although fund manager alpha may be higher in periods of high cross-sectional volatility, this may also be combined with higher dispersion in betas, which may help aid market timing strategies as well as stock-selection strategies. Campbell, Lettau, Malkiel and Xu (2001) and Xu and Malkiel (2003), and Irvine and Pontiff (2009) suggest that the idiosyncratic components of volatility have been increasing over time. In contrast, more recent research by Brandt, Brav, Graham and Kumar (2010) and Bekaert, Hodrick and Zhang (2012) found that idiosyncratic volatility appears to follow a stationary process with occasional shifts into a high-volatility regime. It remains an open question as to whether the systematic components of stock returns are becoming more disparate, particularly around periods of high volatility. Cross-sectional volatility has been used as a proxy for average idiosyncratic volatility in a number of studies (Campbell et al 2001; Goyal and Santa-Clara, 2003; Irvine and Pontiff, 2009). In the special case of time-invariant betas and homogeneous idiosyncratic risk, cross-sectional volatility exactly coincides with average idiosyncratic volatility when the number of stocks is large (Garcia, Mantilla-Garcia, and Martellini, 2013; Goltz, Guobuizaite, and Martellini, 2011).

An understanding of how cross-sectional dispersion in stock returns has arisen is perhaps the key in our ability to evaluate funds managers. Many studies, since the seminal work of Jensen (1968), have reached the conclusion that active funds managers are unable to beat appropriate risk-adjusted benchmarks (e.g. Elton et al, 1993; Carhart, 1997; Bogle, 1998) net of expenses and trading costs. The review article by Jones and Wermers (2011) provides a thorough exposition of the state of research into active management.

The contribution of our paper is to analyse the mathematical properties of cross-sectional dispersion. There are a number of scattered results in both
the academic and practitioner literature, see for example Stivers (2003), Yu and Sharaiha (2007), and Menchero and Morozov (2011). We try to bring these together. We shall do this under a range of assumptions; generally, we shall make assumptions consistent with the academic and practitioner literature. In particular our interest will be in decomposing cross-sectional dispersion into its various components rather like an analysis of variance exercise.

2. Decomposition of Return Dispersion

2.1. Notation and Definitions

We define our \( (n \times 1) \) vector of returns at time \( t \) (monthly unless noted otherwise) by \( R_t = (r_{it}) \) where \( r_{it} \) is the rate of return to asset \( i \) at time \( t \). The usual definition of dispersion at time \( t \), \( d_t \) is equal to the sample standard deviation of the \( n \) asset returns at time \( t \). In detail,

\[
d_t = \sqrt{\frac{\sum_{i=1}^{n} (r_{it} - \bar{r}_t)^2}{n - 1}}
\]

We use the usual symbol for sample mean (equivalent to an equal-weighted market return)

\[
\bar{r}_t = \frac{\sum_{i=1}^{n} r_{it}}{n}
\]

Much of the intuition behind the use of dispersion as a measure of investment opportunity stems from the following trivial result.

Proposition 1. In a world of constant transaction costs, the additional returns to active management is zero if expected return dispersion is zero.

The proof is immediate; expected return dispersion being zero implies that all assets have the same expected returns; then all long portfolios have the same expected returns and all net portfolios will have expected zero returns; thus no choice of assets will lead to any financial advantage.

This result does not necessarily imply that expected return dispersion increasing will lead to increasing investment opportunity. In any case, it is invariably measured in terms of actual current returns rather than expectations.
We shall work with squared dispersion as it leads to much cleaner analysis. We shall use the symbol $D_t$ for this quantity; it is clearly a non-negative random variable. We shall also use the Cross-Sectional Variance ($CSV$) of the series $R_t$, $(CSV(R_t))$. Let $M$ be the usual $n$ by $n$ idempotent matrix associated with the sample variance. We use the notation $R_t \sim (\mu_t, \Omega_t)$ to mean distributed with mean vector $\mu_t$ and covariance matrix $\Omega_t$. Now,

$$D_t = \frac{R_t' M R_t}{n-1} = CSV(R_t)$$ (1)

then it follows that

$$\mathbb{E}(D_t) = \frac{\mu_t' M \mu_t + tr(M \Omega_t)}{n-1}$$ (2)

Given $M = I_n - \frac{i_n i_n'}{n}$, we see that $tr(M \Omega_t) = tr(\Omega_t) - (i_n' \Omega i_n)/n$. Therefore,

$$\mathbb{E}(D_t) = \frac{\mu_t' M \mu_t + tr(\Omega_t) - \frac{i_n' \Omega i_n}{n}}{n-1} = CSV(\mu_t) + \frac{tr(\Omega_t) - \frac{i_n' \Omega i_n}{n}}{n-1}$$ (3)

The expression in equation (3) has three terms; the first can be described by variation (squared dispersion) in time $t$ expected returns; if we have for example, style divergence, then we might expect this to be large. Also, this can be zero whilst, overall, expected dispersion could be very large. At this stage we do not differentiate between alpha- or beta-based returns.

The second term corresponds to the average time $t$ level of variance. If the average stock exhibits large variance at time $t$, expected dispersion of returns is also large.

The third term corresponds to time $t$ correlation and risk. Dispersion was considered as a measure of cross-sectional correlation by Solnik and Roulet (2000). If we assume for the moment that some conditioning information is in the mean, then the remaining “common factors” may be in the covariance matrix. To the extent that we might think these correlations are positive, which is typically the case, this third term will lead to a reduction in dispersion via an increase in correlation. If $\Omega_t$ is diagonal, then the last two terms in (3) collapse to $tr(\Omega_t)/n$. 

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2.2. Dispersion, Sharpe’s One-Factor Model, and the Capital Asset Pricing Model

To refine our results we shall make a number of assumptions. First we assume that returns follow a Sharpe one-factor model; \( R_t = \mu_t + \beta_t f_{1,t} + \varepsilon_t \) where \( \varepsilon_t \sim (0, \Sigma_t) \), \( \Sigma_t \) is diagonal, and \( f_{1,t} \) is the return to the factor. In the case when the factor is the market return, we shall use \( r_{m,t} \) which is assumed to have mean \( \mu_{m,t} \) and variance \( \sigma_{m,t}^2 \). Finally, \( \beta_t \) is the vector of exposures of the \( n \) assets to the common factor at time \( t \).

We give this a capital asset pricing model interpretation. We shall interpret \( R_t \) to be the vector of excess returns relative to the riskless asset. Likewise, \( r_{m,t} \) is excess market return relative to the risk-free asset. In this version \( \mu_{i,t} \) is the alpha of stock \( i \).

