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Competitive Search Equilibrium with Private Information on Monetary Shocks

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Abstract

The relationship between unanticipated inflation and output has been a classic issue in macroeconomics. This paper studies the effects of monetary uncertainty on output in a competitive search environment where there is asymmetric information about monetary shocks. I conclude that in a decentralized environment with asymmetric information, in order to generate real output effects, no real shocks are needed. When the realization of the monetary shock is privately observed by buyers, sellers offer more output when the economy experiences a positive monetary shock. In other words, money is not neutral. This contrasts with the centralized Walrasian market environment, where prices adjust proportionally to changes in the quantity of money, and where nominal shocks by themselves have no real output effects.

In order to get an output effect from a monetary shock, then, it is not enough simply to introduce uncertainty. We need to imagine that the exchange of money for goods takes place in some manner other than in a centralized Walrasian Market.

Robert E. Lucas, Jr., 1996

1 Introduction

The relationship between unanticipated inflation and output has been a classic issue in macroeconomics. This paper studies the effects of monetary uncertainty on output in a competitive search environment where there is asymmetric information about monetary shocks. I conclude that in a decentralized environment with asymmetric information, in order to generate real output effects, no real shocks are needed. When the realization of the monetary shock is privately observed by buyers, sellers offer more output when the economy experiences a positive monetary shock. In other words, money is not neutral.

The study of the effects of unanticipated inflation with differential information can be traced back at least to Lucas and Rapping (1969). Using an overlapping generations model, Lucas (1972) showed that monetary fluctuations lead to real output movement as individuals cannot distinguish real from monetary disturbances. Recently, Katzman, Kennan, and Wallace (2003) examined the inflation-output relationship in a random matching model. In their setting, there is differential information between producers and consumers about realizations of the monetary shock. They concluded that there could be positive- or negative-impact effects on output.

As clearly pointed out by Lucas (1996), to understand the real effects of changes in money, we should construct a decentralized market where prices imperfectly convey information about the current state of the monetary disturbances. With this in mind, I find a search-based approach to be a natural setup for the study. The seminal work of Kiyotaki and Wright (1989) explicitly models the decentralized trading environment that gives rise to a double coincidence-of-wants problem, and thus makes money essential. Since then, search models have been used predominantly to study the microeconomic foundations of monetary theories.

My analysis builds on the recent work by Rocheteau and Wright (2005), who make the first attempt to introduce competitive search equilibrium into monetary search models. The concept of competitive search equilibrium is attractive as it incorporates price competition into an environment where trades take place in bilateral matches. In a competitive search environment, the terms of trade and the relative abundance of sellers over buyers maximize the expected utility of a seller subject to the common expected utility all buyers attain in equilibrium; therefore, a competitive search equilibrium is efficient.

This paper studies the effects of monetary uncertainty in a competitive search environment with asymmetric information about monetary shocks. In such an environment, sellers post prices which do not convey information about the realization of the monetary uncertainty perfectly. I show that when the monetary shock is privately observed by buyers, sellers offer more output to buyers when the economy experiences a positive monetary shock; otherwise, buyers have an incentive to lie about their money holdings. In other words, the model predicts that output increases with unanticipated inflation, and money is not neutral.

In a related paper, Faig and Li (2006) replace the afternoon market with a centralized Walrasian market and introduce real shocks into the model. As pointed out by Lucas (1972), in this case, monetary fluctuations also lead to real output movement, because individuals cannot distinguish real from monetary disturbances. In Faig and Li (2006), the role of money as a medium of exchange is explicitly modelled, and the money demand curve is clearly identified. Their paper provides some new insights into the quantitative study of the welfare costs of expected and unexpected inflation.

The paper is organized as follows. Section 2 describes the environment, which extends the framework of Rocheteau and Wright (2005) to allow for monetary uncertainty. Sections 3 and 4 characterize the competitive search equilibrium with full and private information, respectively. Section 5 generalizes the model by considering a generic distribution of monetary shocks. Numerical analysis is conducted in Section 6 to study the welfare costs of anticipated and unanticipated inflation. Section 7 concludes the chapter.

2 The Environment

Time is discrete and the horizon is infinite. As in Rocheateau and Wright (2005), each period is divided into two subperiods, called morning and afternoon, during which the market structure differs.

There is a continuum of ex-ante identical individuals with measure one. In the morning, individuals produce and trade goods in a centralized Walrasian market, and during the afternoon individuals trade bilaterally in search markets. Individuals can either be buyers (consumers) or sellers (producers) in the afternoon search market, and individuals choose their trading roles at the beginning of each period.

Goods are divisible and non-storable, so the goods produced in the morning cannot be traded in the afternoon and vice versa. There is another object called money that is intrinsically useless, perfectly divisible and storable. The gross growth rate of the money supply at date t is γ_t; that is, $M_{t+1} = \gamma_t M_t$, where M_t is the quantity of money per buyer at the beginning of period t. New money is injected by lump-sum transfers to buyers only, and these transfers take place after the Walrasian day markets are closed and before the search markets are open.

This paper departs from the previous literature by assuming that γ_t is not constant over time. In the basic model, γ_t is assumed to follow a two-state random process: with probability p the realization of the monetary shock is γ^h, and with complementary probability the realization is γ^l, where $\gamma^h > \gamma^l$. Later on, the model is generalized to allow for an N-state distribution. The monetary shock is independent across time.

The instantaneous utility function of an individual who chooses to be a buyer in the afternoon of date t is

$$U_t^b(x_t^b, y_t^b, q_{\gamma_t}) = v(x_t^b) - y_t^b + u(q_{\gamma_t}),$$

where x_t^b and y_t^b are respectively quantity consumed and produced in the morning. q_{γ_t} is the quantity consumed during the afternoon, and it depends on the realization of γ_t.

Similarly, the instantaneous utility function of an individual who chooses to be a seller is

$$U_t^s(x_t^s, y_t^s, q_{\gamma_t}) = v(x_t^s) - y_t^s - c(q_{\gamma_t}),$$

where $c(q_{\gamma_t})$ is the disutility associated with production during the afternoon.

As in Rocheteau and Wright (2005), a key assumption is that individuals have quasi-linear preferences over the goods traded in Walrasian markets. This assumption implies that the distribution of money holdings in the search market is degenerate in equilibrium, and this makes the model analytically tractable.

The lifetime utilities of individuals are $E_t \sum_{t=0}^{\infty} \beta^t U_t^j \left(x_t^j, y_t^j, q_{\gamma_t} \right)$, for $j = b$ and s, and $\beta \in (0,1)$ is the one-period discount factor. The functions v, u, and c are all continuously differentiable and increasing. The functions v and u are concave, while c is convex. Moreover, $v(0) = u(0) = c(0) = c'(0) = 0$, and $u'(0) = \infty$.

To complete the description of the environment, I now describe how the terms of trade are determined in the search market. In the afternoon, goods are traded in a competitive search market. At the beginning of each period, prior to the opening of the search market, every individual who chooses to be a seller posts an offer specifying the quantity traded q_{γ_t} and the nominal payment z_{γ_t}. Since the objective of this paper is to study the effects of unanticipated inflation, I consider the case in which sellers specify q_{γ_t} and z_{γ_t} conditional on the realization of the monetary shock γ_t.

Buyers then direct their search towards the most attractive offer. The set of sellers posting the same offer and the set of buyers directing their search towards them form a submarket. In each submarket, buyers and sellers are randomly matched in pairs, and goods can only be exchanged in these matches.

To avoid unnecessary complications, I assume that individuals experience at most one match and that matching is efficient; that is, the short-side of the market is always served in each submarket. The probability that a buyer meets a suitable seller is then

$$\pi_t^b(n_t) = \min(1, n_t),$$

where n_t is the ratio of sellers over buyers in the submarket the buyer visits. Likewise, the probability that a seller meets a suitable buyer is

$$\pi_t^s(n_t) = \min(1, \frac{1}{n_t}).$$

As summarized in Table 1, the timing of events in a period is as follows. At the beginning of each day, individuals decide the trading role to be played in the search market. Sellers then post their offers for the search market, and buyers choose to direct their search to the most attractive offer. Walrasian markets open, and individuals consume and produce. Based on their expectations about the inflation rates, individuals decide the money balances to be carried into the search market. Walrasian markets close when the morning ends. The government hands out monetary transfers to buyers at noon. Then afternoon comes and the competitive search markets open. In each submarket, buyers and sellers trade according to the pre-specified terms of trade. As a result of trade, sellers produce, buyers consume, and money changes hands from buyers to sellers.

Table 1

Summary of Activities during a Period

<table>
<tr><th colspan="2">Day</th><th rowspan="2">Noon</th><th colspan="2">Afternoon</th></tr>
<tr><th colspan="2">Walrasian Markets</th><th colspan="2">Competitive Search Markets</th></tr>
<tr><td>Buyer-seller choice. Sellers post trading offers. Buyers choose among offers.</td><td>Walrasian markets open. Individuals choose money balances.</td><td>Monetary shock realized.</td><td>Buyers and sellers form submarkets.</td><td>Bilateral trades take place.</td></tr>
</table>

Obviously, this requires a certain amount of commitment. As stated above, the monetary shock takes place after the Walrasian market closes and before the search market opens. It is assumed that sellers cannot change their offers and have to commit to the posted terms. Likewise, buyers always direct their search to the submarket chosen at the beginning of the period.

Our equilibrium concept combines perfect competition in the morning markets with competitive search in the afternoon. In equilibrium, individuals make optimal decisions, taking as given the sequence of prices in the Walrasian markets and the sequence of conditions in the competitive search market to be detailed below. Individuals also have rational expectations about how these prices and conditions evolve over time.

2.1 The Behavior of Buyers

This subsection characterizes the optimal behavior of individuals who choose to be buyers in the search market. I examine the optimal consumption and production decisions, and the buyer's demand for money.

