Effective Mathematics Instruction for Students with Learning Problems: A Balanced Approach

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Background

• Mathematics is an essential aspect of all areas of daily life.

• All children must learn to think mathematically, and they must think mathematically to learn” (NRC, 2001, p.1).
• If the average salaries of a particular group within a population are 16 percent less than the average salary of the entire population, and one wants to give the individuals in that group a raise to bring them up to parity, what should the raise be – 16 percent, something more, or something less?
You know, I don't think math is a science. I think it's a religion. All these equations are like miracles. You take two numbers and when you add them, they magically become one new number! No one can say how it happens. You either believe it or you don't.
Mathematical Proficiency

• The Education Act 1990 (NSW) – every student must be mathematically competent.

• What students should know (e.g., number, patterns and algebra, space and geometry, data, and measurement) and be able to do (i.e., question, reason, communicate, apply strategies, reflect) in mathematics.

Summary: Skill in basic arithmetic is no longer a sufficient mathematics background.
Mathematical Proficiency

1. **Conceptual understanding** – comprehension of mathematical concepts, operations, and relations

2. **Procedural fluency** – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

3. **Strategic competence** – ability to formulate, represent, and solve mathematical problems

4. **Adaptive reasoning** – capacity for logical thought, reflection, explanation, and justification

5. **Productive disposition** – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

NRC (2001, p. 5)
PSST! WHAT'S 7 + 6?
THREE HUNDRED BILLION GAZILLION.

OH, THANKS FOR THE BIG HELP!

THAT'S A THREE, FOLLOWED BY 85 ZEROS.

AH! I KNEW THAT.
## Building Mathematical Proficiency

<table>
<thead>
<tr>
<th>CONTENT STRANDS</th>
<th>WORKING MATHEMATICALLY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1. Question</td>
</tr>
<tr>
<td>1. Number</td>
<td></td>
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<tr>
<td>2. Patterns and Algebra</td>
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<td></td>
<td>Mathematical Proficiency</td>
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<tr>
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<td>Conceptual Understanding</td>
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<td>3. Space and Geometry</td>
<td>Procedural Fluency</td>
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<tr>
<td></td>
<td>Strategic Competence</td>
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<tr>
<td></td>
<td>Adaptive Reasoning</td>
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<tr>
<td>4. Measurement</td>
<td>Productive Disposition</td>
</tr>
<tr>
<td>5. Data</td>
<td></td>
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</tbody>
</table>
Students with learning difficulties evidence problems in learning math or related ideas.

These learning difficulties are not related to IQ, motivation or other factors that influence learning.

Geary (1996)
Students with Learning Difficulties

Difficulties in learning math may be related to a variety of learner characteristics (e.g., slower rate of learning; difficulties with symbol systems). For example, these students

- Cannot remember basic math facts.
- Use immature (and time-consuming) problem-solving procedures to solve simple math problems
- Also have problems learning to read and write

Geary (1996)
Instructional Approaches to Learning

1. Constructivist
2. Direct Instruction
3. Learning Strategies
Constructivist View to Learning

Children actively construct their own solutions to mathematics problems

• True discovery is rare and time consuming.

• Impossible for students to discover all they need to know.

• Discovery learning can be full of errors.

Harris & Pressley (1989)
Direct Instruction

Structured, systematic instruction that is explicit, teacher directed, and includes guided practice and corrective feedback, whether students engage in concrete activities (e.g., using manipulatives) or symbolic activities.

(Lloyd & Keller, 1989)
Learning Strategies

Strategies that help students take control of their own mathematics learning.

For example, students are encouraged to talk to themselves and ask questions about mathematics problems (e.g., What am I missing? Do I need to add or subtract?)
Summary of Instructional Approaches

• Instruction should not be based on extreme positions. That is, students learn solely by “internalizing what a teacher or book says” or “solely by inventing mathematics on their own.”

• Decisions about how to help students reach learning goals can never be made with absolute certainty.

