Economic Crises in Agent Based Models of Housing Markets

Michael Harre
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My interests

Decision Theory

Psychology

Artificial Intelligence and Agents

Economics

Incentivised Social Interactions
Introduction
Introduction

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Introduction

Discontinuous change in economic systems

Economic Theory:
Debreu’s theory as an exemplar

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Outline

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4. A Modern Approach
1. Catastrophe Theory

Ian Stewart’s 1976 general discussion of Catastrophes (Euler’s Arch)

\[ V = x^4 - bx^2 + ax \quad ( + \text{const.}) \]

Equilibrium positions of the arch are given by

\[ 0 = \frac{\partial V}{\partial x} = 4x^3 - 2bx + a. \]

The fold curve \( F \) occurs when

\[ 0 = \frac{\partial^2 V}{\partial x^2} = 12x^2 - 2b, \]

Figure 2.
1. Catastrophe Theory

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\[ 27a^2 = 8b^3 \]
1. Catastrophe Theory

Stochastic Catastrophe Theory

\[
\begin{align*}
\frac{dx_t}{dt} &= -\frac{\partial V(x_t|u_1,u_2)}{\partial x_t} dt \\
\frac{dx_t}{dt} &= -\frac{\partial V_\sigma(x_t|u)}{\partial x_t} dt + \sigma(x_t) dW_t
\end{align*}
\]

\[
\left. \frac{\partial V(x_t|u_1,u_2)}{\partial x_t} \right|_x = -x^3 + u_1x + u_2 = 0 \quad (\equiv \mu(x) \text{ for brevity})
\]

\[
\begin{align*}
p(x|u,\sigma(x)) &= \mathcal{Z} \exp \left( 2 \int_a^{x_t} \frac{\mu(z) - 0.5(d_z\sigma(z)^2)}{\sigma(z)^2} dz \right) \\
p(x|u,\xi) &= \mathcal{Z} \exp \left( \xi \int_a^{x_t} \mu(z) dz \right)
\end{align*}
\]

\[
\begin{align*}
d_z\sigma(z)^2 &= \frac{d}{dz} \sigma(z)^2 \\
2\sigma(z)^{-2} &= \xi
\end{align*}
\]
1. Catastrophe Theory

Stochastic Catastrophe Theory
1. Catastrophe Theory

Stochastic Catastrophe Theory

STOCHASTIC CATASTROPHE MODELS AND MULTIMODAL DISTRIBUTIONS

by Loren Cobb

University of South Florida, Tampa

Nonlinear models such as have been appearing in the applied catastrophe theory literature are almost universally deterministic, as opposed to stochastic (probabilistic). The purpose of this article is to show how to convert a deterministic catastrophe model into a stochastic model with the aid of several reasonable assumptions, and how to calculate explicitly the resulting multimodal equilibrium probability density. Examples of such models from epidemiology, psychology, sociology, and demography are presented. Lastly, a new statistical technique is presented, with which the parameters of empirical multimodal frequency distributions may be estimated.
1. Catastrophe Theory

Stochastic Catastrophe Theory

Transformation invariant stochastic catastrophe theory

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Abstract

Catastrophe theory is a mathematical formalism for modeling nonlinear systems whose discontinuous behavior is determined by smooth changes in a small number of driving parameters. Fitting a catastrophe model to noisy data constitutes a serious challenge, however, because catastrophe theory was formulated specifically for deterministic systems. Loren Cobb addressed this challenge by developing a stochastic counterpart of catastrophe theory (SCT) based on Itô stochastic differential equations. In SCT, the stable and unstable equilibrium states of the system correspond to the modes and the antinodes of the empirical probability density function, respectively. Unfortunately, SCT is not invariant under smooth and invertible transformations of variables—this is an important limitation, since invariance under diffeomorphic transformations is essential in deterministic catastrophe theory. From the Itô transformation rules we derive a generalized version of SCT that does remain invariant under transformation and can include Cobb’s SCT as a special case. We show that an invariant function is obtained by multiplying the probability density function with the diffusion function of the stochastic process. This invariant function can be estimated by a straightforward time series analysis based on level crossings. We illustrate the invariance problem and its solution with two applications.