A buyer who enters the morning market with money balances m_t faces the following budget constraint:

$$x_t^b + \hat{m}_t^b \phi_t = y_t^b + m_t \phi_t,\ \hat{m}_t^b \geq 0; \tag{1}$$

where x_t^b and y_t^b are respectively the consumption and the production of goods during the day; $\hat{m}_t^b$ is the amount of money demanded for the afternoon market; the price of the good traded in the morning is normalized to 1; and the price of money is ϕ_t.

The optimal choice of $\left\{x_t^b, y_t^b, \hat{m}_t^b\right\}$ solves the following maximization program:

$$\begin{aligned} V_t^b(m_t) \quad = \quad & \max_{\left\{x_t^b, y_t^b, \hat{m}_t^b\right\}} v(x_t^b) - y_t^b + \pi_t^b(n_t) E_t \left\{u(q_{\gamma_t}) + \beta V_{+1}^b \left[\hat{m}_t^b - z_{\gamma_t} + (\gamma_t - 1)M_t\right]\right\} \\ & + \left[1 - \pi_t^b(n_t)\right] E_t \left\{\beta V_{t+1}^b \left[\hat{m}_t^b + (\gamma_t - 1)M_t\right]\right\}, \end{aligned} \tag{2}$$

subject to the budget constraint (1). Here, $(\gamma_t - 1)M_t$ denotes the lump-sum transfer from the government. As the monetary shock takes place after the Walrasian market is closed, there is no uncertainty associated with prices in the morning. Buyers adjust their money balances based on their expectations about the realization of the monetary shock.

As in Rocheteau and Wright (2005), the quasi-linearity of preferences implies that

$$x_t^b = x^*, \tag{3}$$

where x^* is implicitly defined by the equality $v'(x^*) = 1$. Also, it implies that the value function is affine:

$$V^b(m_t) = \bar{v}_0^b + m_t \phi_t, \tag{4}$$

where $\bar{v}_0^b$ is independent from m_t.

This paper focusses on a recursive equilibrium in which the only state variable is the money growth rate γ_t. In such an equilibrium, the amount of goods traded in the afternoon market q_{γ_t} is determined by the realization of the shock. In the morning Walrasian market, the value of money ϕ_t declines at the same rate as money increases. Hence, $\phi_t/\phi_{+1} = M_{t+1}/M_t = \gamma_t$. Therefore,

$$E_t\left[V_{t+1}^b(\hat{m}_t^b)\right] = \bar{v}_0^b + \hat{m}_t^b\phi_t E_t(\gamma_t^{-1}). \tag{5}$$

Substituting (1), (3), (4) and (5) into (2), I obtain

$$V_t^b(m_t) = v(x^*) - x^* + m_t\phi_t + \beta\bar{v}_0^b + \beta\phi_t M_t[1 - E(\gamma_t^{-1})] + \bar{S}_t^b. \tag{6}$$

Here, the term $\bar{S}_t^b$ represents the expected trade surplus of a buyer in the afternoon search market, which can be written as

$$\bar{S}_t^b = \max_{\hat{m}_t^b} \pi_t^b(n_t)E_t\left[u(q_{\gamma_t}) - \beta\phi_t\frac{z_{\gamma_t}}{\gamma_t}\right] - \hat{m}_t^b\phi_t\left[1 - \beta E_t(\gamma_t^{-1})\right]. \tag{7}$$

In the expression, $\beta\phi_t z_{\gamma_t}/\gamma_t$ denotes real payments made to the seller. $\hat{m}_t^b\phi_t\left[1 - \beta E_t(\gamma_t^{-1})\right]$ represents the expected opportunity cost of carrying money. The objective of this problem is strictly increasing if $1 - \beta E_t(\gamma_t^{-1}) < 0$, so that there is no solution to (7) in this case. For an equilibrium to exist, the inflation rate must obey $\beta E_t(\gamma_t^{-1}) < 1$. This condition ensures that there is a positive opportunity cost of holding money, and the objective in (7) is decreasing in $\hat{m}_t^b$. Therefore, the buyer should not carry idle money balances.

2.2 The Behavior of Sellers

This subsection characterizes the optimal behavior of individuals who choose to be sellers in the search market. The next section solves the offer sellers post, and the equilibrium ratio of sellers over buyers.

A seller who enters the morning market with money balances m_t faces the following budget constraint:

$$x_t^s + \hat{m}_t^s\phi_t = y_t^s + m_t\phi_t,\ \hat{m}_t^s \geq 0, \tag{8}$$

where x_t^s and y_t^s are respectively the consumption and the production of goods during the day, and $\hat{m}_t^s$ is the amount of money demanded for the afternoon market.

The optimal choice of $\{x_t^s, y_t^s, \hat{m}_t^s\}$ solves the following maximization program:

$$\begin{aligned} V_t^s(m_t) &= \max_{\{x_t^s, y_t^s, \hat{m}_t^s\}} v(x_t^s) - y_t^s + \pi_t^s(n_t) E_t\left[-c(q_{\gamma_t}) + \beta V_{+1}^s(\hat{m}_t^s + z_{\gamma_t})\right] \\ &+ \left[1 - \pi_t^s(n_t)\right] \beta E_t\left[V_{t+1}^s(\hat{m}_t^s)\right] \end{aligned} \tag{9}$$

subject to the budget constraint (8).

Following the same procedure, the seller's problem can be simplified into

$$V_t^s(m_t) = v(x^*) - x^* + m_t\phi_t + \beta\bar{v}_0^s - \hat{m}_t^s\phi_t\left[1 - \beta E_t(\gamma_t^{-1})\right] + \bar{S}_t^s.$$

In the expression, $\hat{m}_t^s\phi_t\left[1 - \beta E_t(\gamma_t^{-1})\right]$ represents the opportunity cost of carrying money. Since $1 - \beta E_t(\gamma_t^{-1})$ is strictly positive, the seller carries no money balances into the search market; that is, $\hat{m}_t^s = 0$. $\bar{S}_t^s$ denotes the expected trade surplus of a seller in the afternoon, and it can be written as

$$\bar{S}_t^s = \max_{\left\{q_{\gamma_t}, z_{\gamma_t}\right\}_{\gamma_t = r^l, r^h}} \pi_t^s(n_t) E_t\left[-c(q_{\gamma_t}) + \beta\phi_t \frac{z_{\gamma_t}}{\gamma_t}\right]. \tag{10}$$

Since individuals choose the trading role that yields maximal utility, the value function of an individual with money balances m_t at the beginning of each period is

$$V_t(m_t) = \max\left\{V_t^b(m_t), V_t^s(m_t)\right\}.$$

3 Equilibrium with Full Information

To serve as a benchmark, this section characterizes a monetary equilibrium with full information; that is, sellers can observe buyers' money balances carried into the search market, and, therefore, can infer the realization of the monetary shock. The next section studies the monetary equilibrium where the monetary shock realizations that buyers experience are private information.

Let $\omega_t = \left(n_t, \left\{q_{\gamma_t}, z_{\gamma_t}\right\}_{\gamma_t = r^l, r^h}\right)$ be the vector that characterizes an active submarket. A competitive search equilibrium is a set $\left\{\bar{S}_t^b, \bar{S}_t^s, \omega_t\right\}$ that satisfies the following:

1. Buyers attain the same expected surplus $\bar{S}_t^b$ in all active submarkets.

2. Sellers attain the same expected surplus $\bar{S}_t^s$ in all active submarkets.

3. The expected surpluses of buyers and sellers are identical and non-negative: $\bar{S}_t^b = \bar{S}_t^s \geq 0$.

4. The list ω_t solves the following program:

$$\bar{S}_t^s = \max_{\left(n_t, \hat{m}_t^b, \left\{q_{\gamma_t}, z_{\gamma_t}\right\}_{\gamma_t = r^l, r^h}\right)} \pi_t^s(n_t) E_t \left[-c(q_{\gamma_t}) + \beta\phi_t \frac{z_{\gamma_t}}{\gamma_t}\right] \tag{11}$$

subject to

$$\bar{S}_t^b = \pi_t^b(n_t) E_t \left[u(q_{\gamma_t}) - \beta\phi_t \frac{z_{\gamma_t}}{\gamma_t}\right] - \hat{m}_t^b \phi_t \left[1 - \beta E_t(\gamma_t^{-1})\right], \tag{12}$$

$$z_{\gamma_t} \leq \hat{m}_t^b + (\gamma_t - 1) M_t, \text{ for } \gamma_t = \gamma^l, \gamma^h, \text{ and} \tag{13}$$

$$n_t, q_{\gamma_t}, z_{\gamma_t} \geq 0, \text{ for } \gamma_t = \gamma^l, \gamma^h. \tag{14}$$

The motivation of these conditions of equilibrium is the following: Since all buyers have identical payoff functions (7), so they must attain the same expected surplus (condition 1). The same is true for sellers. Individuals' freedom to choose the trading roles to be played in the search market implies that in equilibrium individuals must be indifferent between the two trading roles, and these roles must be preferable to no trade (condition 3). Condition 4 requires sellers to choose the offer that maximizes their expected surplus, taking $\bar{S}_t^b$ as given, and realizing that the ratio n_t would endogenously adjust so that (11) and (12) hold.