NRC (2001, p. 11)
"The Bible tells us to study math. It says, 'Go forth and multiply.'"
Mathematics Teaching and Learning

Proficiency in teaching mathematics = effectiveness + versatility

- Effective Instruction and Student Understanding
  - Teachers’ Knowledge and Use of Mathematical Content
  - Teachers’ Attention to and Work With Students
  - Students’ Engagement in and Use of Mathematical Tasks

• NRC (2001)
Mathematics Teaching and Learning: Summary

• “What” teachers teach is just as important as how it is taught.

• Teaching involves more than simply delivering knowledge or information. It entails:
  – a fundamental understanding of the nature and manifestations of learning problems.
  – skills needed to modify and design materials for students with learning difficulties.
Principles for the Successful Teaching of Students at Risk

1. Have high expectations for all students, letting them know that you believe in them.
2. Use “instructional scaffolding.”
3. Make instruction the focus of each class. (Avoid busy work.)
4. Extend students’ thinking and abilities beyond what they already know.
5. Work at gaining in-depth knowledge of your students as well as knowledge of the subject matter.

Ladson-Billings (1995)
Architecture for Teaching and Learning

1. Big ideas
2. Conspicuous strategies
3. Scaffolding
4. Strategic integration
5. Judicious review

Kame‘enui, Carnine, Dixon, Simmons, & Coyne (2002)
Big Ideas

Mathematical concepts, principles, or strategies (heuristics) that:
- form the basis for further mathematical learning.
- map to the content standards
- are sufficiently powerful to allow for broad application
FIGURE 5–1
Volume as the "Big Idea" of Area of the Base Times a Multiple of the Height

Rectangular
Prism
B × h

Wedge
B × h

Cylinder
B × h

Pyramid
Rectangular
B × \frac{1}{3} × h

Triangular
B × \frac{1}{3} × h

Cone
B × \frac{1}{3} × h

Sphere
B × \frac{2}{3} × h
Conspicuous Strategies

Expert actions for problem solving that are made overt through teacher and peer modeling.

In selecting examples, consider teaching:
- the general case for which the strategy works.
- both how and when to apply the strategy.
- when the strategy doesn’t work.
Scaffolding

Overt

Support

Covert

Sequence instruction to avoid confusion of similar concepts
Carefully select and sequence examples
Pre-teach prerequisite knowledge
Provide specific feedback
Integration

Integrate new learning with prior learning and with other disciplines

... It is not necessary that students master place value before they learn a multi-digit algorithm; the two can be developed in tandem (NRC, 2001).
Judicious Review

Review must be:

- Sufficient
- Distributed over time,
- Cumulative
- Varied
A mathematics curriculum should focus on knowledge that is essential for developing mathematics competence. This involves:

(a) Concepts
(b) Skills
(c) Problem Solving

Further, a mathematics curriculum must validate and extend one’s thinking by emphasizing the process of problem solving rather than the answers.
Concepts

Concepts are the building blocks of mathematical knowledge and are fundamental to basic mathematical understanding.

For example, a concept is developed when a student learns that the number 7 is more than the number 6; the concept of one-half; when a number is multiplied by 10, the product is that number followed by a 0.

Activities to teach concepts may involve use of objects and materials.
Developing Fractional Concepts: Using Examples and Nonexamples

When teaching fractional parts (e.g., one-fourths), present examples and nonexamples using drawings of figures that are of different shapes.
Skills/Operations

Skills involve the use and manipulation of symbols that represent mathematical knowledge - something one does.

For example, basic facts, algorithms and/operations in addition, subtraction, multiplication, and division.

Skills tend to develop by degrees and can be improved through instructional activities.

Instruction to develop fluency and recall of math facts may include peer assisted instruction, constant time delay, touch math, or mnemonic strategies.
TODAY'S MY BIRTHDAY, TURNKEY

REALLY?

HOW OLD ARE YOU?

# # # #
• Introduce each fact (e.g., 6 x 4) on a flash card, obtain the student’s attention by saying, “Look (point to fact on card): 6 x 4. Listen, I will read the fact and say the answer: 6 x 4 = 24.”

• “What is 6 x 4?” Immediately prompt the response, which the student then repeats (e.g., “six times four equals twenty-four”).