It follows immediately that, conditionally on the past, but unconditionally on the market factor,

\[
\Omega_t = \Sigma_t + \beta_t \beta_t' \sigma_{m,t}^2
\]

so that

\[
tr(\Omega_t) = tr(\Sigma_t) + \beta_t \beta_t' \sigma_{m,t}^2
\]

and

\[
i_n' \Omega_i n = tr(\Sigma_t) + (\beta_t' i_n)^2 \sigma_{m,t}^2
\]

using the assumed diagonality of \( \Sigma_t \).

Substituting into (3), it follows that

\[
\mathbb{E}(D_t) = \frac{CSV(\mu_t) + tr(\Sigma_t)(1 - \frac{1}{n}) + (\beta_t' i_n)^2 \sigma_{m,t}^2 - \frac{1}{n} (\beta_t' i_n)^2 \sigma_{m,t}^2}{n - 1}
\]

In (7), expected dispersion is increasing in the level of market variance, increasing in the dispersion in beta, and increasing in the average residual (idiosyncratic) risk.

It is also increasing in the dispersion of stock alpha. This model has no factor covariance except through common exposure to the single factor. An overall identical additive increase in beta for all stocks leaves beta dispersion constant and, likewise, dispersion but a scale increase in all betas increases beta dispersion and hence overall dispersion. We see here that changes in factor exposures do not necessarily increase/decrease expected dispersion; it depends upon the nature of the assumed change. Garcia, Mantilla-Garcia
Yu and Sharaiha (2007) derive a similar result by approximating sample data and proxy the market return by an equal-weighted return. They also set the average beta to 1. Whilst this would be true for capitalisation-weighted betas, it is only an approximation (albeit a very close one) for equal-weighted betas. Stivers (2003) presents a similar approach. In both cases, the CAPM is assumed to hold, and therefore alpha-dispersion is implicitly zero.

Within the same structure we might wish to look at the market-conditional version. In this version, \( \mu_t = \beta_t r_{m,t} + \alpha_t; \Omega_t = \Sigma_t \). We define \( CSC(x, y) \) to mean the cross-sectional covariance of sets \( x \) and \( y \). In this case, equation (3) becomes:

\[
E(D_t | r_{m,t}) = \frac{tr(\Omega_t) + CSV(\mu_t)}{n-1} \\
= \frac{tr(\Omega_t)}{n-1} + CSV(\alpha_t) + 2CSC(\alpha_t, \beta_t)r_{m,t} + CSV(\beta_t)r_{m,t}^2 \quad (8)
\]

This tells us that expected conditional dispersion is increasing in the dispersion of alpha, in average idiosyncratic risk, in the dispersion of beta, and in squared market returns. The covariance between alpha and beta may be typically thought to be close to zero and ignored. Ignoring this quantity will lead to slight discrepancies between the sample mean of dispersion and the estimated values of the right hand side. These discrepancies should be small. However, when it is non-zero, for example when managers wrongly confuse stock bets with factor bets, we see that this term may make a positive or negative contribution to dispersion depending upon the signs of the covariance and the market return.\(^1\) This result can be compared with equation 5 and equation A13 in Menchero and Morozov (2011). We note that equation A13 is an exact decomposition in sample values. This expresses cross-sectional dispersion in terms of cross-sectional exposure deviations and correlations between exposures and returns all summed over factors. Whilst this is undeniably a useful contribution it is less amenable to understanding

\(^1\)Stivers (2003) derives a similar result. In expectation, both \( CSV(\alpha_t) \) and \( CSC(\alpha_t, \beta_t) \) are equal to zero. We use realised values of the cross-sectional variation in alpha and the cross-sectional correlation between alpha and beta, which may in effect differ from zero.
cross-sectional deviation in terms of the statistical properties of the underlying population of stocks.

2.3. Dispersion and the Constant-Parameter Covariance Model

Suppose we now wish to compute squared dispersion for the constant parameter covariance model. Here we assume that:

\[ \Omega_t = \sigma^2 I_n (1 - \rho) + \rho \sigma^2 i_n i_n' \]  

(9)

Then,

\[ tr(\Omega_t) = n \sigma^2 \]  

(10)

\[ i_n' \Omega_t i_n = n \sigma^2 + n(n - 1) \rho \sigma^2 \]  

(11)

Returning to (3);

\[ \mathbb{E}(D_t) = \mu_t'M\mu_t + n \sigma^2 - (n - 1) \rho \sigma^2 \frac{n - 1}{n - 1} \]

\[ = CSV(\mu_t) + \sigma^2 \left( 1 - \rho + \frac{1}{n - 1} \right) \]  

(12)

Again, expected dispersion increases with dispersion in alpha, increases with the overall risk level, and decreases with average correlation.

Yu and Sharaiha (2007, p.70-71), and Gorman, Sapra and Weigand (2010, p.66-67) show that if the matrix \( \Omega_t \) is as described in this subsection but defined as the residual covariance matrix, then the cross-sectional volatility is a function of the time-series volatility, that is, it equals \( \sigma \sqrt{1 - \rho} \).

A closely related version of this model is one based on a constant-beta assumption, studied by Garcia, Mantilla-Garcia and Martellini (2013) and Goltz, Guobuzaite and Martellini (2011). The merits of this assumption can be gauged empirically and we look at this in Section 5. They show, theoretically and empirically, that the largest component of CSV is idiosyncratic risk; a result we shall also investigate. The above authors also consider generalisations of CSV based on more general weighting schemes. We shall now show that Equation (3) generalises straightforwardly in this situation.

In our notation, let \( W_t = (w_{it}) \) be an \( n \times 1 \) vector of weights, and \( D_{W_i} \) be a diagonal matrix with \( w_{it} \) as the \( i \)th diagonal element. Then
\[ N_t = D W_t - \frac{W_t W'_t}{W'_t W_t} \]  

is an \( n \times n \) matrix and \[ D_t = \frac{R'_t N_t R_t}{n - 1} \] is our generalised squared dispersion measure.

\[ \mathbb{E}(D_t) = \frac{\mu'_t N_t \mu_t + tr(\Omega_t N_t)}{n - 1} \]

\[ (n - 1)\mathbb{E}(D_t) = \mu'_t N_t \mu_t + tr(\Omega_t D W_t) - \frac{W'_t \Omega_t W_t}{W'_t W_t} \]  

This is a generalisation of the expected dispersion measure from Equation (3).