The program above can be conveniently simplified because a solution must have the following two properties: In all submarkets,

$$n_t = \pi_t^b(n_t) = \pi_t^s(n_t) = 1; \tag{15}$$

otherwise, the seller's expected surplus in (10) can be easily increased while keeping the buyers' expected surplus in (7) constant. Moreover, while sellers care only about the average payment, buyers prefer a payment schedule that exhausts their money balances for a given average payment, so they can reduce the opportunity cost of carrying money. Therefore, the optimal payments must satisfy the following condition:

$$z_{\gamma_t} = \hat{m}_t^b + (\gamma_t - 1)M_t, \ for \ \gamma_t = \gamma^l, \gamma^h. \tag{16}$$

Substituting (15) and (16) into (11) yields

$$\hat{m}_t^b = \frac{\bar{S}_t^s + E_t\left[c(q_{\gamma_t}) - \beta\phi_t M_t(1-\gamma_t^{-1})\right]}{\beta\phi_t E_t(\gamma_t^{-1})}. \tag{17}$$

Since both $\bar{S}_t^b$ and $\bar{S}_t^s$ are monotonic in z_{γ_t}, it is convenient to use a dual formulation of the program described above. Using (15) to (17), the program described by (11) to (14) simplifies into

$$\bar{S}_t^b = \max_{\{q_{\gamma_t}\}_{\gamma_t=r^l,r^h}} E_t\left[u(q_{\gamma_t}) - c(q_{\gamma_t})\right] - \bar{S}_t^s - \left[1 - \beta E_t(\gamma_t^{-1})\right] \frac{\bar{S}_t^s + E_t\left[c(q_{\gamma_t}) - \beta\phi_t M_t(1-\gamma_t^{-1})\right]}{\beta E_t(\gamma_t^{-1})}$$

The following first-order condition that characterizes the equilibrium value of q_{γ_t} should be satisfied:

$$\frac{u'(q_{\gamma_t})}{c'(q_{\gamma_t})} = \frac{1}{\beta E_t(\gamma_t^{-1})}, \text{ for } \gamma_t = \gamma^l, \gamma^h. \tag{18}$$

This expression shows that when the monetary shocks that buyers experience are public information, in equilibrium sellers always hand out the same quantity of goods to buyers, regardless of the realization of the shock. Condition (18) is analogous to the one without monetary shocks, replacing $E_t(\gamma_t^{-1})$ for γ_t^{-1}. As is standard, it states that in equilibrium, the quantity of goods traded should equate the marginal rate of substitution to the expected gross nominal interest rate $1/[\beta E_t(\gamma_t^{-1})]$. It is a simple matter to check that this quantity is less than the efficient amount of output q^*, which is defined by $u'(q^*) = c'(q^*)$. Inflation creates a wedge of $1/\ [\beta E_t(\gamma_t^{-1})]$ between the marginal utility of consumption and the marginal cost of production.

To complete the characterization of a competitive search equilibrium, it is necessary to determine $\hat{m}_t^b$, $\phi_t M_t$ and z_{γ_t}. Since in equilibrium individuals are indifferent between the two trading roles, condition $V_t^b(m_t) = V_t^s(m_t)$ implies:

$$\bar{S}_t^b + \beta\phi_t M_t[1 - E_t(\gamma_t^{-1})] = \bar{S}_t^s, \text{ and} \tag{19}$$

$$\hat{m}_t^b = \frac{u(\bar{q}) + c(\bar{q}) - \beta\phi_t M_t[1 - E(\gamma_t^{-1})]}{\phi_t[1 + \beta E_t(\gamma_t^{-1})]}, \tag{20}$$

where $\bar{q}$ is the equilibrium quantity defined by (18).

Condition (20) together with the money-market clearing condition $\hat{m}_t^b = M_t$ determines $\phi_t M_t$, which is the value of money in real terms. To solve for the nominal payment z_{γ_t}, the equilibrium property that buyers always exhaust their money balances implies that

$$z_{\gamma_t} = \hat{m}_t^b + (\gamma_t - 1)M_t = \frac{u(\bar{q}) + c(\bar{q}) - \beta\phi_t M_t[1 - E(\gamma_t^{-1})]}{\phi_t[1 + \beta E_t(\gamma_t^{-1})]} + (\gamma_t - 1)M_t.$$

The existence of monetary equilibria requires that $\tilde{\gamma} < E(\gamma_t^{-1}) < \beta^{-1}$. As claimed in the previous section, the condition $E(\gamma_t^{-1}) < \beta^{-1}$ ensures there is a positive opportunity cost of carrying money. The condition $E(\gamma_t^{-1}) > \tilde{\gamma}$ ensures the existence of the type of monetary equilibrium that I analyze; that is, both $\hat{m}_t^b$ and z_{γ_t} are positive in equilibrium. Specifically, $\tilde{\gamma}$ is the value of $E(\gamma_t^{-1})$ such that when $E(\gamma_t^{-1})$ is equal to $\tilde{\gamma}$, $\hat{m}_t^b = 0$ and $z_{\gamma_t} \geq 0$ if $\gamma^l \geq 1$, and $z_{\gamma_t} = 0$ and $\hat{m}_t^b > 0$ if $\gamma^l < 1$.[1]

For the purpose of illustration, I complete this section by providing an example of the equilibrium offer that sellers post. Suppose $M_t = 10$ dollars, and $\hat{m}_t^b = 5$ dollars. In equilibrium, a seller will post the following payment schedule:

γ_t	q_{γ_t}	z_{γ_t}	$\hat{m}_t^b + (\gamma_t - 1)M_t$
$\gamma^l = 1.1$	5 *apples*	6 *dollars*	6 *dollars*
$\gamma^h = 1.2$	5 *apples*	7 *dollars*	7 *dollars*.

As indicated in the example, buyers always exhaust their money balances in exchange for the same amount of goods (5 apples), regardless of the realization of the shock. This leads to the conclusion that when monetary uncertainty is publicly observed by all market

[1] If $\gamma^l \geq 1$, $\tilde{\gamma} = 1 - [u(\bar{q}) + c(\bar{q})]/(\beta\phi_t M_t)$. If $\gamma^l < 1$, $\tilde{\gamma} = \left\{1 + \beta - \gamma^l - [u(\bar{q}) + c(\bar{q})]/\phi_t M_t\right\}/(\beta\gamma^l)$.

participants, nominal shocks have no real output effects. In other words, a decentralized market structure (such as a competitive search environment) alone does not lead to monetary non-neutrality.

4 Equilibrium with Private Information

Consider the competitive search equilibrium described in the previous section, but suppose monetary shocks are privately observed by the buyers who experience them. In this case, the trading arrangement under full information is no longer feasible. When the realization of the shock is high, buyers are certainly better off claiming it is low, so that they receive the same quantity of goods but pay less than what they would with truth-revealing. Therefore, with private information on monetary shocks, the offers posted by sellers must be incentive compatible. That is, buyers must have no incentives to lie about their money holdings. The program described by (11) to (14) is then further restricted to satisfy the incentive compatibility constraint:

$$\gamma_t' \in \arg \max_{\gamma_t=\gamma^l,\gamma^h} \left[u(q_{\gamma_t}) - \beta\phi_t \frac{z_{\gamma_t}}{\gamma_t'} \right], \text{ for } \gamma_t' = \gamma^l, \gamma^h. \tag{21}$$

The following program describes the maximization problem with the restriction (21):

$$\bar{S}_t^s = \max_{\hat{m}_t^b, \left\{q_{\gamma_t}, z_{\gamma_t}\right\}_{\gamma_t=r^l,r^h}} E_t \left[-c(q_{\gamma_t}) + \beta\phi_t \frac{z_{\gamma_t}}{\gamma_t} \right] \tag{22}$$

subject to

$$\bar{S}_t^b = E_t \left[u(q_{\gamma_t}) - \beta\phi_t \frac{z_{\gamma_t}}{\gamma_t} \right] - \hat{m}_t^b \phi_t \left[1 - \beta E_t(\gamma_t^{-1}) \right], \tag{23}$$

$$z^l \leq \hat{m}_t^b + (\gamma^l - 1) M_t, \tag{24}$$

$$z^h \leq \hat{m}_t^b + (\gamma^h - 1) M_t, \tag{25}$$

$$u(q^l) - \beta\phi_t \frac{z^l}{\gamma^l} \geq u(q^h) - \beta\phi_t \frac{z^h}{\gamma^l}, \tag{26}$$

$$u(q^h) - \beta\phi_t \frac{z^h}{\gamma^h} \geq u(q^l) - \beta\phi_t \frac{z^l}{\gamma^h}, \text{ and} \tag{27}$$

$$q_{\gamma_t}, z_{\gamma_t} \geq 0, \text{ for } \gamma_t = \gamma^l, \gamma^h, \tag{28}$$

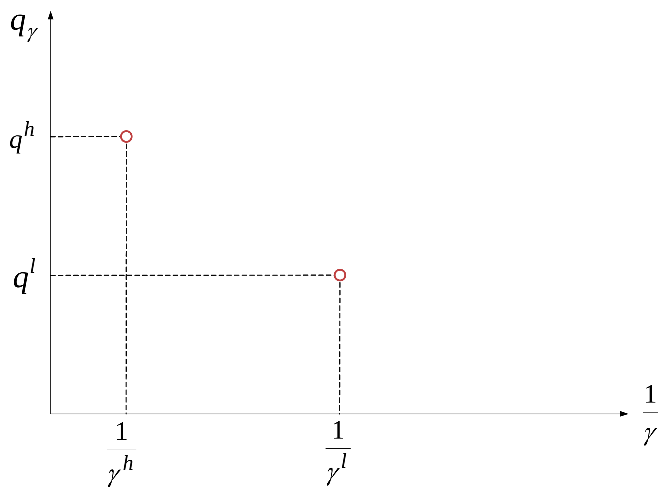
where q^l and q^h are the quantities purchased when the monetary shock realizations are high and low, respectively. z^l and z^h are the corresponding nominal payment required by the seller.

Lemma 1. *Both the equilibrium quantity of goods traded* q_{γ_t} *and the nominal payment* z_{γ_t} *are increasing with the money growth rate. That is,* $q^l < q^h \leq q^*$*, and* $z^l < z^h$*, where* q^* *is the efficient level of output.*

Lemma 1 essentially implies a downward-sloping Phillips curve as indicated by Figure 1. Higher unexpected inflation leads to higher output, and money is not neutral. This contrasts with the full information case, where buyers always receive the same amount of goods, regardless of the realization of the monetary shock.

