Orbital Mechanics and Space Geometry

AERO4701 Space Engineering 3
– Week 2
Overview

• First Hour
  – Co-ordinate Systems and Frames of Reference (Review)
  – Kepler’s equations, Orbital Elements
• Second Hour
  – Orbit Types and Applications
  – Orbital Perturbations
Co-ordinate Systems

- Heliocentric
- Earth-Centered Inertial (ECI)
Co-ordinate Systems

• Earth-Centered, Earth-Fixed (ECEF)

• Local Geocentric Vertical (LGCV)

\[
\begin{bmatrix}
    r_x \\
    r_y \\
    r_z
\end{bmatrix} =
\begin{bmatrix}
    R \cos(\lambda_i)\cos(\phi) \\
    R \cos(\lambda_i)\sin(\phi) \\
    R \sin(\lambda_i)
\end{bmatrix}
\]

\[
R = \sqrt{r_x^2 + r_y^2 + r_z^2}
\]

\[
\lambda_i = \sin^{-1}\left(\frac{r_z}{R}\right)
\]

\[
\phi = \tan^{-1}\left(\frac{r_y}{r_x}\right)
\]

Geocentric Latitude

Longitude

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Co-ordinate Systems

- Earth is not spherical, ellipsoidal bulge due to rotation

- Geodetic latitude more commonly used for ground station co-ordinates

\[
\begin{bmatrix}
    r_x \\
    r_y \\
    r_z
\end{bmatrix}_{\text{ECEF}} = \begin{bmatrix}
    (N + h)\cos(\lambda)\cos(\phi) \\
    (N + h)\cos(\lambda)\sin(\phi) \\
    (N(1 - e^2) + h)\sin(\lambda)
\end{bmatrix}
\]

\[
N = \frac{a}{\sqrt{1 - e^2\sin^2(\lambda)}}
\]

- Leads to a Local Geodetic Vertical (LGV) Frame where frame axes are more equally aligned with Earth surface
Co-ordinate Transformations

\[
C_{ECEF}^{ECI} = \begin{bmatrix}
\cos(\omega_{ie}t) & \sin(\omega_{ie}t) & 0 \\
-\sin(\omega_{ie}t) & \cos(\omega_{ie}t) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\omega_{ie} = 7.292115 \times 10^{-5} \text{ rad/s}
\]

\[
t_i - \text{time since Vernal Equinox alignment}
\]

\[
C_{ECEF}^{LGCV} = \begin{bmatrix}
-\sin(\lambda^i)\cos(\phi) & -\sin(\phi) & -\cos(\lambda^i)\cos(\phi) \\
-\sin(\lambda^i)\sin(\phi) & \cos(\phi) & -\cos(\lambda^i)\sin(\phi) \\
\cos(\lambda^i) & 0 & -\sin(\lambda^i)
\end{bmatrix}
\]

\[
C_{ECEF}^{LGV} = \begin{bmatrix}
-\sin(\lambda)\cos(\phi) & -\sin(\phi) & -\cos(\lambda)\cos(\phi) \\
-\sin(\lambda)\sin(\phi) & \cos(\phi) & -\cos(\lambda)\sin(\phi) \\
\cos(\lambda) & 0 & -\sin(\lambda)
\end{bmatrix}
\]
2D Orbital Mechanics – Kepler’s Laws of Planetary Motion

- Law 1 - The orbit of a planet/comet about the Sun is an ellipse with the Sun's center of mass at one focus.
- Law 2 - A line joining a planet/comet and the Sun sweeps out equal areas in equal intervals of time.
- Law 3 - The squares of the periods of the planets are proportional to the cubes of their semi-major axes.

\[ \frac{P_a^2}{P_b^2} = \frac{a_a^3}{a_b^3} \]

\( P \) – Orbit Period
\( a \) – Semi-major axis
Orbital Mechanics – Ellipse Geometry

- $a =$ semimajor axis $= (r_a + r_p)/2$
- $b =$ semiminor axis
- $e =$ eccentricity $= (r_a - r_p)/(r_a + r_p)$
- $\theta =$ true anomaly
- $r_a =$ apogee radius $= a(1 + e)$
- $r_p =$ perigee radius $= a(1 - e)$
- $p =$ apogee radius $= b^2/a = r_p(1 + e)$
  $= r_a(1 - e) =$ semilatus rectum
- $\gamma =$ flight-path angle
  $= \pi/2 - \beta$

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2D Orbital Mechanics – Newton’s Law of Gravitation

• Kepler’s Equations based on restricted two body problem

\[
F = \frac{Gm_1 m_2}{r^2}, \quad -\frac{GMmr}{r^3} = m\ddot{r}
\]

\[
\ddot{r} + \frac{\mu r}{r^3} = 0, \quad \mu = GM
\]

• Conservation of mechanical energy

\[
\dot{r} \cdot \ddot{r} + \frac{\mu r \cdot \dot{r}}{r^3} = 0
\]