2.4. Dispersion and the Two-Factor Conditional Model

Suppose instead that returns are generated using a two factor model, \( R_t = \mu_t + \varepsilon_t, \) where \( \mu_t = \beta_1 f_{1,t} + \beta_2 f_{2,t} \) and where it is assumed that \( f'_t = (f_{1,t}, f_{2,t}) \sim (0, \Sigma_{f_t}) \). Then we can decompose expected conditional squared dispersion as follows:

\[ \mu'_t M \mu_t = f^2_{1,t} \beta'_1 M \beta_1 + 2 f_{1,t} f_{2,t} \beta'_1 M \beta_2 + f^2_{2,t} \beta'_2 M \beta_2 \]  

We can see immediately that if we use equation (2)

\[ \frac{\mu'_t M \mu_t}{n - 1} = CSV(\beta_1) f^2_{1,t} + 2 CSC(\beta_1, \beta_2) f_{1,t} f_{2,t} + CSV(\beta_2) f^2_{2,t} \]  

with obvious generalisations to higher dimensions. This is of course the dispersion of conditional means; unconditionally we have

\[ \Omega_t = \Sigma_t + \beta_t \Sigma_{f_t} \beta'_t \]  

Where \( \Sigma_{f_t} \) is the \( k \times k \) factor covariance matrix at time \( t \) whilst \( \beta_t \) is an \( n \times k \) matrix of exposures. Neither \( tr(\Omega_t) \) nor \( \beta'_t \Omega_t \beta_t \) are especially informative in the general case.
2.5. Analysis of Variance

It is interesting to observe that CSV arises naturally in an analysis of variance framework. We shall replace returns \( r_{it} \) by \( y_{it} \) in this section. Suppose we have, over all assets and all points in time

\[
S = \sum_{t=1}^{T} \sum_{i=1}^{n} (y_{it} - \bar{y})^2
\]  
(19)

where

\[
\bar{y} = \frac{\sum_{t=1}^{T} \sum_{i=1}^{n} y_{it}}{nT} \]
(20)

Now \( \bar{y}_i \) is the average return for asset \( i \).

Also,

\[
S = \sum_{t=1}^{T} \sum_{i=1}^{n} (y_{it} - \bar{y}_i + \bar{y}_i - \bar{y})^2
\]

\[
= \sum_{t=1}^{T} \sum_{i=1}^{n} (y_{it} - \bar{y}_i)^2 + T \sum_{i=1}^{N} (\bar{y}_i - \bar{y})^2
\]
(21)

And so, by symmetry, letting \( \bar{y}_t \) be the average return at time \( t \):

\[
S = \sum_{t=1}^{T} \sum_{i=1}^{n} (y_{it} - \bar{y}_t + \bar{y}_t - \bar{y})^2
\]

\[
= \sum_{t=1}^{T} \sum_{i=1}^{n} (n - 1) CSV_t + n \sum_{t=1}^{T} (\bar{y}_t - \bar{y})^2
\]
(22)

where \( CSV(y_t) = CSV_t \) is the cross-sectional variance at time \( t \) which equals

\[
CSV_t = \sum_{i=1}^{n} \frac{(y_{it} - \bar{y}_t)^2}{n - 1}
\]
(23)

Let the variance of the cross-sectional mean (VCSM) be given by

\[
VCSM = \sum_{t=1}^{T} \frac{(\bar{y}_t - \bar{y})^2}{T - 1}
\]
(24)
Hence, our two-factor decomposition of the variance becomes

\[ S = \sum_{i=1}^{T} (CSV_i)(n - 1) + n(T - 1)VCSM \]  \hspace{1cm} (25)

This analysis leads to a valuable insight into the links between periods of high time series volatility and high cross-sectional volatility. High cross-sectional volatility is a necessary condition for high overall volatility but it is not a sufficient condition as high variation in returns period-to-period (VCSM) will also lead to high overall volatility. Yu and Sharaiha (2007, p.69) present a similar interpretation of the relationship between cross-sectional and time-series volatility, but their arguments are in terms of population moments and their assumptions seem stronger than ours.

3. Dispersion in Dispersion

In order to assess when measures of dispersion are unreliable, we need to consider what the second moment of dispersion is. This approach could be criticised as our analysis to date strongly suggests that the distributions involved in the analysis of cross-sectional volatility do not involve normal distributions, except perhaps possibly under asymptotic conditions. However, other work exploring CSV routinely makes the normality assumption (e.g. Gorman, Sapra, and Weigand, 2010; Yu and Sharaiha, 2007, p. 70; Goltz, Guobuzaite, and Martellini, 2011, p. 9), inter alia. Again, we could consider the second moment of the standard deviation which would involve fractional moments and either approximations or knowledge of the actual distribution; or, we consider the variance of the second moment, which is what we shall do to avoid writing long expressions involving cross-moments of order 4; we shall assume here that our returns are normally distributed; then (returning to the one-factor model of Section 2.2):

\[ R_t = \mu_t + \varepsilon_t; \quad \text{where} \quad \varepsilon_t \sim \mathcal{N}(0, \Omega_t) \]

\[
\begin{align*}
\text{Var}(D_t) &= \text{Var}\left(\frac{(\mu_t + \varepsilon_t)'M(\mu_t + \varepsilon_t)}{(n - 1)^2}\right) \\
\text{Var}(D_t) &= \frac{4\text{Var}\{\varepsilon_t'M\mu_t\}}{(n - 1)^2} + \frac{\text{Var}\{\varepsilon_t'M\varepsilon_t\}}{(n - 1)^2}
\end{align*}
\]  \hspace{1cm} (26) \hspace{1cm} (27)
Where we have used the fact that that third-order homogeneous polynomials of normals with zero means have zero expected values. Equation (5) can be easily computed using the standard result that the expected value of polynomials of multivariate normals can be computed by considering all pair-wise expectations. This leads to

\[
\text{Var}(D_t) = \frac{4\mu'_t M\Omega_t M\mu_t}{(n-1)^2} + \frac{2tr\{\Omega_t M\Omega_t M\}}{(n-1)^2} \quad (28)
\]

Garcia, Mantilla-Garcia and Martellini (2013) provide a similar analysis to the above, although their specific assumptions differ slightly.