Figure 1

A Downward-sloping Phillips Curve



Lemma 2. *The liquidity constraint is binding in states in which the money growth rate is low. That is,* $z^l = \hat{m}_t^b + (\gamma^l - 1)M_t$.

Lemma 3.

(a) *The incentive compatibility constraint is not binding in the low-value states, so buyers are strictly better off to truthfully reveal their money balances when the realization of the monetary shock is* γ^l.

(b) *The incentive compatibility constraint is binding in the high-value states, so buyers are indifferent between revealing the truth or not when the realization of the monetary shock is* γ^h.

These results are quite intuitive. First, q^h cannot exceed the efficient level of output q^*; otherwise, the seller's marginal benefit of offering $(q_{\gamma_t}, z_{\gamma_t})$ would be negative. Second, the liquidity constraint should be binding in low-value states; if not, it means the buyer is carrying idle money balances. Third, buyers cannot have an incentive to lie about their money holdings when the monetary shock realization is γ^l; therefore, the incentive compatibility constraint is never binding in the low-value states. Last, the equilibrium trading offers posted by sellers must make buyers indifferent between revealing the truth or not at high-value states; otherwise, sellers can always offer less output in exchange for less money.

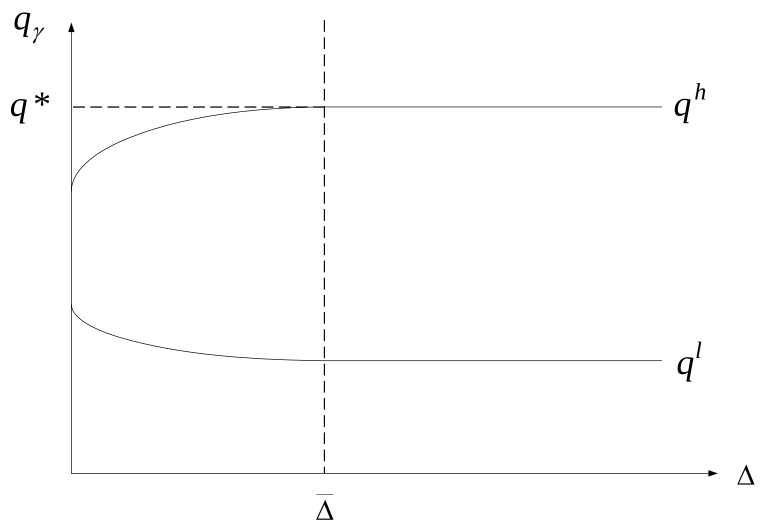
Using Lemma 1, 2 and 3, I solve for the equilibrium solution to the program described above. Proposition 1 presents the main theoretical results:

Proposition 1. *Define* $\Delta = \gamma^h - \gamma^l$. *For a given* $E(\gamma_t^{-1})$, *there exists a critical value of* Δ, *called* $\bar{\Delta}$, *such that two different types of equilibria arise as* Δ *increases. If* $\Delta \leq \bar{\Delta}$, $d(q^h)/d(\Delta) > 0$, $d(q^l)/d(\Delta) < 0$*; that is, output dispersion increases with monetary uncertainty; instead, if* $\Delta > \bar{\Delta}$, $d(q^h)/d(\Delta) = 0$, $d(q^l)/d(\Delta) = 0$.

Figure 2 shows the equilibrium values of output for a given $E(\gamma_t^{-1})$ at date t.

Figure 2

Equilibrium Output



As illustrated by Figure 2, if $\Delta \leq \bar{\Delta}$, an increase in monetary uncertainty leads to a more dispersed output profile. In this regime, both the liquidity constraints (24) and (25) are binding; that is,

$$z^l = \hat{m}_t^b + (\gamma^l - 1)M_t, \text{ and}$$

$$z^h = \hat{m}_t^b + (\gamma^h - 1)M_t.$$

The equilibrium solution of q_{γ_t} is characterized by the following equations:

$$u(q^h) - u(q^l) = \beta\phi_t M_t \frac{\gamma^h - \gamma^l}{\gamma^h}, \tag{29}$$

$$\frac{u'(q^h)}{c'(q^h)} = \frac{p}{\beta E(\gamma_t^{-1})(p + \theta^h)}, \text{ and} \tag{30}$$

$$\frac{u'(q^l)}{c'(q^l)} = \frac{1-p}{\beta E(\gamma_t^{-1})(1 - p - \theta^h)}, \tag{31}$$

where θ^h is the Lagrangian multiplier associated with the incentive compatibility constraint (27), and p is the probability that the realization of the shock is γ^h.

The system implies that $q^l < q^h \leq q^*(q^h = q^*$ at $\bar{\Delta}$) and the following comparative statics:

$$\frac{d(q^h)}{d(\Delta)} > 0, \; \frac{d(q^l)}{d(\Delta)} < 0.$$

If $\Delta > \bar{\Delta}$, a different type of equilibrium arises. In this case, the liquidity constraint (25) is slack, while constraint (24) is still binding. That is,

$$z^l = \hat{m}_t^b + (\gamma^l - 1)M_t, \text{ and}$$

$$z^h < \hat{m}_t^b + (\gamma^h - 1)M_t.$$

The optimal solution of q_{γ_t} must satisfy the following equations:

$$\frac{u'(q^h)}{c'(q^h)} = 1, \text{ and}$$

$$\frac{u'(q^l)}{c'(q^l)} = \frac{1-p}{\beta E(\gamma_t^{-1}) - p}.$$

Therefore, if $\Delta > \bar{\Delta}$, $q^l < q^h = q^*$, $d(q^h)/d(\Delta) = 0$, and $d(q^l)/d(\Delta) = 0$. Sellers always hand out the efficient amount of output to buyers in the high-value states. Similarly, buyers would receive the same amount of goods from sellers (which is less than q^*) as long as the realization of the monetary shock is low, regardless of the value of γ^l.

5 Generalization

This section studies a general model in which the monetary shock γ_t follows an N-state random process ($N > 2$). I show that the results are robust to an arbitrary finite number of states.

Consider the competitive search environment described in the previous section, but suppose now that γ_t can take on N values, and probability $(\gamma_t = \gamma^i) = p^i$, $i = 1, 2, ..., N$, where $\sum_{i=1}^{N} p^i = 1$.

Again, I start with the case in which the monetary shocks that buyers experience are public information. It is not difficult to show that the same conclusion is reached: sellers always hand out the same amount of goods to buyers (which is less than q^*), regardless of the realization of the monetary shock. The equilibrium value of q_{γ_t} is characterized by the same first-order condition (18) as in the two-state case.

I now proceed to investigate the monetary equilibrium with private information. In this case, the offers posted by sellers should be incentive compatible. Similar to the monetary equilibrium characterized in the two-state model, the following program defines the sellers' maximization problem:

$$\bar{S}_t^s = \max_{\hat{m}_t^b, \{q_{\gamma_t}, z_{\gamma_t}\}_{\gamma_t = r^1, \ldots, r^N}} E_t \left[-c(q_{\gamma_t}) + \beta \phi_t \frac{z_{\gamma_t}}{\gamma_t} \right], \tag{32}$$

subject to

$$\bar{S}_t^b = E_t \left[u(q_{\gamma_t}) - \beta \phi_t \frac{z_{\gamma_t}}{\gamma_t} \right] - \hat{m}_t^b \phi_t \left[1 - \beta E_t(\gamma_t^{-1}) \right], \tag{33}$$

$$z_{\gamma_t} \leq \hat{m}_t^b + (\gamma^i - 1) M_t, \text{ for } \gamma_t = \gamma^1, ..., \gamma^N, \tag{34}$$

$$q_{\gamma_t}, z_{\gamma_t} \geq 0, \text{ for } \gamma_t = \gamma^1, ..., \gamma^N, \tag{35}$$

$$u(q_{\gamma^i}) - \beta \phi_t \frac{z_{\gamma^i}}{\gamma^i} \geq u(q_{\gamma^j}) - \beta \phi_t \frac{z_{\gamma^j}}{\gamma^i}, \text{ for } i \neq j. \tag{36}$$

The following lemmas generalize the equilibrium properties I established in the previous section.

Lemma 1 (Generalized). *Both the equilibrium quantity of goods traded q_{γ_t} and the nominal payment z_{γ_t} are increasing with the money growth rate. That is, $q_{\gamma^1} < q_{\gamma^2} < ... < q_{\gamma^{N-1}} < q_{\gamma^N} \leq q^*$, and $z_{\gamma^1} < z_{\gamma^2} < ... < z_{\gamma^{N-1}} < z_{\gamma^N}$, where q^* is the efficient level of output.*

Lemma 2 (Generalized). *The liquidity constraint is binding in states in which the money growth rate is at the lowest level. That is, $z_{\gamma^1} = \hat{m}_t^b + (\gamma^1 - 1) M_t$.*

Lemma 3 (Generalized).

(a) For $i < j$, the incentive compatibility constraint is not binding at γ^i; that is, $u(q_{\gamma^i}) - \beta\phi_t z_{\gamma^i}/\gamma^i > u(q_{\gamma^j}) - \beta\phi_t z_{\gamma^j}/\gamma^i$, so buyers are strictly better off revealing truthfully the actual money growth rate rather than claim a higher one.