• All subsequent training on the target set of twenty-five fact cards would use a 5-sec delay.
  - Present the fact card, say: “Read the fact. Say the answer.”
  - Give the student 5 seconds after reading the fact aloud to state the correct product. (The 5-second constitutes a wait period, after which provide the correct answer).

Mattingly & Bott (1990)
Touch Math

Touch Math method simplifies and clarifies all areas of computation, develops left/right directionality, reduces number reversals, reinforces number values, eliminates guesswork, and helps to develop positive self-image. It favors tactile learners, as well as auditory and visual.

Caution: Students using touch math to solve computational problems often fail to develop a good sense of the underlying concepts. If you use this strategy, students should have a good understanding of the concepts underlying the operations.
<table>
<thead>
<tr>
<th>Pegwords and Corresponding Numbers for Groups A and B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Group “A” pegwords</strong></td>
</tr>
<tr>
<td>4 = door</td>
</tr>
<tr>
<td>6 = sticks</td>
</tr>
<tr>
<td>7 = heaven</td>
</tr>
<tr>
<td>8 = gate</td>
</tr>
<tr>
<td>9 = line</td>
</tr>
<tr>
<td>16 = sitting</td>
</tr>
<tr>
<td><strong>Group “B” pegwords</strong></td>
</tr>
<tr>
<td>4 = door</td>
</tr>
<tr>
<td>6 = sticks</td>
</tr>
<tr>
<td>7 = heaven</td>
</tr>
<tr>
<td>8 = gate</td>
</tr>
<tr>
<td>9 = line</td>
</tr>
<tr>
<td>24 = twin doors</td>
</tr>
</tbody>
</table>

Greene (1999)
Greene (1999)

Mnemonic Strategy for Multiplication Facts

6 = sticks; 7 = heaven; 42 = warty shoe

\[ 6 \times 7 = 42 \]

"6x7=42; sticks in heaven with a warty shoe"

FIGURE 1 Example of a mnemonic condition flashcard.
Problem Solving

Problem solving is the application of math concepts and skills. It involves the selection and use of known concepts or skills in a new or different setting.

For example: When measuring a board of lumber, what concepts and skills are involved?
Functional Skills

Functional skills should be integrated into the math curriculum, beginning in the early grades (Halpern, 1981). Topics may include:

- basic operations
- consumer skills (i.e., doing cost comparisons, balancing a checkbook)
- telling time and estimating time intervals
- general estimation
- budgeting and money management; banking skills; calculating simple interest
- computing taxes
- evaluating insurance and benefit plans.

See Jitendra & Nolet (1995)
1. Number

1. Whole Numbers: Students develop a sense of the relative size of numbers and the role of place value in their representation
   - Early Stage 1: Counts to 30, and orders, reads, and represents numbers in the range of 0 to 20.
Number sense refers to a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons (Case, 1998).
Number Sense and Conceptual Understanding

- Charlie: "My math teacher is crazy".
- Mother: "Why?"
- Charlie: "Yesterday she told us that five is 4+1; today she is telling us that five is 3 + 2."
Teaching Students with Special Needs

• Teaching number sense is analogous to teaching phonemic awareness to young children. Developing readers need phonemic awareness instruction and developing mathematicians need number sense instruction.

• Explicitly teach to develop a rich understanding of number, a relational understanding, which involves many different ideas, relationships, and skills.
  – Use modeling,
  – pose process questions,
  – encourage thinking about numbers, and in general
  – create a classroom environment that nurtures number sense.

Gersten & Chard (1999)
1. Counting tells how many things are in the set.

2. Numbers are related to each other through a wide variety of relationships.

3. Numbers have different values when connected to real objects and measures.

(Van de Walle, 2004)
Content Connections

• Category 1: Content that interacts with and enhances the development of number
  - Measurement
  - Data
  - Operation meanings

• Category 2: Content that is directly affected by how well early number concepts have been developed
  - Basic facts
  - Place value
  - Computation

Van de Walle (p. 115, 2004)
BIG IDEA #1: Counting

To make counting meaningful, it is important to promote knowledge and awareness of patterns, classification, cardinality principle, and conservation of number.

Counting Sets Activity

Have children count several sets where the number of objects is the same but the objects are very different in size.