4. Empirical Evidence on Dispersion

4.1. Data Description

To gain further insight into our calculations, we shall present some empirical results. Our main data source is the 100 value-weighted portfolios formed on size and book to market equity ratios, taken from the data library on Ken French’s website. We calculate dispersion based on the 100 portfolios, rather than individual stocks. Following Stivers and Sun (2010) and Maio (2013), this portfolio-level return dispersion (RD) may be less noisy than a firm-level RD, as the influence of extreme individual stock returns is mitigated. Stivers and Sun (2010) also note that the portfolio-level RD metric performs similarly, but generally better, than firm-level RD metrics.

We calculate the return dispersion at the monthly level by taking the standard deviation of monthly returns of the 100 portfolios, over the period from July 1963 to December 2010. For 89 out of 57,000 portfolio-month observations, there are no stocks in the intersection of the size/value deciles (for example, large growth stocks). We exclude these returns from the analysis. Strictly speaking \( n = 100 \) is occasionally \( n = 99 \), for example.

The histogram of cross-sectional volatility shows that there is a large degree of skewness in dispersion as calculated from the 100 portfolios. This is consistent with the sample variance having a distribution which is approximately scale gamma, for example, if all assets were cross-sectionally i.i.d. normal, then we would expect a chi-squared distribution which is a special case of the scale gamma family. However, dispersion does not typically follow a predictable pattern with the market as a whole; there is a correlation of 0.089 between return dispersion and average return of the 100 portfolios. This is however, statistically different from zero. Since it is a characteristic
of normal random variables that the sample mean and sample standard deviation are independent for i.i.d. data, we are left wondering if this small non-zero correlation reflects the cross-sectional correlations rather than non-normality. This can be seen to be a joint characteristic function problem in linear and quadratic normal forms so that, using standard results in, for example, Knight (1985).

Indeed, if \( Z \sim \mathcal{N}(\mu, \Omega) \) then the characteristic function of a linear and
quadratic form in \( Z \) is given by

\[
E \{ \exp(itu'Z + isZ'AZ) \} = \quad (29)
\]

\[
\det(I - 2isA\Omega)^{-\frac{i}{2}} \exp \left\{ \frac{1}{2} (\mu + it\Omega a)'(\Omega - 2is\Omega A\Omega)^{-1}(\mu + it\Omega a) - \frac{1}{2} \mu'\Omega^{-1}\mu \right\}
\]

If we differentiate the expression in (29) we find that the covariance between \( a'Z \) and \( ZA'Z \) is given by

\[
(\mu A'\Omega a + a'\Omega A\mu) \quad (30)
\]

In our case \( A \) is the idempotent projection matrix \( M = I_n - i_n i_n'/n \) where \( I_n \) is the unit diagonal matrix. We see that a sufficient condition for the covariance to be zero is when \( M\mu = 0 \). But this is when the cross-sectional volatility of means is zero. So the non-zero correlation is informing us that there is mean dispersion in our portfolios.

4.2. Calculating Beta Dispersion

We record daily measurements of the same variables as discussed in Section 4.1 over the period July 1, 1963 to December 31, 2010 from Ken French’s website. These daily data are used to construct monthly parameter estimates.

We need to distinguish between daily returns and monthly returns in the calculation of beta. We calculate the beta as the monthly loading on the realised market risk premium in month \( t \). Let the excess return on portfolio \( i \) for day \( q \) be \( r_{i,q} \) and the market excess return on day \( q \) be \( r_{m,q} \). The average daily excess return over month \( t \) is given by \( \bar{r}_i \). For each month \( t \) of \( Q_t \) days we take the covariance of portfolio \( i \) with the market as

\[
\text{cov}(r_{i,t}, r_{m,t}) = \frac{\sum_{q=1}^{Q_t} (r_{i,q} - \bar{r}_i)(r_{m,q} - \bar{r}_m)}{Q_t} \quad (31)
\]

and then take the estimated beta for portfolio \( i \) in month \( t \), \( \hat{\beta}_{i,t} \) as

\[
\hat{\beta}_{i,t} = \frac{\text{cov}(r_{i,t}, r_{m,t})}{\text{Var}(r_{m,t})} \quad (32)
\]

This means that we can calculate the beta-dispersion (standard deviation
and variance, respectively as $\sigma_{\beta,t}$ and $\sigma^2_{\beta,t}$) in month $t$ as

$$
\sigma_{\beta,t} = \sqrt{\frac{\sum_{i=1}^{n}(\hat{\beta}_{i,t} - \bar{\beta}_t)^2}{n - 1}}, \quad \sigma^2_{\beta,t} = \frac{\sum_{i=1}^{n}(\hat{\beta}_{i,t} - \bar{\beta}_t)^2}{n - 1}
$$

(33)

where $\bar{\beta}_t$ is the average of the estimated stock betas calculated in month $t$ based on daily data.

### 4.3. Calculating Alpha Dispersion

The monthly alpha is the unpriced abnormal return on portfolio $i$ as a deviation from the expected return on the capital asset pricing model. Using the estimated value of beta from (32) shows us that:

$$
\hat{\alpha}_{i,t} = \bar{r}_{i,t} - \hat{\beta}_{i,t}(\bar{r}_{m,t})
$$

(34)

In other words, we impute alpha from the capital asset pricing model using our estimated beta and monthly data, for each month $t$ for portfolio $i$. Alpha-dispersion is calculated in a similar fashion to beta-dispersion, as the cross-sectional standard deviation of the alphas for month $t$:

$$
\sigma_{\alpha,t} = \sqrt{\frac{\sum_{i=1}^{n}(\hat{\alpha}_{i,t} - \bar{\alpha}_t)^2}{n - 1}}, \quad \sigma^2_{\alpha,t} = \frac{\sum_{i=1}^{n}(\hat{\alpha}_{i,t} - \bar{\alpha}_t)^2}{n - 1}
$$

(35)

where $\bar{\alpha}_t$ is the average of the estimated stock alphas calculated in month $t$ based on daily data.