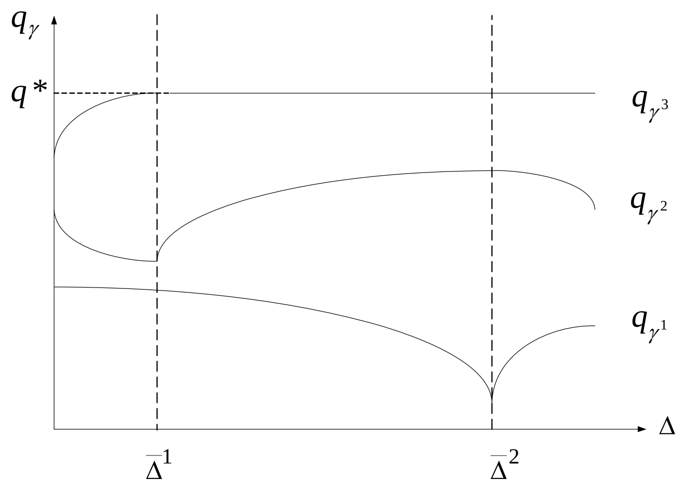
(b) For $i < j$, the incentive compatibility constraint is binding at γ^j; that is, $u(q_{\gamma^j}) - \beta\phi_t z_{\gamma^j}/\gamma^j = u(q_{\gamma^i}) - \beta\phi_t z_{\gamma^i}/\gamma^j$, so buyers are indifferent between revealing the actual realization of the monetary shock or claiming a lower one.

Following the same procedure, I solve for the equilibrium solution to the generalized program described by conditions (32) to (36). Proposition 2 summarizes the main results.

Proposition 2. *Define $\Delta = \gamma^N - \gamma^1$. For a given $E(\gamma_t^{-1})$ and for given values of $\gamma^2, \gamma^3, ..., \gamma^{N-1}$, there exist $N-1$ critical values of Δ, called $\bar{\Delta}^1, \bar{\Delta}^2, ..., \bar{\Delta}^{N-1}$, where $\bar{\Delta}^1 < \bar{\Delta}^2 < ... < \bar{\Delta}^{N-1}$, such that N different types of equilibria arise as $\gamma_\triangle$ increases. If $\bar{\Delta}^k < \Delta \leq \bar{\Delta}^{k+1}$ $(k = 1, 2, ..., N-2)$, the liquidity constraint is binding at $\gamma_t = \gamma^1, \gamma^2, ..., \gamma^{N-k}$, and is slack at $\gamma_t = \gamma^{N-k+1}, ..., \gamma^N$. If $\Delta > \bar{\Delta}^{N-1}$, the liquidity constraint is binding only at $\gamma_t = \gamma^1$; and if $\Delta \leq \bar{\Delta}^1$, the liquidity constraint is binding at all $\gamma_t = \gamma^1, \gamma^2, ..., \gamma^N$.*

Figure 3

Equilibrium Output (N=3)



For example, when $N = 3$, for given values of $E(\gamma_t^{-1})$ and γ^2, there exist two critical values of Δ, called $\bar{\Delta}^1$ and $\bar{\Delta}^2$, such that if $\Delta \leq \bar{\Delta}^1$, the liquidity constraint is binding at all γ_t. If $\bar{\Delta}^1 < \Delta \leq \bar{\Delta}^2$, the liquidity constraint is binding at γ^1 and γ^2, but not at γ^3. If $\Delta > \bar{\Delta}^2$, the liquidity constraint is binding only at γ^1. Figure 3 depicts the equilibrium values of q_{γ_t}.

6 Numerical Analysis

This section analyzes a quantified version of the model. An artificial two-state economy is constructed to match sample values for the U.S. economy between 1892-2005. My objective is to study the implications of the model about the welfare costs of anticipated and unanticipated inflation. Specifically, I compare the social welfare achieved through two alternative monetary policies that generate the same expected money growth rate $E(\gamma)$: a stochastic policy, and a stationary monetary policy. I show that the welfare cost of implementing the stochastic policy relative to the stationary policy is equivalent to 0.055 percent of GDP.

To parametrize the model, the following functional forms are adopted:

$$u(q_t) = \frac{q_t^{1-\alpha}}{1-\alpha},$$

$$c(q_t) = \frac{q_t^{1+\eta}}{1+\eta}, \text{ and}$$

$$v(x_t) = B\ln(1+x_t).$$

These functional forms are widely used in the literature. With an isoelastic utility function $u(q_t)$, the average commercial margin is increasing with the curvature parameter α. Intuitively, a large curvature parameter α implies that buyers are inclined to consume a small quantity of goods with high frequency because the marginal utility of consumption is decreasing rapidly in q_t. As a result, sellers require a large compensation, in the form of a large commercial margin, for sacrificing their time to be sellers.

As described in the simple model, the money growth rate γ_t is assumed to follow a two-state random process. Since the results depend only on the difference between γ^h and γ^l, I normalize γ^l to be 1. The distribution of the monetary shock is characterized as follows,

$$\gamma_t = \begin{cases} \gamma^h = 1+\delta \text{ with probability } p, \\ \gamma^l = 1 \text{ with probability } 1-p. \end{cases}$$

To calibrate the five parameters of the model $(\alpha, B, \eta, \delta, p)$, I use the following moments from annual time series of the United States from 1892 to 2005:[2] (1) the average money growth rate is 4.14%; (2) the standard deviation of the money growth rate is 0.0465; (3) the standard deviation of the money-induced innovation in $\ln(GDP)$ is 0.0193; (4) the average commercial margin is 0.28; (5) the average velocity of the circulation of money is 4.87.

The model's counterpart of the average commercial margin is $p * [q^h - c(q^h)]/q^h + (1-p) * [q^l - c(q^l)]/q^l$. The velocity of the circulation of money is GDP over $M1^*$, where $M1^*$ is $M1$ in circulation inside the United States. The counterpart of the average annual velocity

[2] See Faig and Li (2006) for the calculation of these moments.

of the circulation of money is equal to:

$$\text{Velocity} \equiv \frac{GDP}{M} = p\frac{x^b + x^s + q^h}{\beta\phi_t[\hat{m}_t^b + (\gamma^h - 1)M_t]/\gamma^h} + (1-p)\frac{x^b + x^s + q^l}{q^l}.$$

Observations (1) and (2) jointly determine the two distribution parameters δ and p, while observations (3) to (5) identify the remaining three parameters. In addition, the discount factor β is calibrated to match the observed average real rate of interest, which is 3 per cent. Finally, the length of the period is set to be one year. Table 2 reports the calibrated parameter values.

Table 2

Parameter Values

Parameter	α	B	η	δ	p	β
Values	0.337	2.718	0.341	0.094	0.442	1/1.03

Using the calibrated parameters, the welfare cost of inflation is then estimated as follows. Social welfare is measured as $W = E\left[v(x^b) - y^b + v(x^s) - y^s + u(q) - c(q)\right]$. This measure of welfare is calculated at the Friedman rule (subscript F) and at the stochastic monetary policy described above (subscript US). The welfare cost of adopting this stochastic policy is $W_F - W_{US}$, which due to the quasi-linear preferences is measured in units of the general good. So, $W_F - W_{US}$ is how many units of the general good would have to be given to United States residents to compensate them for adopting policy US instead of policy F. Table 3 reports this welfare cost as a percentage of GDP. For comparison purposes, it also reports the welfare cost associated with an alternative stationary monetary policy (subscript $\overline{US}$) with a constant γ_t equal to the average money growth rate.

As we can see in Table 3, the welfare cost of implementing the stochastic policy relative to the Friedman rule is equivalent to 0.131 percent of GDP, and the deadweight-loss associated with the stationary monetary policy is 0.077 percent of GDP. The stochastic policy is more welfare costly than the stationary policy. This result is not surprising and is consistent with the consensus view in the literature. The stochastic policy leads to fluctuations in output, with a concave utility function and a convex cost function, output fluctuations reduce social welfare. However, as the third row of Table 3 indicates that switching from the stochastic monetary policy to the stationary one is equivalent to a 0.055 per cent increase in GDP, this numerical result suggests the extra welfare cost created by unanticipated inflation is quantitatively negligible.[3]

Table 3

Welfare Cost of Inflation

Welfare Cost	$\frac{(W_F - W_{US})}{GDP_F}$	$\frac{(W_F - W_{\overline{US}})}{GDP_F}$	$\frac{(W_{\overline{US}} - W_{US})}{GDP_{US}}$	$\frac{(W_F - W_{10})}{GDP_F}$
Values (%)	0.131	0.077	0.055	0.23

To facilitate a comparison with the existing literature, I also computed the social welfare achieved under a stationary monetary policy that generates a 10-percent inflation (subscript 10). The last row of Table 3 shows that the welfare loss of this policy relative to the Friedman rule is equivalent to 0.23 percent of GDP. This number is substantially smaller than the welfare cost as reported in Lagos and Wright (2005). One possible explanation is that this paper uses competitive search as the pricing mechanism, while Lagos and Wright (2005) assumes that price is determined by Nash bargaining. As proved by Rocheteau and Wright (2006), Friedman rule yields first best in a competitive search equilibrium, and inflation implies second-order welfare losses. In contrast, since inflation creates inefficiency in both the intensive margin (amount of goods traded) and the extensive margin (number of trades) under bargaining there are first-order welfare losses associated with inflation in that case. Another possible reason is that the semi-elasticity of money demand with respect to interest rate predicted by the model is well below that of the U.S.. In other words, the model generates a flatter money demand curve, which implies that expenditures are less responsive to monetary fluctuations. As a result, welfare cost of inflation becomes smaller.

[3] Faig and Li (2006) get similar quantitative results of the welfare cost of unanticipated inflation.

7 Conclusion

Monetary uncertainty and asymmetric information are put into a competitive search environment to study the effects of unanticipated inflation on output. I show that in a decentralized market with asymmetric information on monetary uncertainty, in order to generate real output effects, no real shocks are needed. When the growth rate of the money supply is privately observed by buyers, sellers post a price schedule that is incentive compatible. In equilibrium, output increases with unexpected inflation, and monetary shocks by themselves have real output consequences.