Discuss how they are alike or different. "Were you surprised that they were the same amount? Why or why not?"

(Van de Walle, 2004)
Count and Rearrange Activity

Have children count a set. Then rearrange the set and ask, “How many now?”

If they see no need to count over, you can infer that they have connected the cardinality to the set regardless of its arrangement.

If they choose to count again, discuss why they think the answer is the same.

(Van de Walle, 2004)
Summary: Counting

• Informal Strategies (Gersten & Chard, 1999)
  • Parents can help children develop early number sense using various activities such as asking them to:
    • ascend and count four steps and then count and descend two steps
    • count forks and knives when setting the table.

• Formal Strategies
  • Rational counting (Stein, Silbert, Carnine, 1997)
  • “Min” strategy: Starting with the larger number and counting on when trying to find the answer to either 3+8 or 8+3, whether using one’s fingers, manipulatives, or stick figures (Gersten & Chard, 1999).
BIG IDEA #2: Number Relationships

Development of number relationships requires knowledge and awareness of comparison of quantities and group recognition. Group recognition:

(1) saves time,
(2) is the forerunner of some powerful number ideas (i.e., knowing early order relations, such as 3 > than 2),
(3) helps develop more sophisticated counting skills, and
(4) accelerates the development of addition and subtraction skills.

Number Relationships

1. Spatial relationships
2. One and two more, one and two less
3. Anchors or “benchmarks of 5 and 10
4. Part-part-whole relationship

Van de Walle (2004)
Figure 6.5  Four relationships to be developed involving small numbers.
Learning Patterns Activity

To introduce the patterns, provide each student with about 10 counters and a piece of construction paper as a mat. Hold up a dot plate for about 3 seconds.

“Make the pattern you saw using the counters on the mat. How many dots did you see? How did you see them?

Discuss the configuration of the patterns and how many dots. Do this with a few new patterns each day.

(Van de Walle, 2004)
One-Less-Than Dominoes Activity

Use the dot pattern dominoes or a standard set to play “one-less-than” dominoes. Play in the usual way, but instead of matching ends, a new domino can be added if it has an end one less than the end on the board.

A similar game can be played for two less, one more, or two more.

Van de Walle (2004)
Ten-Frame Flash Activity

• Flash ten-frame cards to the class or group, and see how fast the children can tell how many dots are shown.

• Variations:
  • Saying the number of spaces on the card instead of the number of dots
  • Saying one more than the number of dots (or two more, and also less than)
  • Saying the “ten fact” – for example, “Six and four make ten”

(Van de Walle, 2004)
Ten Frame

[Diagram of a ten frame with some dots]
### Part-Part-Whole: Two out of Three Activity

Make lists of three numbers, two of which total the whole. Here is an example list for the number 5.

- 2-3-4
- 5-0-2
- 1-3-2
- 3-1-4
- 2-2-3
- 4-3-1

1. Write the list on the board or overhead.
2. Have children take turns selecting the two numbers that make the whole.
3. Challenge children to justify their answers.

(Van de Walle, 2004)
Summary: Teaching Number Sense

- Math instruction that emphasizes memorization and repeated drill is limited (Gersten & Chard, 1999).
- Strategies that support mathematical understanding are more generalizable.
- Teaching number sense shifts the focus from computation to mathematical understanding and better helps students with learning difficulties.
I. Number

2. Addition and Subtraction: Students develop facility with number facts and computation with progressively larger numbers in addition and subtraction and an appreciation of the relationship between those facts.

• **Stage 1:** Uses a range of mental strategies and informal recording methods for addition and subtraction involving one- and two-digit numbers.
Number Outcomes

Essential to learning addition and subtraction, include understanding:

- Basic facts
- Place value
- Problem structures and properties (laws)
- Algorithms

Understanding (e.g., meanings of operations) should occur at all levels: concrete, semiconcrete (representational), and abstract.

Knowing basic number combinations - the single digit addition and multiplication pairs and their counterparts for subtraction and division - is essential.” (NCTM, Principles and Standards, p. 32)
Addition and Subtraction Basic Facts

• Basic facts are simple closed number sentences used in computation. They must be understood and memorized; these are the tools of computation.