### 4.4. Cross-Sectional Covariance between Alpha and Beta

The cross-sectional correlation between alpha and beta is expected to be somewhat close to zero according to the properties of OLS. Returning to $y, x$ notation, if $y_i = \alpha + \beta X_i + \nu_i, i = 1, \ldots, n$, then we have

$$
\text{cov}(\hat{\alpha}, \hat{\beta}) = -\frac{\bar{X}}{n \sum X_i^2 - \bar{X}^2}, \quad \text{and}
$$

(36)

$$
\text{corr}(\hat{\alpha}, \hat{\beta}) = -\frac{\bar{X}}{\sqrt{n \sum X_i^2}}
$$

(37)

For realistic market values where market standard deviation is 25% p.a. and average excess market return is 5% p.a. with $n = 100$, we get a correlation
of about −0.01. Financial theory suggests no connection between these two parameters. However, as first noted by Black (1972, 1993) and Black, Jensen and Scholes (1972) the security market line appears to be flatter than suggested by the CAPM for U.S. equities. These authors, among others, find that low-beta U.S. stocks performed better than predicted by the CAPM over the period 1931-1968, whilst high-beta stocks performed worse. More recent empirical evidence presented by Baker, Bradley and Wurgler (2011) over the period 1968-2008 showed low-beta U.S. stocks outperformed high-beta U.S. stocks in absolute terms, a result later confirmed in an extended period ending in 2012 and for international markets by Baker, Bradley and Taliaferro (2013) and Asness, Frazzini and Pedersen (2013). The relatively good performance of lower-risk assets extends to many different asset classes, according to Frazzini and Pedersen (2013). Possible causes of this anomaly have been suggested; it may be due to an aversion to leverage (Asness, Frazzini, and Pedersen, 2012) or delegated portfolio management with benchmarked institutional investors (Brennan, 1993; Baker, Bradley, and Wurgler, 2011) where mutual fund managers have an incentive to invest in high-beta stocks due to convexity in the flow-performance relationship (e.g. Chevalier and Ellison, 1997; Sirri and Tufano, 1998). Other explanations for the low-beta outperformance have also been put forward.

The sample cross-sectional covariance between alpha and beta for month $t$, $\hat{CSC}(\alpha_t, \beta_t)$ can be calculated as:

$$\hat{CSC}(\alpha_t, \beta_t) = \frac{\sum_{i=1}^{n}(\hat{\alpha}_{i,t} - \bar{\alpha}_t)(\hat{\beta}_{i,t} - \bar{\beta}_t)}{n-1}$$ (38)

This corresponds to a sample analogue of a population parameter. For $CSV$ we could also calculate $\hat{CSV}(\alpha_t)$ and $\hat{CSV}(\beta_t)$.

4.5. Idiosyncratic (Residual) Component of Volatility

Each individual excess daily portfolio return can be calculated as:

$$r_{i,q} = \hat{\alpha}_{i,t} + \hat{\beta}_{i,t} r_{m,q} + \hat{\varepsilon}_{i,q}$$ (39)

for $q = 1, \ldots, Q_t$, where $\hat{\varepsilon}_{i,q}$ is the residual component of the fitted regression line. The idiosyncratic portfolio variance $\hat{\sigma}^2_{i,t}$ for month $t$ is then calculated as:

$$\hat{\sigma}^2_{i,t} = \frac{\sum_{q=1}^{Q_t} \hat{\varepsilon}_{i,q}^2}{Q_t - 2}$$ (40)
We assume that the one-factor model is valid so that the error covariance matrix \((\Omega_t)\) is diagonal. Under this assumption we can find the total proportion of monthly dispersion that is due to idiosyncratic components of variance as the trace of the covariance matrix.

\[
tr(\hat{\Omega}_t) = \sum_{i=1}^{n} \hat{\sigma}_{i,t}^2
\]  

So we have decomposed the expected squared market volatility into those components as predicted by the single index model:

\[
E(D_t|r_{m,t}) = CSV(\alpha_t) + CSV(\beta_t)(r_{m,t}^2) + 2SC(\alpha_t, \beta_t)r_{m,t} + \frac{tr(\hat{\Omega}_t)}{n-1}
\]  

Because our calculations involve some judicious use of time-series parameter estimates, we do not expect equation (42) to be exact since we can estimate the left hand side by the first sample moments of dispersion. Indeed, we would expect some covariation between our estimated parameters and the error terms due to the mix of time-series and cross-sectional estimation. We shall return to this point in subsection Section 4.6.

4.6. Decomposing Sample Data

To present results in line with other researchers, we present a sample version of our decomposition. This allows for non-zero cross-sectional covariance between our estimated parameters and residuals. We present the formulae below.

Firstly, we use (32) and (34) in the estimation of our residuals:

\[
r_{i,t} = \hat{\alpha}_{i,t} + \hat{\beta}_{i,t}r_{m,t} + \hat{\varepsilon}_{i,t}
\]  

The average returns, \(\bar{r}_t\), and average values of parameters, \(\bar{\alpha}_t\) and \(\bar{\beta}_t\), are used to calculate the empirical dispersion metrics as follows:

\[
\bar{r}_t = \bar{\alpha}_t + \bar{\beta}_t r_{m,t} + \bar{\varepsilon}_t
\]  

which leads to our estimation of each portfolio’s cross-sectional deviation for each month:

\[
(r_{i,t} - \bar{r}_t) = (\hat{\alpha}_{i,t} - \bar{\alpha}_t) + (\hat{\beta}_{i,t} - \bar{\beta}_t) r_{m,t} + (\hat{\varepsilon}_{i,t} - \bar{\varepsilon}_t)
\]  

16
The result follows from squaring both the left-hand-side and the right-hand-side of (45), summing over \(i = 1, \ldots, n\), and then dividing by \(n - 1\):

\[
CSV(R_t) = CSV(\alpha_t) + CSV(\beta_t)r_{m,t} + CSV(\varepsilon_t) + 2\hat{CSC}(\alpha_t, \beta_t)r_{m,t}
\]

\[
+ 2\hat{CSC}(\alpha_t, \varepsilon_t) + 2\hat{CSC}(\beta_t, \varepsilon_t)r_{m,t}
\]

where, as mentioned above, \(\hat{CSC}(\alpha_t, \varepsilon_t)\) and \(\hat{CSC}(\beta_t, \varepsilon_t)\) are the estimated values of cross-sectional variation between alpha and the residual component of the regression, and beta and the residual components of the regression, respectively. The term \(\hat{CSC}(\alpha_t, \varepsilon_t)\) is expected to be zero by the properties of OLS, but may be non-zero if the CAPM performs poorly for a subsection of stocks. The last term, \(\hat{CSC}(\beta_t, \varepsilon_t)\) is similarly expected to be close to zero. We will use (46) to investigate the relative measures of each of these sample terms in cross-sectional variation in the next section.