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1. Catastrophe Theory

Stochastic Catastrophe Theory

This split between the original Thom formulation and the more generalized Arnol’d formulation also shows up in regard to the question of whether or not the systems must have a potential function, for which there is a necessary symmetry condition that all cross-partial derivatives must be equal. Again, the broader singularity theory does not require this, and forms that resemble the elementary catastrophes can appear within this theory even while not fulfilling the stricter assumption about the existence of a potential function. This would become a central issue in the later controversy over catastrophe theory.  

In Sandholm [8], potential games are defined using standard concepts from multivariable calculus. A population game $F$ is called a potential game if there is a scalar-valued function $f$, the potential function, whose gradient always equals the vector of payoffs; put differently, the payoff to strategy $i$ must always be given by the $i$th partial derivative of $f$. By standard results, $F$ is a potential game in this sense if and only if its derivative matrices $DF(x)$ are symmetric, so that corresponding cross partial derivatives of $F$ are equal.

1. Catastrophe Theory

Stochastic Catastrophe Theory: Evolutionary Game Theory

1. Linear fitness. We wish to include interactions between the types. The simplest possibility consists in considering replicator equations with linear fitness. With an interaction matrix $A = (a_{ij})_{i,j \in I}$, we consider the equation

$$\dot{p}_i = p_i \left( \sum_j a_{ij} p_j - \sum_k p_k \sum_j a_{kj} p_j \right).$$

(6.84)

While a constant fitness function could always be represented as a gradient field, in the linear case, by (2.38), we need the following condition:

$$a_{ij} + a_{jk} + a_{ki} = a_{ik} + a_{kj} + a_{ji}.$$ 

(6.85)

In particular, this is satisfied if the matrix $A$ is symmetric, and a potential function then is

$$V(p) = \frac{1}{2} \sum_{ij} a_{ij} p_i p_j.$$ 

Replicator equation: $p_i = \text{probability of state } i$
1. Catastrophe Theory

Stochastic Catastrophe Theory: Evolutionary Game Theory

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In particular, this is satisfied if the matrix \( A \) is symmetric, and a potential function then is

\[
V(p) = \frac{1}{2} \sum_{ij} a_{ij} p_i p_j.
\]

\[
- \frac{\partial V(x_t | u_1, u_2)}{\partial x_t} \bigg|_{x} = -x^3 + u_1 x + u_2
\]

Quoted from: Nihat Ay et al Information Geometry (2017, pg. 331)
2. Catastrophe Theory in Economics
2. Catastrophe Theory in Economics

The assumptions made below are, in several respects, weaker and closer to economic reality than A. Wald’s [23]. Unlike his models, ours presents an integrated system of production and consumption which takes account of the circular flow of income. The proof of existence is also simpler than his. Neither the uniqueness nor the stability of the competitive solution is investigated in this paper. The latter study would require specification of the dynamics of a competitive market as well as the definition of equilibrium.
2. Catastrophe Theory in Economics

Debreu and Singularity Theory: Econometrica article (1970)

Debreu stated: “A mathematical model which attempts to explain economic equilibrium must have a nonempty set of solutions ...” eventually concluding: “Our main result asserts that, under assumptions we will shortly make explicit ... every economy has a finite set of equilibria.”

Debreu knew of Thom’s work, but argued that so called ‘critical economies’ occurred negligibly often, i.e. the subset of all economic states that are exactly critical has measure zero.