• Addition and subtraction facts involve two one-digit addends.

• There are 200 basic addition and subtraction facts.

• Once understanding and basic facts are mastered, the specific operation can be expanded readily using place value.

Ashlock (1998)
A Three-Step Approach to Fact Mastery

1. Develop a good understanding of the operations and of number relationships.

If you didn’t know how much $6 + 8$ was, how could you find out. Try to find some easy ways that you could do quickly in your head.

Van de Walle (2004)
A Three-Step Approach to Fact Mastery

2. Help students develop efficient strategies for fact retrieval.

Examples of strategies for addition facts include:

- one-more-than and two-more-than facts
- facts with zero, doubles, near-doubles or “doubles-plus one facts”
- make-ten.
- doubles plus two or two apart (3 + 5, 4 + 6),
- make-ten extended (7 + 4 = 7 and three more is ten and 1 left is 11)
- counting on
- ten frame facts (model all of the facts from 5 + 1 to 5 + 5 and the respective turn arounds).

Van de Walle (2004)
Figure 1. Addition & Subtraction Number Family Table

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 | 39 | 42 | 45 | 48 | 51 | 54 | 57 | 60 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 | 52 | 56 | 60 | 64 | 68 | 72 | 76 | 80 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 | 95 | 100 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 72 | 78 | 84 | 90 | 96 | 102 | 108 | 114 | 120 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 | 77 | 84 | 91 | 98 | 105 | 112 | 119 | 126 | 133 | 140 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 | 104 | 112 | 120 | 128 | 136 | 144 | 152 | 160 |
| 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 | 99 | 108 | 117 | 126 | 135 | 144 | 153 | 162 | 171 | 180 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 | 140 | 150 | 160 | 170 | 180 | 190 | 200 |
Alternative Teaching Sequence for Addition Facts

<table>
<thead>
<tr>
<th></th>
<th>(1) number + 1 and number + 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2) doubles</td>
<td>2+2 3+3 4+4 5+5 6+6 7+7 8+8 9+9</td>
</tr>
<tr>
<td>(3) doubles + 1</td>
<td>2+3 3+4 4+5 5+6 6+7 7+8 8+9</td>
</tr>
<tr>
<td>(4) doubles + 2 (shared doubles)</td>
<td>2+4 3+5 4+6 5+7 6+8 7+9 2+6 3+7</td>
</tr>
<tr>
<td>(5) number + 10</td>
<td></td>
</tr>
<tr>
<td>(6) number + 9</td>
<td></td>
</tr>
<tr>
<td>(7) number + 8</td>
<td></td>
</tr>
</tbody>
</table>

Remaining facts: 2+5 2+7 3+6 4+7
A Three-Step Approach to Fact Mastery

• Two approaches to develop specific strategies:
  • use **simple story problems** that would focus attention on using a strategy to solve it.
  • present a **special collection of facts** for which a particular type of strategy is appropriate. Suggest an approach to show how these facts might all be alike in some way and see if students are able to use it on similar facts. Then have them discuss how these facts might all be alike.

Van de Walle (2004)
Introduce subtraction facts after students have mastered half of the basic addition facts.

Stein, Carnine, & Silbert (1997)
A Three-Step Approach to Fact Mastery

3. Then provide drill in the use and selection of those strategies once they have been developed. Drill (using flash cards and fact games) with a focus on in-place strategy to make it more automatic is critical.

*Note.* Remember drill and practice are not synonymous. Drill refers to repetitive, non problem-based exercises designed to improve skills or procedures already acquired. Practice refers to different problem-based tasks or experiences, spread over numerous class periods, each addressing the same basic ideas.

Van de Walle (2004)
When to Use Drill

Two Criteria:

• An efficient strategy for the skill to be drilled is already in place.
• Automaticity with the skill or strategy is a desired outcome

Never use drill as a means of helping children learn!