I conclude that neither the decentralized trading environment nor the asymmetric information structure alone is sufficient to generate monetary non-neutrality. In a centralized Walrasian market, prices fully reveal monetary shocks if those are the only shocks in the economy. As a result, prices adjust proportionally to changes in the quantity of money. On the other extreme, in a decentralized trading environment with complete information on the realization of monetary shocks, buyers receive the same amount of goods, regardless of the money growth rate.

Based on the simple model of this paper, much more work can be done in the future. For example, to integrate the current framework with a standard real business-cycle model, one interesting extension would be to introduce neoclassical firms and capital as well as technology shocks, and to study the implications about the U.S. business cycle. Another extension could be to replace the competitive search market by an alternative imperfect market: price is determined by bilateral bargaining, which is assumed in much of the microfoundations literature.

References

[1] Faig, Miquel and Belen Jerez (2005), "A Theory of Commerce," *Journal of Economic Theory*, 122, 60-99.

[2] Faig, Miquel and Belen Jerez (2006a), "Inflation, Prices and Information in Competitive Search," *Advances in Macroeconomics*, forthcoming.

[3] Faig, Miquel and Belen Jerez (2006b), "Precautionary Balances and the Velocity of Circulation of Money," *Journal of Money, Credit and Banking*, forthcoming.

[4] Faig, Miquel and Zhe Li (2006), "The Welfare Costs of Expected and Unexpected Inflation," manuscript.

[5] Katzman, Brett, John Kennan and Neil Wallace (2003), "Output and Price Level Effects of Monetary Uncertainty in a Matching Model," *Journal of Economic Theory*, 108, 217-277.

[6] Lagos, Ricardo and Randall Wright (2005), "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113, 463-484.

[7] Lucas, Robert E., Jr. and Leonard Rapping (1969), "Price Expectations and the Phillips Curve," *American Economic Review*, 59, 342-350.

[8] Lucas, Robert E., Jr. (1972), "Expectations and the Neutrality of Money," *Journal of Economic Theory*, 4, 103-124.

[9] Lucas, Robert E., Jr. (1996), "Nobel Lecture: Monetary Neutrality," *Journal of Political Economy*, 104, 661-682.

[10] Rocheteau, Guillaume and Randall Wright (2005), "Money in Search Equilibrium, in Competitive Equilibrium, in Competitive Search Equilibrium," *Econometrica*, 73, 175-202.

Appendix

1. Proof of Lemma 1. Define $v_\gamma \equiv u(q_\gamma) - \beta\phi\frac{z_\gamma}{\gamma}$, $v_{\gamma',\gamma} \equiv u(q_{\gamma'}) - \beta\phi\frac{z_{\gamma'}}{\gamma}$ (the actual inflation rate is γ, but the buyer claims it is γ'). The incentive compatibility constraint (21) implies that $v_\gamma \geq v_{\gamma',\gamma}$ and $v_{\gamma'} \geq v_{\gamma,\gamma'}$. Since $v_{\gamma',\gamma} = v_{\gamma'} + \beta\phi\frac{z_{\gamma'}}{\gamma'} - \beta\phi\frac{z_{\gamma'}}{\gamma} = v_{\gamma'} + \beta\phi z_{\gamma'}(\frac{1}{\gamma'} - \frac{1}{\gamma})$. Therefore, $v_\gamma \geq v_{\gamma',\gamma}$ can be rewritten as $v_\gamma - v_{\gamma'} \geq \beta\phi z_{\gamma'}(\frac{1}{\gamma'} - \frac{1}{\gamma})$. Similarly, I can rewrite $v_{\gamma'} \geq v_{\gamma,\gamma'}$ as $v_\gamma - v_{\gamma'} \leq \beta\phi z_\gamma(\frac{1}{\gamma'} - \frac{1}{\gamma})$. Altogether, I can rewrite the incentive compatibility constraint as $\beta\phi z_{\gamma'}(\frac{1}{\gamma'} - \frac{1}{\gamma}) \leq v_\gamma - v_{\gamma'} \leq \beta\phi z_\gamma(\frac{1}{\gamma'} - \frac{1}{\gamma})$. This implies that for all $\gamma' < \gamma$, z_γ is a non-decreasing function of γ. Consequently, q_γ should also be a non-decreasing function of γ; otherwise, the incentive compatibility constraint is violated.

It is a simple matter to show that q^h should be strictly greater than q^l. Suppose that in equilibrium sellers post a trading offer $w = (q^h, z^h; q^l, z^l)$ with $q^h = q^l$ and $z^h = z^l$; however, there always exists another trading offer $w' = (q^{h'}, z^{h'}; q^l, z^l)$ with $q^{h'} > q^h$ and $z^{h'} > z^h$, such that buyers are strictly better off with offer w' while sellers are indifferent between the two offers. That is, w' pareto dominates w, and w cannot be the equilibrium trading offer. Therefore, in equilibrium q_γ and z_γ should both be strictly increasing with γ.

2. Proof of Lemma 2. Since both (22) and (23) are monotonic in z_t^γ, the solution of (22) to (28) is the same as the solution of a dual program that maximizes (23) subject to (22) and the remaining constraints of the original problem. Therefore, the Lagrangian of the maximization problem is as follows: (I drop the time subscripts in what follows)

$$
\begin{aligned}
L \;=\; & p\left[u(q^h) - \beta\phi\frac{z^h}{\gamma^h}\right] + (1-p)\left[u(q^l) - \beta\phi\frac{z^l}{\gamma^l}\right] - \hat{m}^b\phi[1 - \beta E(\gamma_t^{-1})] \\
& +\lambda\left\{p\left[-c(q^h) + \beta\phi\frac{z^h}{\gamma^h}\right] + (1-p)\left[-c(q^l) + \beta\phi\frac{z^l}{\gamma^l}\right] - \bar{S}^s\right\} \\
& +\mu^l\left[\hat{m}^b + (\gamma^l - 1)M - z^l\right] + \mu^h\left[\hat{m}^b + (\gamma^h - 1)M - z^h\right] \\
& +\theta^l\left[u(q^l) - \beta\phi\frac{z^l}{\gamma^l} - u(q^h) + \beta\phi\frac{z^h}{\gamma^l}\right] \\
& +\theta^h\left[u(q^h) - \beta\phi\frac{z^h}{\gamma^h} - u(q^l) + \beta\phi\frac{z^l}{\gamma^h}\right],
\end{aligned}
$$

where μ^l and μ^h are the Lagrangian multiplier associated with the liquidity constraint

(24) and (25), respectively. θ^l and θ^h are respectively the Lagrangian multiplier associated with the incentive compatibility constraint (26) and (27). λ is the Lagrangian multiplier associated with constraint (22). I obtain the following first-order conditions:

$$q^l : \left(1 - p + \theta^l - \theta^h\right) u'(q^l) = \lambda(1-p)c'(q^l), \tag{37}$$

$$q^h : \left(p - \theta^l + \theta^h\right) u'(q^h) = \lambda p c'(q^h), \tag{38}$$

$$z^l : \frac{(\lambda - 1)\,\beta\phi(1-p)}{\gamma^l} - \mu^l - \theta^l \frac{\beta\phi}{\gamma^l} + \theta^h \frac{\beta\phi}{\gamma^h} = 0, \tag{39}$$

$$z^h : \frac{(\lambda - 1)\,\beta\phi p}{\gamma^h} - \mu^h + \theta^l \frac{\beta\phi}{\gamma^l} - \theta^h \frac{\beta\phi}{\gamma^h} = 0, \tag{40}$$

$$\hat{m}^b : \mu^l + \mu^h = \phi[1 - \beta E(\gamma_t^{-1})], \tag{41}$$

$$\lambda : p\left[-c(q^h) + \beta\phi\frac{z^h}{\gamma^h}\right] + (1-p)\left[-c(q^l) + \beta\phi\frac{z^l}{\gamma^l}\right] - \bar{S}^s > 0; = 0 \text{ if } \lambda > 0, \tag{42}$$

$$\mu^l : \hat{m}^b + (\gamma^l - 1)M - z^l > 0; = 0 \text{ if } \mu^l > 0, \tag{43}$$

$$\mu^h : \hat{m}^b + (\gamma^h - 1)M - z^h > 0; = 0 \text{ if } \mu^h > 0, \tag{44}$$

$$\theta^l : u(q^l) - \beta\phi\frac{z^l}{\gamma^l} - u(q^h) + \beta\phi\frac{z^h}{\gamma^l} > 0; = 0 \text{ if } \theta^l > 0, \text{ and} \tag{45}$$

$$\theta^h : u(q^h) - \beta\phi\frac{z^h}{\gamma^h} - u(q^l) + \beta\phi\frac{z^l}{\gamma^h} > 0; = 0 \text{ if } \theta^h > 0. \tag{46}$$

To solve the FOCs systems (37) to (46), substitute conditions (39) and (41) into (40), I obtain

$$\lambda = \frac{1}{\beta E(\gamma_t^{-1})}.$$

I now proceed to show $\mu^l > 0$. Suppose $\mu^l = 0$, then condition (39) implies that

$$\frac{(\lambda-1)\,\beta\phi(1-p)}{\gamma^l} = \theta^l\frac{\beta\phi}{\gamma^l} - \theta^h\frac{\beta\phi}{\gamma^h}. \tag{47}$$

Since $\lambda > 1$, the two possible solutions are $\theta^l > 0$, $\theta^h = 0$ and $\theta^l > 0$, $\theta^h > 0$. In the case of $\theta^l > 0$, $\theta^h = 0$, conditions (37) and (38) imply that $q^l > q^h$, which contradicts Lemma 1. In the case of $\theta^l > 0$, $\theta^h > 0$, conditions (45) and (46) imply that $q^l = q^h$ and $z^l = z^h$, which again contradicts Lemma 1. This completes the proof of Lemma 2.