While this is true, it misses the point of an economy transitioning through a critical state.
2. Catastrophe Theory in Economics

Debreu and Singularity Theory: Nobel prize (1983)

ECONOMIC THEORY IN
THE MATHEMATICAL MODE

Nobel Memorial lecture, 8 December, 1983

by
GERARD DEBREU

“The explanation of equilibrium given by a model of the economy would be complete if the equilibrium were unique, and the search for satisfactory conditions guaranteeing uniqueness has been actively pursued … However, the strength of the conditions that were proposed made it clear by the late sixties that global uniqueness was too demanding a requirement and that one would have to be satisfied with local uniqueness. Actually, that property of an economy could not be guaranteed even under strong assumptions about the characteristics of the economic agents. But one can prove, as I did in 1970, that, under suitable conditions, in the set of all economies, the set of economies that do not have a set of locally unique equilibria is negligible.”
2. Catastrophe Theory in Economics

Debreu and Singularity Theory: Nobel prize (1983)

ECONOMIC THEORY IN THE MATHEMATICAL MODE

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The $i^{th}$ consumer is characterized by his demand function $f_i$ ... [T]he economy is described by the list $e = (e_1, ..., e_m)$ of the $m$ endowment-vectors ... [and] the price vector $p$

$$F(p, e) = \sum_{i=1}^{m} [f_i (p, p \cdot e_i) - e_i].$$

For which equilibrium means the “excess demand” is zero: $F(p, e) = 0$
2. Catastrophe Theory in Economics

Debreu and Singularity Theory: Nobel prize (1983)

ECONOMIC THEORY IN
THE MATHEMATICAL MODE

Nobel Memorial lecture, 8 December, 1983

by
GERARD DEBREU

“Now let $T$ be the projection from $M$ into $P^m$ and define a critical economy $e$ as an economy such that it is the projection of a point $(e,p)$ of $M$ where the Jacobian of $T$ is singular, geometrically where the tangent linear manifold of dimension $l \times m$ does not project onto $P^m$. By Sard’s theorem the set of critical economies is closed and of Lebesgue measure zero. A regular economy, outside the negligible critical set, not only has a discrete set of equilibria; it also has a neighborhood in which the set of equilibria varies continuously as a function of the parameters defining the economy.”
2. Catastrophe Theory in Economics


To build up the qualitative picture of the flow, we shall take as hypotheses a number of observed qualitative features of stock exchanges and currencies, and translate each feature into mathematics.
2. Catastrophe Theory in Economics


Two types of ‘agents’:
- Fundamentalists
- Chartists

Proportion of market that are chartists: \( C \)

Fundamentalist excess demand: \( F \)

Market index: \( J \)

\[ J^3 - (C - C_0)J - F = 0 \]
2. Catastrophe Theory in Economics


**Hypothesis 3.** If $C$ is large this introduces an instability into the market. What does ‘instability’ mean mathematically? ...we are postulating in Hypothesis 3 that it is dynamically unstable for the index to remain constant. Any slight perturbation of the index up or down (by external forces) will at once be amplified by the chartists.

... The critical consequence of **Hypothesis 3** is that for large $C$ and small $F$ the function $J(C, F)$ is 2-valued, and so the attractor surface $S$ is 2-sheeted.
2. Catastrophe Theory in Economics

The rise and fall of catastrophe theory applications in economics:
Was the baby thrown out with the bathwater?

J. Barkley Rosser Jr.

The most important criticisms of catastrophe theory applications in general were by Zahler and Sussman (1977), Sussman and Zahler (1978a, b), and Kolata (1977). Responses appeared in *Science* and *Nature* in 1977, with a more vigorous and extended set of defenses appearing in *Behavioral Science* (Oliva and Capdeville, 1980; Guastello, 1981), with the first of these being the source of the line that ‘the baby was thrown out with the bathwater.’ More balanced overviews came from mathematicians (Guckenheimer, 1978; Arnol’d, 1992).
2. Catastrophe Theory in Economics

The rise and fall of catastrophe theory applications in economics: Was the baby thrown out with the bathwater?