Van De Walle (2004)
(1) Specify performance criterion for introducing new facts

(2) Provide intensive practice on newly introduced facts

(3) Schedule systematic practice on previously introduced facts

(4) Ensure adequate allotted time

(5) Use a record-keeping system

(6) Establish a motivation system \(\text{(Stein, Carnine, & Silbert, 1997)}\)

**Establishing Mastery Criterion**

1. Determine students’ writing ability by giving a 1 min. timing.

2. Compute student’s writing rate by counting the number of digits written during this 1 min. period.

3. Multiply the number derived in step 2 by 2/3 to estimate the number of digits a student should write as answers during a 1-minute fact timing.
Basic Fact Remediation in the Upper Grades

1. More drill is never the answer.
2. Find out which specific facts are known and unknown.
3. Diagnose to determine strengths, weaknesses, and current strategies - investigate what command of number relationships and operations the students have.
4. Provide hope.
5. Build success into the remediation plan.

Failure to master basic facts should never be a barrier to doing real mathematics.

Van de Walle (2004)
Relational understanding of basic place value requires an integration of new and difficult-to-construct concepts of grouping by tens (the base-ten concept) with procedural knowledge of how groups are recorded in our place-value scheme, how numbers are written, and how they are spoken.

(Van de Walle, 2004)
Counting

- By ones
- By groups and singles
- By tens and ones

Oral Names

Standard:
Thirty-two

Base-Ten:
Three tens and two

Written Names

32

Van de Walle (p. 181, 2004)

Base-Ten Concepts

Standard and equivalent groupings meaningfully used to represent quantities
“Equally essential [with basic facts] is computational fluency - having and using efficient and accurate methods of computing. Fluency might be manifested in using a combination of mental strategies and jottings on paper or using an algorithm with paper and pencil, particularly when the numbers are large, to produce accurate results quickly.” (NCTM, Principles and Standards, p. 32)
<table>
<thead>
<tr>
<th>Big Idea</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Place value:</strong> The understanding that in our number system, the &quot;place&quot; a number holds in a sequence of numbers gives information about that number.</td>
<td>In the number 265, the 2 at the beginning of the number is a hundreds number. We know that the placement of the 2 tells us that there are two units of 100, or two 100s, in that number. Similarly, the location of the 5 tells us that there are 5 units of 10 in the number.</td>
</tr>
<tr>
<td><strong>Expanded notation:</strong> The awareness by learners that you can reduce a number to its constituent units.</td>
<td>The number 213 is composed of two 100s, one 10, and three 1s, which can be represented in an equation as $100 + 10 + 1 + 1 + 1 + 1 = 213$ or as $200 + 10 + 3 = 213$.</td>
</tr>
<tr>
<td><strong>Commutative property:</strong> The order in which numbers are placed in the equation can be changed without affecting the outcome.</td>
<td>Addition and multiplication are commutative: In addition, $3 + 4 = 7$ and $4 + 3 = 7$. In multiplication, $4 \times 5 = 20$ and $5 \times 4 = 20$.</td>
</tr>
<tr>
<td>$a + b = b + a$</td>
<td>Subtraction and division are not commutative: In subtraction, $5 - 3 = 2$ and $3 - 5 = -2$. In division, $6 \div 3 = 2$ and $3 \div 6 = 0.5$.</td>
</tr>
<tr>
<td><strong>Associative property:</strong> The groupings in which numbers are placed in the equation can be changed without affecting the outcome.</td>
<td>Addition and multiplication are associative: In addition, $(2 + 4) + 5 = 11$ and $2 + (4 + 5) = 11$. In multiplication, $(3 \times 2) \times 5 = 30$ and $3 \times (2 \times 5) = 30$.</td>
</tr>
<tr>
<td>$(a + b) + c = a + (b + c)$</td>
<td>Subtraction and division are not associative: In subtraction, $(15 - 3) - 5 = 7$ and $15 - (3 - 5) = 13$. In division, $(32 \div 8) \div 2 = 2$ and $32 \div (8 \div 2) = 8$.</td>
</tr>
<tr>
<td><strong>Distributive property:</strong> Numbers in an equation involving multiple operations can be distributed.</td>
<td>$5 \times (3 + 2) = (5 \times 3) + (5 \times 2)$</td>
</tr>
<tr>
<td>$a \times (b + c) = (a \times b) + (a \times c)$</td>
<td>$(8 + 4) \div 2 = (8 \div 2) + (4 \div 2)$</td>
</tr>
</tbody>
</table>
| **Equivalence:** The quantity to the left of the equal sign (=) is the same as the quantity to the right. | $32 + 15 = 47$  
$16 + 16 + 15 = 47$  
$8 + 8 + 8 + 8 + (5 \times 3) = 20 + 20 + 7$ |
| **Rate of composition/decomposition of numbers (Ma, 1999):** A form of number sense. The rate of composition (or decomposition) of sets of numbers in our base 10 system is simply 10. | When you have accumulated 10 ones you have one 10. When you have accumulated 10 tens you have one 100 and so on. This concept is sometimes referred to as *unitizing*, that is, creating a tens unit from 10 ones. Similarly, when you remove a one from a 10 you have 9 ones; that is, you have decomposed the ten. |
Problem Structures and Properties