5. Empirical Results

We examine the relationship between our metric for dispersion (the variance of cross sectional monthly returns from the 100 portfolios sorted on size and book-to-market) and the components of dispersion from equation (46), where portfolio 10 is formed in periods of the highest dispersion and portfolio 1 is formed in the period of the lowest dispersion. The period covered is July 1963 to December 2010, a total of 570 months; each decile contains 57 months of returns. The top decile of dispersion contains periods of recent volatility; 26 (45.6%) of top-decile months are between August 1998 and July 2002 (around 8% of the total time period) which is the period of the Nasdaq tech bubble, and a further 10 (17.5%) months are between July 2008 and August 2009, around the global economic recession.

The main results are reported in Table 1.

In the periods of the highest actual dispersion there is the highest expected dispersion by our calculations. The pattern in the dispersion of alphas is confirmed, similar to previous studies (e.g. De Silva, Sapra, and Thorley, 2002; Ankrim and Ding, 2002; Connor and Li, 2009); high dispersion tends to coincide with periods of high dispersion in the ex-post value of alpha. There is a strong degree of convexity in this relationship, the highest decile of dispersion exhibits alpha-dispersion (0.072) that is more than twice as large as decile 9 (0.031). The fifth decile of dispersion realises about half (0.016) the alpha dispersion of decile 9. The impact of alpha dispersion manifests
Table 1: Decomposing dispersion into actual components. This table shows deciles of dispersion, decomposed into total components. The first number in each cell is the average value of each component of dispersion in each decile, while the second number is the proportion of the total cross-sectional dispersion made up by that component. The second column presents the average cross-sectional variance of alphas. The third column presents the variance in the cross-sectional betas, multiplied by the concurrent squared excess market return. The fourth column is the variance in cross-sectional error terms. Columns five, six, and seven are the cross-variation terms, for the three terms in the decomposition. The final term is the average cross-sectional variation in each decile.

<table>
<thead>
<tr>
<th>Time Decile</th>
<th>CSV((\alpha_t))</th>
<th>CSV((\beta_t)) (\times r_{m,t}^2)</th>
<th>CSV((\hat{\varepsilon}))</th>
<th>2CSV((\alpha,\varepsilon)) (\times r_{m,t})</th>
<th>2CSV((\beta,\varepsilon)) (\times r_{m,t})</th>
<th>2CSV((\alpha,\beta)) (\times r_{m,t})</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 (High)</td>
<td>0.072</td>
<td>5.412</td>
<td>33.268</td>
<td>3.004</td>
<td>-1.308</td>
<td>-0.072</td>
<td>40.376</td>
</tr>
<tr>
<td></td>
<td>0.18%</td>
<td>13.40%</td>
<td>82.40%</td>
<td>7.44%</td>
<td>-3.24%</td>
<td>-0.18%</td>
<td>100.00%</td>
</tr>
<tr>
<td>9</td>
<td>0.031</td>
<td>2.897</td>
<td>14.158</td>
<td>1.302</td>
<td>-2.528</td>
<td>-0.112</td>
<td>15.748</td>
</tr>
<tr>
<td></td>
<td>0.20%</td>
<td>18.40%</td>
<td>89.90%</td>
<td>8.27%</td>
<td>-16.05%</td>
<td>-0.71%</td>
<td>100.00%</td>
</tr>
<tr>
<td>8</td>
<td>0.026</td>
<td>2.190</td>
<td>11.200</td>
<td>1.062</td>
<td>-2.599</td>
<td>-0.125</td>
<td>11.754</td>
</tr>
<tr>
<td></td>
<td>0.22%</td>
<td>18.63%</td>
<td>95.29%</td>
<td>9.04%</td>
<td>-22.11%</td>
<td>-1.06%</td>
<td>100.00%</td>
</tr>
<tr>
<td>7</td>
<td>0.021</td>
<td>2.272</td>
<td>8.873</td>
<td>0.846</td>
<td>-2.456</td>
<td>-0.120</td>
<td>9.436</td>
</tr>
<tr>
<td></td>
<td>0.22%</td>
<td>24.08%</td>
<td>95.03%</td>
<td>8.97%</td>
<td>-26.03%</td>
<td>-1.17%</td>
<td>100.00%</td>
</tr>
<tr>
<td>6</td>
<td>0.018</td>
<td>1.810</td>
<td>7.864</td>
<td>0.741</td>
<td>-2.415</td>
<td>-0.114</td>
<td>7.904</td>
</tr>
<tr>
<td></td>
<td>0.23%</td>
<td>22.90%</td>
<td>99.49%</td>
<td>9.38%</td>
<td>-30.55%</td>
<td>-1.44%</td>
<td>100.00%</td>
</tr>
<tr>
<td>5</td>
<td>0.016</td>
<td>1.527</td>
<td>6.532</td>
<td>0.625</td>
<td>-1.759</td>
<td>-0.080</td>
<td>6.861</td>
</tr>
<tr>
<td></td>
<td>0.23%</td>
<td>22.26%</td>
<td>95.30%</td>
<td>9.11%</td>
<td>-35.64%</td>
<td>-1.77%</td>
<td>100.00%</td>
</tr>
<tr>
<td>4</td>
<td>0.014</td>
<td>1.471</td>
<td>5.786</td>
<td>0.547</td>
<td>-1.836</td>
<td>-0.088</td>
<td>5.894</td>
</tr>
<tr>
<td></td>
<td>0.24%</td>
<td>24.96%</td>
<td>98.17%</td>
<td>9.28%</td>
<td>-31.15%</td>
<td>-1.49%</td>
<td>100.00%</td>
</tr>
<tr>
<td>3</td>
<td>0.012</td>
<td>0.920</td>
<td>5.150</td>
<td>0.488</td>
<td>-1.433</td>
<td>-0.068</td>
<td>5.069</td>
</tr>
<tr>
<td></td>
<td>0.24%</td>
<td>18.15%</td>
<td>101.60%</td>
<td>9.63%</td>
<td>-28.27%</td>
<td>-1.34%</td>
<td>100.00%</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
<td>0.918</td>
<td>4.393</td>
<td>0.413</td>
<td>-1.371</td>
<td>-0.066</td>
<td>4.297</td>
</tr>
<tr>
<td></td>
<td>0.23%</td>
<td>21.36%</td>
<td>103.23%</td>
<td>9.61%</td>
<td>-31.91%</td>
<td>-1.54%</td>
<td>100.00%</td>
</tr>
<tr>
<td>1 (Low)</td>
<td>0.008</td>
<td>0.624</td>
<td>3.385</td>
<td>0.322</td>
<td>-0.94</td>
<td>-0.044</td>
<td>3.355</td>
</tr>
<tr>
<td></td>
<td>0.24%</td>
<td>18.60%</td>
<td>100.89%</td>
<td>9.60%</td>
<td>-28.02%</td>
<td>-1.31%</td>
<td>100.00%</td>
</tr>
<tr>
<td>High - Low</td>
<td>0.064</td>
<td>4.788</td>
<td>29.883</td>
<td>2.681</td>
<td>-0.368</td>
<td>-0.028</td>
<td>37.02</td>
</tr>
<tr>
<td></td>
<td>0.17%</td>
<td>12.93%</td>
<td>80.72%</td>
<td>7.24%</td>
<td>-0.99%</td>
<td>-0.08%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>
mainly in the very extreme periods of dispersion. The proportion of overall dispersion due to the dispersion in alpha is very low. Alpha dispersion appears to make up around 0.2% of total dispersion in our model for all deciles; the proportion is in decile 10 at 0.18%. The second component of total dispersion is the product of beta-dispersion and squared excess market returns, \( CSV(\beta) \times r_m^2 \). This term varies positively and in convex fashion with the overall dispersion. In decile 10 of dispersion, this term takes a value (5.472) of nearly twice that of dispersion decile 9 (2.897). However, the proportion of total dispersion that is explained by \( CSV(\beta) \times r_m^2 \) is only 13.4% in decile 10, compared with between 18% and 24% in all other deciles. In Table 2, we present the terms in the decomposition individually; dispersion in beta is close to constant across the deciles of dispersion. The variation across deciles is exclusively due to the variation in squared market returns across dispersion deciles.