J. Barkley Rosser Jr.

The critics succeeded in pointing out some dirty bathwater. The most salient points include: (1) excessive reliance on qualitative methods, (2) inappropriate quantization in some applications, and (3) the use of excessively restrictive or narrow mathematical assumptions. The third point in turn has at least three sub-points: (a) the necessity for a potential function to exist for it to be properly used, (b) that the necessary use of gradient dynamics does not allow the use of time as a control variable as was often done in many applications, and (c) that the set of elementary catastrophes is only a limited subset of the possible range of bifurcations and catastrophes. These arguments relate to applications of catastrophe theory in general rather than to economics specifically.
2. Catastrophe Theory in Economics

Financial Markets: Plerou et al.

\( Q_B = \) buyer initiated transactions
\( Q_S = \) seller initiated transactions

\[ \Omega(t) \equiv Q_B - Q_S = \sum_{i=1}^{N} q_i a_i \]
\[ \Sigma(t) \equiv \langle |q_i a_i - \langle q_i a_i \rangle| \rangle \]

The order parameter \( \Psi = \Psi(\Sigma) \) is given by the values of the maxima of \( P(\Omega) \).

\[
\Psi(\Sigma) = \begin{cases} 
0 & [\Sigma < \Sigma_c] \\
\Sigma - \Sigma_c & [\Sigma \gg \Sigma_c]
\end{cases}
\]

2. Catastrophe Theory in Economics

Financial Markets: Data for the S&P500 (log-linear)
2. Catastrophe Theory in Economics

Financial Markets: Data for the S&P500 (log-linear)
2. Catastrophe Theory in Economics

Housing Markets: US Housing Market Collapse

Figure 2: The impact of the boom and bust in the US housing market was regionally heterogeneous, seasonally adjusted monthly house price indices for the nine Census Bureau Divisions of the United States, indices set to 100 on January 1st 1991.
2. Catastrophe Theory in Economics

Housing Markets: US Housing Market Collapse

Figure 12: Pitchfork bifurcation w.r.t. the policy parameter $R$. Bold curves denote stable steady states; dotted curves are unstable steady states. **Left Panel:** supercritical pitchfork bifurcation with stable fundamental steady state for $R > R_{\text{crit}}$ and unstable fundamental steady state surrounded by two stable non-fundamental steady states for $R < R_{\text{crit}}$. **Right Panel:** subcritical pitchfork bifurcation with stable fundamental steady state surrounded by a corridor of stability bounded by two unstable non-fundamental steady states for $R > R_{\text{crit}}$ and (globally) unstable fundamental steady state with exploding dynamics for $R < R_{\text{crit}}$. The dots represent the estimated values $R - R_{\text{crit}}$ for the eight countries.
2. Catastrophe Theory in Economics

Housing Markets: US Housing Market Collapse

Can a stochastic cusp catastrophe model explain housing market crashes?

Cees Diks, Juanxi Wang

Fig. 18: Bifurcations showing the predicted equilibria as a function of the interest rate r for the different countries. Red scatter represents the stable equilibrium (up or bottom sheet). Blue scatter represents the unstable equilibrium (middle sheet).
Figure 3: Plot 1.: The *Cusp catastrophe* with three contours shown for fixed $u_2$ but variable $u_1$ control parameters. A Pitchfork bifurcation is shown in red and two fold bifurcations are shown in black and blue. Plot 2.: Two distinct regions $A$ and $B$ can be discerned in the projection of the equilibrium surface onto the control plane $\{u_1, u_2\}$, region $A$ has one equilibrium point and region $B$ has three equilibrium points.
3. The Quantal Response Equilibrium

Fig. 1. Hysteresis and the cyclical collapse of a system that is drifting across an equilibrium surface. $Q$ can be thought of as a behavioural outcome, dictated by the system structure and its parameters.
3. The Quantal Response Equilibrium