3. Proof of Lemma 3. To show $\theta^l = 0$ and $\theta^h > 0$, first, consider the case in which $\theta^l = \theta^h = 0$, conditions (37) and (38) imply that $q^l = q^h$; second, if $\theta^l = \theta^h > 0$, conditions (45) and (46) imply that $q^l = q^h$. Last, if $\theta^l > 0$ and $\theta^h = 0$, conditions (37) and (38) imply that $q^l > q^h$. Therefore, the only possible solution is $\theta^l = 0$, $\theta^h > 0$.

4. Proof of Proposition 1. According to Lemma 1, 2 and 3, I only need to consider the following two cases.

Case 1: $\mu^l > 0$**,** $\mu^h = 0$, $\theta^l = 0$, $\theta^h > 0$.

According to condition (40), $\mu^l = \phi[1 - \beta E(\gamma_t^{-1})]$. Condition (40) implies that $\theta^h = p[\frac{1}{\beta E(\gamma_t^{-1})} - 1]$. Substituting θ^h into condition (37), I obtain $u'(q^l)/c'(q^l) = \frac{1-p}{\beta E(\gamma_t^{-1})-p}$. Since $\beta E(\gamma_t^{-1}) < 1$, $u'(q^l)/c'(q^l) > 1$; therefore, $q^l < q^*$. Substituting θ^h into condition (38), I obtain $u'(q^h) = c'(q^h)$; therefore, $q^h = q^*$.

Case 2: $\mu^l > 0$**,** $\mu^h > 0$, $\theta^l = 0$, $\theta^h > 0$.

In this case, condition (37) implies that $u'(q^l)/c'(q^l) = \frac{1-p}{\beta E(\gamma_t^{-1})(1-p-\theta^h)}$, and condition (38) implies that $u'(q^h)/c'(q^h) = \frac{p}{\beta E(\gamma_t^{-1})(p+\theta^h)}$. In addition, conditions (39) and (40) imply that $\theta^h\frac{\beta\phi}{\gamma^h} = \frac{p/\gamma^h}{\beta E(\gamma_t^{-1})}(1-\beta E(\gamma_t^{-1}))\phi - \mu^h$; therefore, $\theta^h \leq p(\frac{1}{\beta E(\gamma_t^{-1})} - 1)$. As a result, $q^l < q^h \leq q^*$.

At $\bar{\Delta}$, $\theta^h = p(\frac{1}{\beta E(\gamma_t^{-1})} - 1)$ and $q^h = q^*$; therefore, case 1 coincides with case 2.

5. Proof of Proposition 2. I omit the proofs of the generalized versions of Lemmas 1, 2 and 3, which are essentially the same as in the 2-state model.

To solve the maximization problem, condition (36) implies that there are $N*(N-1)$ incentive compatibility constraints. However, once redundant constraints are eliminated, the system of constraints can be simplified as follows:

$$u(q_{\gamma^1}) - \beta\phi_t \frac{z_{\gamma^1}}{\gamma^1} \geq u(q_{\gamma^2}) - \beta\phi_t \frac{z_{\gamma^2}}{\gamma^1}, \tag{48}$$

$$u(q_{\gamma^1}) - \beta\phi_t \frac{z_{\gamma^1}}{\gamma^1} \geq u(q_{\gamma^3}) - \beta\phi_t \frac{z_{\gamma^3}}{\gamma^1}, \tag{49}$$

$$\vdots$$

$$u(q_{\gamma^1}) - \beta\phi_t \frac{z_{\gamma^1}}{\gamma^1} \geq u(q_{\gamma^N}) - \beta\phi_t \frac{z_{\gamma^N}}{\gamma^1}, \tag{50}$$

$$u(q_{\gamma^2}) - \beta\phi_t \frac{z_{\gamma^2}}{\gamma^2} \geq u(q_{\gamma^1}) - \beta\phi_t \frac{z_{\gamma^1}}{\gamma^2}, \tag{51}$$

$$u(q_{\gamma^3}) - \beta\phi_t \frac{z_{\gamma^3}}{\gamma^3} \geq u(q_{\gamma^2}) - \beta\phi_t \frac{z_{\gamma^2}}{\gamma^3}, \tag{52}$$

$$\vdots$$

$$u(q_{\gamma^{N-1}}) - \beta\phi_t \frac{z_{\gamma^{N-1}}}{\gamma^{N-1}} \geq u(q_{\gamma^{N-2}}) - \beta\phi_t \frac{z_{\gamma^{N-2}}}{\gamma^{N-1}}, \text{ and} \tag{53}$$

$$u(q_{\gamma^N}) - \beta\phi_t \frac{z_{\gamma N}}{\gamma^N} \geq u(q_{\gamma^{N-1}}) - \beta\phi_t \frac{z_{\gamma^{N-1}}}{\gamma^N}. \tag{54}$$

Therefore, dropping time subscripts, the Lagrangian of the maximization problem is as follows:

$$\begin{aligned}
L &= p^1\left[u(q_{\gamma^1}) - \beta\phi\frac{z_{\gamma^1}}{\gamma^1}\right] + p^2\left[u(q_{\gamma^2}) - \beta\phi\frac{z_{\gamma^2}}{\gamma^2}\right] + ... \\
&+p^N\left[u(q_{\gamma^N}) - \beta\phi\frac{q_{\gamma^N}}{\gamma^1}\right] - \hat{m}^b\phi[1-\beta E(\gamma_t^{-1})] \\
&+\lambda\left\{p^1\left[-c(q_{\gamma^1}) + \beta\phi\frac{z_{\gamma^1}}{\gamma^1}\right] + p^2\left[-c(q_{\gamma^1}) + \beta\phi\frac{z_{\gamma^2}}{\gamma^2}\right] + ...\right. \\
&\left.+p^N\left[-c(q_{\gamma^N}) + \beta\phi\frac{z_{\gamma^N}}{\gamma^N}\right] - \bar{S}^s\right\} \\
&+\mu^1\left[\hat{m}^b + (\gamma^1-1)M - z_{\gamma^1}\right] + \mu^2\left[\hat{m}^b + (\gamma^2-1)M - z_{\gamma^2}\right] + ... \\
&+\mu^N\left[\hat{m}^b + (\gamma^N-1)M - z_{\gamma^N}\right] \\
&+\theta^{1,2}\left[u(q_{\gamma^1}) - \beta\phi_t\frac{z_{\gamma^1}}{\gamma^1} - u(q_{\gamma^2}) + \beta\phi_t\frac{z_{\gamma^2}}{\gamma^1}\right] \\
&+\theta^{1,3}\left[u(q_{\gamma^1}) - \beta\phi_t\frac{z_{\gamma^1}}{\gamma^1} - u(q_{\gamma^3}) + \beta\phi_t\frac{z_{\gamma^3}}{\gamma^1}\right] + ... \\
&+\theta^{1,N}\left[u(q_{\gamma^1}) - \beta\phi_t\frac{z_{\gamma^1}}{\gamma^1} - u(q_{\gamma^N}) + \beta\phi_t\frac{z_{\gamma^N}}{\gamma^1}\right] \\
&+\theta^2\left[u(q_{\gamma^2}) - \beta\phi_t\frac{z_{\gamma^2}}{\gamma^2} - u(q_{\gamma^1}) + \beta\phi_t\frac{z_{\gamma^1}}{\gamma^2}\right] \\
&+\theta^3[u(q_{\gamma^3}) - \beta\phi_t\frac{z_{\gamma^3}}{\gamma^3} - u(q_{\gamma^2}) + \beta\phi_t\frac{z_{\gamma^2}}{\gamma^3}] + ... \\
&+\theta^N[u(q_{\gamma^N}) - \beta\phi_t\frac{z_{\gamma N}}{\gamma^N} - u(q_{\gamma^{N-1}}) + \beta\phi_t\frac{z_{\gamma^{N-1}}}{\gamma^N}],
\end{aligned}$$

where μ^i is the multiplier associated with the liquidity constraint $z_{\gamma^i} \leq \hat{m}^b + (\gamma^i - 1)M$, and θ^i is the Lagrangian multiplier associated with the incentive compatibility constraint $u(q_{\gamma^i}) - \beta\phi_t\frac{z_{\gamma i}}{\gamma^i} = u(q_{\gamma^{i-1}}) - \beta\phi_t\frac{z_{\gamma i-1}}{\gamma^i}$ (for $1 < i \leq N$). I obtain the following first-order conditions:

$$q_{\gamma^1} : \left(p^1 + \theta^{1,2} + ... + \theta^{1,N} - \theta^2\right) u'(q_{\gamma^1}) = \lambda p c'(q_{\gamma^1}), \tag{55}$$

$$q_{\gamma^2} : \left(p^2 - \theta^{1,2} + \theta^2 - \theta^3\right) u'(q_{\gamma^2}) = \lambda p^2 c'(q_{\gamma^2}), \tag{56}$$

$$\vdots$$

$$q_{\gamma^N} : \left(p^N - \theta^{1,N} + \theta^N\right) u'(q_{\gamma^N}) = \lambda p^2 c'(q_{\gamma^N}),$$

$$z_{\gamma^1} : \frac{(\lambda - 1)\,\beta\phi p^1}{\gamma^1} - \mu^1 - (\theta^{1,2} + ... + \theta^{1,N})\frac{\beta\phi}{\gamma^1} + \theta^2\frac{\beta\phi}{\gamma^2} = 0, \tag{57}$$

$$\vdots$$

$$z_{\gamma^N} : \frac{(\lambda - 1)\,\beta\phi p^N}{\gamma^N} - \mu^N + \theta^{1,N}\frac{\beta\phi}{\gamma^1} - \theta^N\frac{\beta\phi}{\gamma^N} = 0, \tag{58}$$

$$\hat{m}^b : \mu^1 + ... + \mu^N = \phi[1 - \beta E(\gamma_t^{-1})], \tag{59}$$

$$\lambda : p^1\left[-c(q_{\gamma^1}) + \beta\phi\frac{z_{\gamma^1}}{\gamma^1}\right] + ... + p^N\left[-c(q_{\gamma^N}) + \beta\phi\frac{z_{\gamma^N}}{\gamma^N}\right] - \bar{S}^s > 0; = 0 \text{ if } \lambda > 0, \tag{60}$$