<table>
<thead>
<tr>
<th>Time Decile</th>
<th>CSV(\beta)</th>
<th>CSV(\alpha, \beta)</th>
<th>CSV(\beta, \epsilon)</th>
<th>( r_m )</th>
<th>( \rho(CSV(\beta, \epsilon), r_m) )</th>
<th>( r_m^2 )</th>
<th>( r_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 (High)</td>
<td>0.120</td>
<td>0.002</td>
<td>0.042</td>
<td>-0.957</td>
<td>-0.122</td>
<td>63.699</td>
<td>0.367</td>
</tr>
<tr>
<td>9</td>
<td>0.112</td>
<td>-0.006</td>
<td>-0.124</td>
<td>0.961</td>
<td>-0.461</td>
<td>29.232</td>
<td>0.364</td>
</tr>
<tr>
<td>8</td>
<td>0.114</td>
<td>-0.004</td>
<td>-0.084</td>
<td>0.252</td>
<td>-0.572</td>
<td>22.438</td>
<td>0.460</td>
</tr>
<tr>
<td>7</td>
<td>0.122</td>
<td>-0.005</td>
<td>-0.100</td>
<td>1.320</td>
<td>-0.587</td>
<td>23.529</td>
<td>0.487</td>
</tr>
<tr>
<td>6</td>
<td>0.120</td>
<td>-0.004</td>
<td>-0.094</td>
<td>0.416</td>
<td>-0.753</td>
<td>17.086</td>
<td>0.500</td>
</tr>
<tr>
<td>5</td>
<td>0.105</td>
<td>-0.002</td>
<td>-0.046</td>
<td>0.092</td>
<td>-0.714</td>
<td>16.443</td>
<td>0.422</td>
</tr>
<tr>
<td>4</td>
<td>0.146</td>
<td>-0.006</td>
<td>-0.120</td>
<td>0.864</td>
<td>-0.673</td>
<td>11.658</td>
<td>0.472</td>
</tr>
<tr>
<td>3</td>
<td>0.119</td>
<td>-0.004</td>
<td>-0.091</td>
<td>0.568</td>
<td>-0.775</td>
<td>8.369</td>
<td>0.447</td>
</tr>
<tr>
<td>2</td>
<td>0.106</td>
<td>-0.004</td>
<td>-0.088</td>
<td>0.544</td>
<td>-0.815</td>
<td>9.027</td>
<td>0.435</td>
</tr>
<tr>
<td>1 (Low)</td>
<td>0.111</td>
<td>-0.004</td>
<td>-0.095</td>
<td>0.396</td>
<td>-0.845</td>
<td>5.439</td>
<td>0.462</td>
</tr>
<tr>
<td>High - Low</td>
<td>0.009</td>
<td>0.006</td>
<td>0.136</td>
<td>-1.353</td>
<td>0.723</td>
<td>58.26</td>
<td>-0.095</td>
</tr>
</tbody>
</table>

The vast majority of dispersion is made up of the idiosyncratic component, \( CSV(\epsilon) \), as expected from the results of Campbell, Lettau, Malkiel and Xu (2001). Similar to the alpha-dispersion component, the idiosyncratic component is convex across deciles; decile 10 has nearly two-and-a-half times the idiosyncratic variance of decile 9. Proportionally, however, an increasing amount of the total dispersion tends to be made up of the idiosyncratic component in lower dispersion deciles; in deciles 1, 2, and 3 more than 100% of the total dispersion is due to the idiosyncratic component of volatility.

The remaining three terms in the dispersion decomposition contain cross-
sectional covariance terms. The cross-sectional covariation between $\alpha$ and $\varepsilon$ is positive for all deciles of dispersion, and varies monotonically with the dispersion deciles. From Table 2, the covariation of beta and epsilon is generally negative (for deciles 1 - 9) and for decile 10 only, the average excess market return is negative. Hence $2 \hat{CSC}(\beta, \varepsilon) \times r_{m,t}$ is negative in all deciles. This term has a relatively large impact on the overall dispersion; a reduction in dispersion of between 22% and 31% occurs in deciles 1 - 8 due to this term. The cross-sectional covariation between alpha and beta is similarly positive in only decile 10; in combination with the market return, the term reduces overall dispersion in all deciles of dispersion, but proportionally more so in the low-dispersion deciles.

The column titled $\rho(CSC(\beta, \varepsilon), r_{m,t})$ is the within-decile correlation between the monthly cross-sectional covariation of betas and residuals with the monthly market excess return. This term is increasingly negative with decreasing deciles of dispersion. In the highest dispersion decile, market returns are negative on average. Investors move out of high beta stocks into other forms of exposure, thus increasing the idiosyncratic term (readers should be reminded that we are assuming a one-factor model). As a consequence, a negative correlation is observed. For the other nine deciles, market returns on average are positive, and the opposite phenomenon occurs as investors divest themselves of non-market exposures and increase their betas.