A game is a function

\[ g : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}^n, \]

where \( g(s_1, \ldots, s_n) = (g_1(s_1, \ldots, s_n), \ldots, g_n(s_1, \ldots, s_n)) \), and \( g_i(s_1, \ldots, s_n) \) is player \( i \)'s payoff when strategies \( (s_1, \ldots, s_n) \) are played.
3. The Quantal Response Equilibrium

A game is a function

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\[
\text{player } i \text{'s expected payoff } = \sum_{s_1 \in S_1} \ldots \sum_{s_n \in S_n} g_i(s_1, \ldots, s_n) p_1(s_1) \ldots p_n(s_n)
\]

\[ = g_i(p_1, \ldots, p_n) = g_i(p_i, p_{-i}). \]
3. The Quantal Response Equilibrium

A game is a function

\[ g : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}^n, \]

where \( g(s_1, \ldots, s_n) = (g_1(s_1, \ldots, s_n), \ldots, g_n(s_1, \ldots, s_n)) \), and \( g_i(s_1, \ldots, s_n) \) is player \( i \)'s payoff when strategies \((s_1, \ldots, s_n)\) are played.

player \( i \)'s expected payoff =

\[
\sum_{s_1 \in S_1} \ldots \sum_{s_n \in S_n} g_i(s_1, \ldots, s_n) \ p_1(s_1) \ldots p_n(s_n)
\]

\[ = g_i(p_1, \ldots, p_n) = g_i(p_i, p_{-i}). \]

Nash proved that there exists an \( n \)-tuple \((p_1^*, \ldots, p_n^*)\) such that

\[ g_i(p_1^*, \ldots, p_n^*) \geq g_i(p_i, p_{-i}^*) \quad \text{for all } i \text{ and } p_i. \]
3. The Quantal Response Equilibrium

Nash Equilibrium:

\[
u_{i,j}^a = \begin{bmatrix} u_{1,1}^a & u_{1,2}^a \\ u_{2,1}^a & u_{2,2}^a \end{bmatrix}, \quad u_{i,j}^b = \begin{bmatrix} u_{1,1}^b & u_{1,2}^b \\ u_{2,1}^b & u_{2,2}^b \end{bmatrix}
\]

\[
E(u^a) = \sum_{i,j} p_i q_j u_{i,j}^a, \quad E(u^b) = \sum_{i,j} p_i q_j u_{i,j}^b.
\]

\[
p_i^* = \arg\max_{p_i} \sum_{i,j} p_i q_j^* u_{i,j}^a \forall i \quad \text{and} \quad q_j^* = \arg\max_{q_j} \sum_{i,j} p_i^* q_j u_{i,j}^b \forall j.
\]

\[
p_i \in [0,1] \text{ and } q_j \in [0,1] \quad p_1 + p_2 = 1, \; q_1 + q_2 = 1
\]

Nash Equilibrium in \( p \) and \( q \): not unique
3. The Quantal Response Equilibrium

Quantal Response Equilibrium:

\[ p_i^* = \max_{p_i} S(p_i) = \max_{p_i} \left( - \sum_i p_i \ln(p_i) \right) \]

subject to the constraints:

\[ p_i \geq 0 \forall i, \sum_i p_i = 1, \sum_{i,j} p_i q_j u_{i,j}^a = E(u^a). \]
3. The Quantal Response Equilibrium

Quantal Response Equilibrium:

\[ p_i^* = \max_{p_i} S(p_i) = \max_{p_i} \left( - \sum_i p_i \ln(p_i) \right) \]

subject to the constraints:

\[ p_i \geq 0 \forall i, \quad \sum_i p_i = 1, \quad \sum_{i,j} p_i q_j u_{i,j}^a = E(u^a). \]

\[ \mathcal{L}(q_i) = S(p_i) + \beta a \sum_{i,j} p_i q_j u_{i,j}^a + \beta_0 \sum_i p_i, \]

\[ \frac{\partial \mathcal{L}(p_i)}{\partial p_i} = - \ln(p_i^*) + \beta a \sum_j q_j u_{i,j}^a + \beta_0 - 1 = 0, \]

\[ p_i^* = \mathcal{Z}_a^{-1} \exp \left( \beta_a \sum_j q_j^* u_{i,j}^a \right), \]