The most prevalent problem structures in elementary school are change, group, compare, equal groups (or vary), and multiplicative comparison. Each structure has three numbers. Any one of the three numbers can be the unknown in a story situation.

Learning mathematical properties can help children connect addition and subtraction concepts and multiplication and division concepts and promote computational fluency.

For example: A student who memorizes that $7 \times 3 = 21$ but sees $3 \times 7$ as a new problem to memorize needs to understand a basic structure -- the commutative property of multiplication.

Algorithms are steps used to solve a math problem.

When the number of digits or complexity of computation increases, the need for a traditional algorithm increases.

Traditional algorithms require an understanding of regrouping.
Working With Models

1. Display the problem at the top of the place value mat:

   \[
   \begin{align*}
   & 27 \\
   \quad + & 54 \\
   \end{align*}
   \]

2. Think aloud as you attempt to answer the question.

   Teacher: You are going to learn a method of adding that most “big people” learned when they were in school. Here’s one rule: You begin in the ones column. This is the way people came up with a long time ago, and it worked for them. Okay, I will start with the ones place. We have seven ones and 4 ones on our two ten-frames in the ones place. I am going to fill the first ten-frames by moving some ones in the second ten-frame. I filled up 10. There’s 11. That’s 10 and one. Can I make a trade? Yes, because now I have a ten and an extra. So, I am going to trade the ten ones for a 10. I have one left in the ones column, which is not enough to trade tens. So the answer is 8 tens and a one – 81.
3. Next, have students work through similar problems with you prompting them (ask leading questions) as needed.

4. Peer or partner group work or independent work with you monitoring it.

5. Finally, students should be able to do it independently and provide explanations.

Have students explain what they did and why. Let students use overhead models or magnetic pieces to help with their explanations.

Van de Walle (2004)
Display the problem on the place chart (see above) next to the place mat

Lead (after you model how to solve the problem) by asking students to work through the problem and prompt them as needed.

**Teacher:** How much is in the ones column? (14) Will you need to make a trade? (yes) How many tens will you make? (1) How many ones will be left? (4) Good! Make the trade now.
Let’s stop now and record exactly what we have done. You had 14 ones, and you made 1 ten and 4. Write a small “1” at the top of the tens column to show the ten you put there and a “4” in the answer space of the ones column for the 4 ones left.

Look at the tens column on your mat. You have 1 ten on top, three from 36, and 4 more from the 48. See how your paper shows the same thing?

Now add all the tens together. Write how many tens that is in the answer space for the tens column.
3. Fractions: Students develop an understanding of the parts of a whole, and the relationships between the different representations of fractions

- **Stage 2**: Models, compares, and represents commonly used fractions and decimals, adds and subtracts decimals to two decimal places, and interprets everyday percentages.
How old are you, Hurricane?

Seven... and one quarter!

Why is it so important, when you're young, to add fractions to years?

How old are you?

Thirty-nine... minus three and three-quarters!
Equivalent Fraction

Outcome: Recognize and generate equivalent forms of commonly used fractions.