The results in Table 3 show the impact of cross-sectional dispersion on the total volatility experienced by the market over the time period from July 1963 to December 2010. Using the Analysis of Variance decomposition from (25), it is noticeable that the overall variation in the 100-portfolio returns sorted on size and book-to-market value over this time period is made up of 71.32% variation due to the changing cross-sectional means over time, with the remaining 28.68% due to cross-sectional variation in asset returns each month.

In terms of investment strategy, our results indicate that high CSV corresponds to better opportunities for investors who can time the market by differentiating between high and low beta stocks. In particular, this decile corresponds to large falls in the market (on average) and so the investment opportunity here is to switch from high beta stocks to low beta stocks. The CSV of beta in this decile is the largest, although not excessively greater than others. In the case of low CSV, this corresponds to circumstances where investors may be well-served by focussing more on idiosyncratic stock considerations.
Table 3: Components of Analysis of Variance Decomposition

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
<th>Total (Proportion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VCSM</td>
<td>27.12</td>
<td>56,900</td>
</tr>
<tr>
<td>$N(T - 1)$</td>
<td>56,900</td>
<td>1,543,128</td>
</tr>
<tr>
<td>Product</td>
<td>1,543,128</td>
<td>71.32%</td>
</tr>
<tr>
<td>$\sum_{t=1}^{T} CSV$</td>
<td>6269.07</td>
<td>99</td>
</tr>
<tr>
<td>$(N - 1)$</td>
<td>99</td>
<td>620,637</td>
</tr>
<tr>
<td>Product</td>
<td>620,637</td>
<td>28.68%</td>
</tr>
<tr>
<td>Total</td>
<td>2,163,765</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

6. Discussion

In this paper we have explored various theoretical decompositions of cross-sectional dispersion in stock returns. In the most general form, from equation (2), we have shown that dispersion is expected to be high when the variation between stock returns (squared dispersion) is high, when average stock variance is high, and when the correlation between stocks is relatively low. Incorporation of a one-factor asset pricing model for stock returns, see (7), allows us to analyse the source of cross-sectional variation of stock returns; expected dispersion is increasing in the level of market variance, increasing in the dispersion in beta, and increasing in the average residual (idiosyncratic) risk.

The market-conditional version presented in equation (8), which does not assume that the CAPM holds (allowing for non-zero alphas), shows that cross-sectional dispersion in alphas and the cross-sectional covariation between alpha and beta are both positively related to expected cross-sectional volatility. Allowing for non-zero constant correlations between assets in the covariance matrix, as in equation (12), a negative relationship between average correlation and cross-sectional volatility is predicted under the single-index model’s assumptions. We also look at extensions of our decomposition to two-factor models.
We next consider an Analysis of Variance approach. The total volatility, over both time-series and cross-section, exhibited by all assets is decomposed using equation (25) into cross-sectional volatility at each point in time, plus a factor relating to the variance of the cross-sectional means of asset returns. In other words, holding the returns of all assets fixed at any point in time, but variable over time leads to volatility, but total volatility is amplified by cross-sectional variation in asset returns at a single point in time. The empirical results in Table 3 show that around 71% of volatility is due to variation in cross-sectional means, with the remaining 29% due to cross-sectional variance in portfolio returns around the mean.

We utilise the model of conditional dispersion in (42) and show how to decompose sample return data into a one-factor model. Our decomposition, which extends existing results, specifically accounts for cross-sectional co-variation in terms involving the idiosyncratic component of stock returns. The empirical results present a demonstration of the decomposition of sample data. We calculate monthly realised values of alphas, betas, and residual risk using daily observations of 100 portfolio returns over the period July 1963 to December 2010 sorted on size and book-value to market-value. Our parameter estimates reveal that the majority of cross-sectional variance in stock returns is due to the idiosyncratic component of risk, rather than priced or unpriced components from a single-index model. This confirms earlier empirical work (Campbell, Lettau, Malkiel, and Xu, 2001, inter alia).

We examine the components of the decomposition by the level of actual dispersion reveals that in periods of low dispersion. The impact of the high idiosyncratic component of risk is mitigated due to negative cross-sectional covariation between beta and the residual variation. Whilst variation in both unpriced returns due to alpha-dispersion, and priced returns from beta-dispersion are positive determinants of overall cross-sectional variation; these are relatively minor components at 0.24% and 18.60%, respectively. In the highest dispersion decile, both alpha-dispersion and beta-dispersion are higher in magnitude than in low-dispersion periods. In particular, alpha-dispersion is nine times higher in the top decile of dispersion than in the lowest decile. Beta-dispersion is relatively constant across the deciles (see Table 2), but squared market returns are around nine times higher in the top decile of dispersion relative to the lowest decile. Hence, the priced component explains a lot more of the overall cross-sectional variance. In the highest decile of dispersion, both of these components exhibit a lower proportion of the cross-sectional volatility than the lowest deciles of dispersion.
(0.18% vs. 0.24% and 13.40% vs. 18.60%, respectively). Interestingly, a large difference between the top and bottom deciles in the proportion of overall volatility is due to the cross-sectional covariation of betas with the residuals. This component is negative (reduces cross-sectional volatility) in all ten deciles. In the top decile of dispersion, the cross-sectional variation of beta with the residual contributes -3.24% to the total, which is a far smaller proportional reduction than in any of the other nine deciles, which average a 26.64% reduction. This appears to be a major contributing source, along with the idiosyncratic component of risk, in explaining total cross-sectional volatility.

The theoretical decomposition and empirical analysis presented in this paper suggests that portfolio managers should be aware of the cross-sectional variation in stock returns, and implement investment strategies accordingly. The empirical decomposition presented could be easily extended to multifactor models, which would reduce the share of dispersion attributed to the idiosyncratic component of volatility. As our empirical results show a negative correlation between the within-decile market returns and the cross-sectional covariation in beta and the residual component of risk, it would be helpful to understand the changing nature of the relationship with the incorporation of additional pricing factors. Alternatively, the decomposition of dispersion could be explored in a predictive setting. For example, Stivers and Sun (2010) find that cross-sectional return dispersion helps to forecast the value premium at a horizon of six months, whilst Maio (2013) shows that dispersion is a powerful tool to forecast month-ahead aggregate market returns. It would be interesting to see whether the components of dispersion aid forecast accuracy in such settings.


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Connor, Gregory, and Sheng Li (2009) ‘Market Dispersion and the Profitability of Hedge Funds.’ Mimeo, National University of Ireland