D. Wolpert M. Harre (2013)
3. The Quantal Response Equilibrium

\[ Q_r = f(Q_c, \beta_r) = 1 - \frac{2e^{\beta_r E(u_{r|top})}}{e^{\beta_r E(u_{r|top})} + e^{\beta_r E(u_{r|bottom})}} \]

\[ Q_c = g(Q_r, \beta_c) = 1 - \frac{2e^{\beta_c E(u_{c|left})}}{e^{\beta_c E(u_{c|left})} + e^{\beta_c E(u_{c|right})}} \]

\[ Q_r = \tanh \left[ 2\beta_r (f_r(U_r) + f_{c,r}(U_r)Q_c) \right] \]

\[ Q_c = \tanh \left[ 2\beta_c (f_c(U_c) + f_{c,r}(U_c)Q_r) \right] \]

\[ G(Q_c, \beta_c, \beta_r) \triangleq Q_c - g(Q_r, \beta_c) \]

\[ = Q_c - g(f(Q_c, \beta_r), \beta_c) \]

\[ F(Q_r, \beta_r, \beta_c) \triangleq Q_r - f(Q_c, \beta_r) \]

\[ = Q_r - f(g(Q_r, \beta_c), \beta_r) \]

\( Q_r \) in \([-1,1]\)
and
\( Q_c \) in \([-1,1]\)
3. The Quantal Response Equilibrium

\[ 0 = \frac{\partial Q_r}{\partial \beta_r} - \left( \frac{\partial f}{\partial \beta_r} + \frac{\partial f}{\partial g} \frac{\partial g}{\partial Q_r} \frac{\partial Q_r}{\partial \beta_r} \right) \]

\[ \frac{\partial f}{\partial \beta_r} = \frac{\partial Q_r}{\partial \beta_r} \left( 1 - \frac{\partial g}{\partial Q_r} \frac{\partial f}{\partial g} \right) \]

\[ \frac{\partial Q_r}{\partial \beta_r} = \frac{\frac{\partial f}{\partial \beta_r}}{1 - \frac{\partial g}{\partial Q_r} \frac{\partial f}{\partial g}} \]

\[ \frac{\partial Q_r}{\partial \beta_r} = \frac{f_2(Q_c, \beta_r)}{1 - g_1(Q_r, \beta_c) f_1(Q_c, \beta_r)} = \frac{f_2}{1 - g_1 f_1} \]
3. The Quantal Response Equilibrium

\[ 0 = \frac{\partial Q_r}{\partial \beta_r} - \left( \frac{\partial f}{\partial \beta_r} + \frac{\partial f}{\partial g} \frac{\partial g}{\partial Q_r} \frac{\partial Q_r}{\partial \beta_r} \right) \]

\[ \frac{\partial f}{\partial \beta_r} = \frac{\partial Q_r}{\partial \beta_r} \left( 1 - \frac{\partial g}{\partial Q_r} \frac{\partial f}{\partial g} \right) \]

\[ \frac{\partial Q_r}{\partial \beta_r} = \frac{\frac{\partial f}{\partial \beta_r}}{1 - \frac{\partial g}{\partial Q_r} \frac{\partial f}{\partial g}} \]

\[ \frac{\partial Q_r}{\partial \beta_r} = \frac{f_2(Q_c, \beta_r)}{1 - g_1(Q_r, \beta_c)f_1(Q_c, \beta_r)} = \frac{f_2}{1 - g_1f_1} \]