$$\mu^1 : \hat{m}^b + (\gamma^1 - 1)M - z_{\gamma^1} > 0; = 0 \text{ if } \mu^1 > 0, \tag{61}$$

$$\vdots$$

$$\mu^N : \hat{m}^b + (\gamma^N - 1)M - z_{\gamma^N} > 0; = 0 \text{ if } \mu^N > 0, \tag{62}$$

$$\theta^{1,2} : u(q_{\gamma^1}) - \beta\phi\frac{z_{\gamma^1}}{\gamma^1} - u(q_{\gamma^2}) + \beta\phi\frac{z_{\gamma^2}}{\gamma^1} > 0; = 0 \text{ if } \theta^{1,2} > 0, \tag{63}$$

$$\vdots$$

$$\theta^{1,N} : u(q_{\gamma^1}) - \beta\phi\frac{z_{\gamma^1}}{\gamma^1} - u(q_{\gamma^N}) + \beta\phi\frac{z_{\gamma^N}}{\gamma^1} > 0; = 0 \text{ if } \theta^{1,N} > 0, \tag{64}$$

$$\theta^2 : u(q_{\gamma^2}) - \beta\phi\frac{z_{\gamma^2}}{\gamma^2} - u(q_{\gamma^1}) + \beta\phi\frac{z_{\gamma^1}}{\gamma^2} > 0; = 0 \text{ if } \theta^2 > 0, \tag{65}$$

$$\vdots \tag{66}$$

$$\theta^N : u(q_{\gamma^N}) - \beta\phi\frac{z_{\gamma^N}}{\gamma^N} - u(q_{\gamma^{N-1}}) + \beta\phi\frac{z_{\gamma^{N-1}}}{\gamma^N} > 0; = 0 \text{ if } \theta^N > 0. \quad (67)$$

Following the same procedure, I can show that

$$\lambda = \frac{1}{\beta E(\gamma_t^{-1})}.$$

From Lemmas 2 and 3, it is clear that $\mu^1 > 0$, $\theta^{1,2} = \theta^{1,3} = ... = \theta^{1,N} = 0$, and $\theta^2 > 0, \theta^3 > 0, ..., \theta^N > 0$. Therefore, when I solve for the solution to the system described above, I only need to consider the following N possible cases.

Case 1: $\mu^1 > 0$, $\mu^2 = \mu^3 = ... = \mu^N = 0$. The equilibrium solution of q_{γ_t} is characterized by the following equations:

$$\frac{u'(q_{\gamma^N})}{c'(q_{\gamma^N})} = 1,$$

$$\frac{u'(q_{\gamma^i})}{c'(q_{\gamma^i})} = \frac{p^i}{\beta E(\gamma_t^{-1})(p^i + \theta^i - \theta^{i+1})}, \text{ for } 1 < i \leq N-1.$$

$$\frac{u'(q_{\gamma^1})}{c'(q_{\gamma^1})} = \frac{p^1}{\beta E(\gamma_t^{-1})(p^1 - \theta^2)},$$

$$\theta^N = [\frac{1}{\beta E(\gamma_t^{-1})} - 1]p^N, \text{and}$$

$$\theta^i = [\frac{1}{\beta E(\gamma_t^{-1})} - 1]\gamma^i(\frac{p^i}{\gamma^i} + \frac{p^{i+1}}{\gamma^{i+1}} + ... + \frac{p^N}{\gamma^N}), \text{ for } 1 < i \leq N-1,$$

where θ^i is the Lagrangian multiplier associated with the incentive compatibility constraint $u(q_{\gamma^i}) - \beta\phi_t\frac{z_{\gamma^i}}{\gamma^i} = u(q_{\gamma^{i-1}}) - \beta\phi_t\frac{z_{\gamma^{i-1}}}{\gamma^i}$ (for $1 < i \leq N$).

Case 2: $\mu^1 > 0$, $\mu^2 > 0$, $\mu^3 = ... = \mu^N = 0$. The equilibrium solution of q_{γ_t} is characterized by the following equations:

$$\frac{u'(q_{\gamma^N})}{c'(q_{\gamma^N})} = 1,$$

$$\frac{u'(q_{\gamma^i})}{c'(q_{\gamma^i})} = \frac{p^i}{\beta E(\gamma_t^{-1})(p^i + \theta^i - \theta^{i+1})}, \text{ for } 1 < i \leq N-1.$$

$$\frac{u'(q_{\gamma^1})}{c'(q_{\gamma^1})} = \frac{p^1}{\beta E(\gamma_t^{-1})(p^1 - \theta^2)},$$

$$u(q_{\gamma^2}) - u(q_{\gamma^1}) = \beta\phi_t M_t \frac{\gamma^2 - \gamma^1}{\gamma^2}$$

$$\theta^N = [\frac{1}{\beta E(\gamma_t^{-1})} - 1]p^N, \text{and}$$

$$\theta^i = [\frac{1}{\beta E(\gamma_t^{-1})} - 1]\gamma^i(\frac{p^i}{\gamma^i} + \frac{p^{i+1}}{\gamma^{i+1}} + ... + \frac{p^N}{\gamma^N}), \text{ for } 2 < i < N-1.$$

Case 3: $\mu^1 > 0$**,** $\mu^2 > 0$, $\mu^3 > 0, \mu^4 = ... = \mu^N = 0$. The same logic as Case 3 applies here.

$$\vdots$$

Case N-1: $\mu^1 > 0$**,** $\mu^2 > 0$, $..., \mu^{N-1} > 0, \mu^N = 0$. The optimal solution of q_{γ_t} must satisfy the following equations:

$$\frac{u'(q_{\gamma^N})}{c'(q_{\gamma^N})} = 1,$$

$$u(q_{\gamma^i}) - u(q_{\gamma^{i-1}}) = \beta\phi_t M_t \frac{\gamma^i - \gamma^{i-1}}{\gamma^i}, \text{ for } 1 < i \leq N-1,$$

$$\frac{u'(q_{\gamma^i})}{c'(q_{\gamma^i})} = \frac{p^i}{\beta E(\gamma_t^{-1})(p^i + \theta^i - \theta^{i+1})}, \text{ for } 1 < i \leq N-1,$$

$$\frac{u'(q_{\gamma^1})}{c'(q_{\gamma^1})} = \frac{p^1}{\beta E(\gamma_t^{-1})(p^1 - \theta^2)}, \text{ and}$$

$$\theta^N = [\frac{1}{\beta E(\gamma_t^{-1})} - 1]p^N.$$

Case N: $\mu^1 > 0$**,** $\mu^2 > 0$, $..., \mu^{N-1} > 0, \mu^N > 0$. The optimal solution of q_{γ_t} must satisfy the following equations:

$$u(q_{\gamma^i}) - u(q_{\gamma^{i-1}}) = \beta\phi_t M_t \frac{\gamma^i - \gamma^{i-1}}{\gamma^i}, \text{ for } 1 < i \leq N,$$

$$\frac{u'(q_{\gamma^i})}{c'(q_{\gamma^i})} = \frac{p^i}{\beta E(\gamma_t^{-1})(p^i + \theta^i - \theta^{i+1})}, \text{ for } 1 < i < N,$$

$$\frac{u'(q_{\gamma^N})}{c'(q_{\gamma^N})} = \frac{p^N}{\beta E(\gamma_t^{-1})(p^N + \theta^N)}, \text{ and}$$

$$\frac{u'(q_{\gamma^1})}{c'(q_{\gamma^1})} = \frac{p^1}{\beta E(\gamma_t^{-1})(p^1 - \theta^2)}.$$

In this case, the system implies that for a given value of $E(\gamma_t^{-1})$ and for given values of $\gamma^2, \gamma^3, ..., \gamma^{N-1}$,

$$\frac{d(\theta^N)}{d(\Delta)} > 0, \frac{d(q_{\gamma^N})}{d(\Delta)} > 0.$$

To summarize the results, if $\Delta \leq \bar{\Delta}^1$, the equilibrium is characterized by Case N. If $\bar{\Delta}^1 \leq \Delta < \bar{\Delta}^2$, $\theta^N = [\frac{1}{\beta E(\gamma_t^{-1})} - 1]p^N$, and the liquidity constraint is not binding only at γ^N, the equilibrium now switches to the one characterized by Case N-1. As Δ increases and reaches the next critical values, the equilibrium switches again. Finally, if $\gamma^N \geq \bar{\Delta}^{N-1}$, the liquidity constraint is binding only at γ^1, and the equilibrium is now characterized by Case 1.

6. Data Sources. The interest rate is the short-term commercial paper rate. For 1892-1971, it is taken from Friedman and Schwartz (1982), Table 4.8, Column 6. For 1972-2004, it is taken from the DRI series FYCP90 (averaged).

Money is $M1^* = M1$- currency outside the country. M1 is the stock at the end of June of each year. For 1892-1928, the source of M1 is the United States Bureau of the Census (1960), Series X267. For 1929-1958, it is the series constructed by the St. Louis FED that extends modern M1back in time: http://reserarch.stlouisfed.org/aggreg. For 1959-2004, it is the DRI series FZM1. Currency in circulation abroad is from the FED *Flow of Funds Table* L-204 in the file ltab204d.prn downloaded from http://www.federalreserve.gov/releases/z1/current/data.htm.

For 1892-1928, GDP is calculated from the real GDP series in Kendrick (1961) and the implicit price deflator in Friedman and Schwartz (1982), Table 4.8, Column 4. For 1929-2004, it is taken from BEA NIPA Table 1.1.5 downloaded from www.bea.doc.gov/bea/dn/nipaweb.