1. Display question to class: Generate an equivalent fraction for 4/6.

2. Think aloud as you attempt to answer the question. It is important that you model your thinking rather than tell your students how to solve the problem.
Teacher: $\frac{4}{6} = \frac{2}{3}$. How do I know that $\frac{4}{6} = \frac{2}{3}$? Ok, I think I will use models to help me and start with the fraction $\frac{4}{6}$. If I have a set of 6 things and I take 4 of them, then that would be $\frac{4}{6}$. Now I need to show how this $\frac{4}{6} = \frac{2}{3}$.

I can make the 6 into groups of 2. So then there would be 3 groups, and the 4 would be 2 groups out of the 3 groups. That means it’s $\frac{2}{3}$. 
Equivalent Fractions

Teacher: Another way I could answer this question is to draw a picture and start with the 2/3. I will draw a square cut into 3 parts and shade in 2, then that would be 2/3 shaded.

Now I need to show how this $\frac{2}{3} = \frac{4}{6}$. If I cut all 3 of these parts in half, that would be 4 parts shaded and 6 parts in all. That’s $\frac{4}{6}$, and it would be the same amount.
Equivalent Fractions

3. Next, have students work through similar problems with you prompting them (asking leading questions) as needed.

4. Then have students do peer or partner group work or independent work with you monitoring it.

5. Finally, students should be able to do it independently and provide explanations.

Note: Students should learn to develop an understanding of equivalent fractions and also develop from that understanding a conceptually based algorithm.

Van de Walle (2004)
Activity: Missing-Number Equivalencies

\[
\frac{5}{3} = \underline{\square} \quad \frac{2}{3} = \underline{\square}
\]

\[
\frac{8}{12} = \underline{\square} \quad \frac{9}{12} = \underline{\square}
\]

Van de Walle (2004)
II. Patterns and Algebra

• Patterns and Algebra: Students develop skills in creating, describing and recording number patterns as well as an understanding of the relationships between numbers.

• Stage 4: Uses algebraic techniques to solve linear equations and simple inequalities
Solve It!!

You are going to build a square garden and surround its border with square tiles. Each tile is 1 foot by 1 foot. For example, if the dimensions of the garden are 10 feet by 10 feet, then you will need 44 tiles for the border.

How many tiles would you need for a garden that is \( n \) feet by \( n \) feet?

Explain your answer and illustrate using a diagram.
Equations

$3B + 7 = B - C$

In the above expression or equation, the quantity on the left is the same as the quantity on the right.

Van de Walle (2004)
Equations and Inequalities

Names for Numbers Activity

• Challenge students to find different ways to express a particular number, say 10. Give a few simple examples, such as $5 + 5$ or $12 - 2$.

• Encourage the use of two or more different operations. “How many names for 8 can you find using only numbers less than 10 and at least three operations?”

• In your discussion, emphasize that each expression is a way of representing or writing a number.

Van De Walle (2004)
On the board, draw a simple two-pan balance. In each pan, write a numeric expression [e.g., \((3 \times 9) + 5; 6 \times 8\)], and ask which pan will go down or whether the two will balance.

Challenge students to write expressions for each side of the scale to make it balance [e.g., \((3 \times 4) + 2; 2 \times 7\)].

For each, write a corresponding equation to illustrate the meaning of \(=\).

Note that when the scale “tilts,” either a “greater than or “less than” symbol (\(>\) or \(<\)) is used.
(3 x 4) + 2 < 2 x 7

(3 x 9) + 5 < 6 x 8

5 x 7 < (4 + 9) x 3

Van De Walle (p. 431, 2004)
Finally, teach students that solving an equation means to find values of the variable that make the equation true.

To help students develop skills of solving equations in one variable, maintain the image of the balance pans.

The balance makes it reasonably clear to students that if you add or subtract values from one side, you must add or subtract like values from the other side to keep the scales balanced.

Van De Walle (2004)
Subtract 4 from both sides and multiply right-hand expression.

Subtract 3x from both sides.

Divide both sides by -9.

Check

Both sides = $3\frac{1}{3}$
Summary

- Teach conceptual understanding (from concrete to semiconcrete to abstract)
- Develop procedural fluency and accuracy
- Teach strategies
- Provide opportunities for working mathematically (applying strategies, questioning, reasoning, communicating, reflecting)
References

References