\[ \frac{\partial Q_r}{\partial \beta_r} = \frac{f_2}{1 - f_1g_1}, \quad \frac{\partial Q_r}{\partial \beta_c} = \frac{f_1g_2}{1 - f_1g_1} \]

\[ \frac{\partial Q_c}{\partial \beta_r} = \frac{g_1f_2}{1 - f_1g_1}, \quad \frac{\partial Q_c}{\partial \beta_c} = \frac{g_2}{1 - f_1g_1} \]

\[ \frac{1}{1 - f_1g_1} \begin{pmatrix} f_2 & f_1g_2 \\ g_1f_2 & g_2 \end{pmatrix} \]
3. The Quantal Response Equilibrium

Curves are values of $\beta_i$ for which: $1 = f_i g_i$
The bifurcation set in $Q_r \times Q_c$ partitions the possible states of a market/economy.
3. The Quantal Response Equilibrium

\[ Q_r = \tanh \left[ 2\beta_r (f_r(U_r) + f_{c,r}(U_r)Q_c) \right] \]
\[ Q_c = \tanh \left[ 2\beta_c (f_c(U_c) + f_{c,r}(U_c)Q_r) \right] \]

**Figure 4.** Perturbed QRE solutions for \( \delta_c = \delta_r \in \{0.2, 0, -0.2\} \) from left to right with a \( \beta \) pair \( \beta_c = \beta_r = 2 \), the equilibrium strategy is where the black dot is, see Equations (38)–(39).
3. The Quantal Response Equilibrium

\[ Q_r = \tanh \left[ 2\beta_r (f_r(U_r) + f_{c,r}(U_r)Q_c) \right] \]
\[ Q_c = \tanh \left[ 2\beta_c (f_c(U_c) + f_{c,r}(U_c)Q_r) \right] \]

Variations in \( \beta_{row} \) results in an induced bifurcation in \( Q_{col} \).
### 4. A Modern Treatment

\[
Q_r = \tanh \left[ 2\beta_r (f_r(U_r) + f_{c,r}(U_r)Q_c) \right]
\]

\[
Q_c = \tanh \left[ 2\beta_c (f_c(U_c) + f_{c,r}(U_c)Q_r) \right]
\]

\[
\langle \sigma_i \rangle = \tanh \left( \xi_i (h_i + \sum_{j \in \mathcal{V}_i} J_{i,j} \langle \sigma_j^e \rangle) \right)
\]

Brock and Durlauf


Modelling for “neighborhood effects” in economic markets
4. A Modern Treatment

\[
Q_r = \tanh \left[ 2\beta_r (f_r(U_r) + f_{c,r}(U_r)Q_c) \right]
\]
\[
Q_c = \tanh \left[ 2\beta_c (f_c(U_c) + f_{c,r}(U_c)Q_r) \right]
\]
\[
\langle \sigma_i \rangle = \tanh \left( \xi_i (h_i + \sum_{j \in V_i} J_{i,j} \langle \sigma^e_{j} \rangle) \right)
\]

Brock and Durlauf

"endogenous effects, wherein the propensity of an individual to behave in some way varies with the behaviour of the group … exogenous (contextual) effects, wherein the propensity of an individual to behave in some way varies with the exogenous characteristics of a group … correlated effects, wherein individuals in the same group tend to behave similarly because they have similar individual characteristics or face similar institutional environments”

Manski (1993, pg. 532)
4. A Modern Treatment

\[ Q_r = \tanh \left[ 2\beta_r (f_r(U_r) + f_{c,r}(U_r)Q_c) \right] \]

\[ Q_c = \tanh \left[ 2\beta_c (f_c(U_c) + f_{c,r}(U_c)Q_r) \right] \]

\[ \langle \sigma_i \rangle = \tanh \left( \xi_i (h_i + \sum_{j \in \mathcal{V}_i} J_{i,j} \langle \sigma_j^e \rangle) \right) \]

Brock and Durlauf


Chapter 54

INTERACTIONS-BASED MODELS

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