Integrate

\[
\frac{d}{dt} \left( \frac{\dot{r} \cdot \ddot{r}}{2} \right) + \frac{\mu}{r^3} \frac{d}{dt} \left( \frac{r \cdot r}{2} \right) = 0, \quad \frac{d(r \cdot r)}{dt} = 2r \dot{r}
\]
2D Orbital Mechanics – Position Solution

- Conservation of angular momentum

\[
\mathbf{r} \times \dot{\mathbf{r}} + \mathbf{r} \times \frac{\mu \mathbf{r}}{r^3} = 0, \quad \mathbf{r} \times \ddot{\mathbf{r}} = \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}})
\]

\[
\frac{d}{dt} \mathbf{H} + 0 = 0, \quad \mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}} = \text{const.}
\]

\[
\dot{\mathbf{r}} \times \mathbf{H} = \frac{\mu}{r^3} (\mathbf{H} \times \mathbf{r}), \quad \mathbf{H} \times \dot{\mathbf{r}} = (r^2 \dot{\theta}) \mathbf{\hat{r}}
\]

\[
\frac{d}{dt}(\mathbf{r} \times \mathbf{H}) = \frac{\mu}{r^3} (r^2 \dot{\theta}) r \dot{\theta} = \mu \dot{\theta} \mathbf{\hat{r}} = \mu \frac{d}{dt}(\mathbf{\hat{r}})
\]

Integrate \( \dot{\mathbf{r}} \times \mathbf{H} = \mu \mathbf{\hat{r}} + \mathbf{B} \), \( \mathbf{B} = \text{const.} \)

\[
r \cdot (\dot{\mathbf{r}} \times \mathbf{H}) = r \cdot (\mu \mathbf{\hat{r}} + \mathbf{B}) = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{H} = \mathbf{H} \cdot \mathbf{H} = H^2
\]

\[
H^2 = \mu r + \mu B \cos \theta
\]

\[
r = \frac{H^2/\mu}{1 + (B/\mu) \cos \theta}
\]

\[
H^2/\mu = p = \text{semilatus rectum}
\]

\[
B/\mu = e = \left(1 + \frac{2\varepsilon H^2}{\mu^2}\right)^{1/2} = \text{eccentricity}
\]

\[
\theta = \text{true anomaly}
\]

- Function for shape of satellite orbit is a general equation of conic sections such as the circle, ellipse, parabola and hyperbola
2D Orbital Mechanics – Velocity Solution

- Radial and tangential components of velocity:

\[ v_r = \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{dr}{d\theta} = \frac{dr}{d\theta} \cdot \frac{H}{r^2} \]

\[ \frac{dr}{d\theta} = \frac{d}{d\theta} \left[ p(1 + e \cos \theta)^{-1} \right] \]

\[ v_r = \frac{He \sin \theta}{p} \]

\[ v_r = \sqrt{\frac{\mu}{p}} e \sin \theta \]

\[ v_n = r \dot{\theta} = r \left( \frac{H}{r^2} \right) \]

\[ v_n = \sqrt{\frac{\mu}{p}} (1 + e \cos \theta) \]

- Total velocity:

\[ v = (v_n^2 + v_r^2)^{1/2} = \left[ \frac{\mu}{p} (1 + e^2 + 2e \cos \theta) \right]^{1/2} \]

\[ v_p = \sqrt{\frac{\mu}{p}} (1 + e) \]

Perigee

\[ v_a = \sqrt{\frac{\mu}{p}} (1 - e) \]

Apogee

- Velocity greatest at perigee, slowest at apogee

\[ e \to 1, v_a \to 0, r_a \to 2a; \quad \therefore \quad \varepsilon = -\frac{\mu}{2a} \]

- General equation for satellite velocity

\[ v = \sqrt{\frac{\mu}{r - \frac{1}{a}}} \]

\[ \rightarrow \] from conservation of mechanical energy
Orbit Types and Escape Velocity

- For circular orbit: \( v_c = \sqrt{\mu/r} \)

- Escape velocity (parabolic orbit): \( v_{\text{esc}} = \sqrt{2} v_c \)
Example – Orbital Position and Velocity

- Initial low altitude orbit
Example – Orbital Position and Velocity

- Initial low altitude orbit
- Want to transfer to higher orbit
Example – Orbital Position and Velocity

- Initial low altitude orbit
- Want to transfer to higher orbit
Example – Orbital Position and Velocity

- Initial low altitude orbit
- Want to transfer to higher orbit

\[ v_1 = \sqrt{\frac{\mu}{R+h_1}} \quad v_2 = \sqrt{\frac{\mu}{R+h_2}} \]
Example – Orbital Position and Velocity

- Initial low altitude orbit
- Want to transfer to higher orbit

\[ v_1 = \sqrt{\frac{\mu}{R+h_1}}, \quad v_2 = \sqrt{\frac{\mu}{R+h_2}} \]

Transfer ellipse

\[ e = \frac{(r_d - r_p)/(r_d + r_p)}{h_2 - h_1} \]
\[ = \frac{h_2 - h_1}{h_1 + h_2 + 2R} \]

\[ v_p = \sqrt{\frac{\mu}{p}(1+e)}, \quad v_a = \sqrt{\frac{\mu}{p}(1-e)} \]

\[ \Delta v_1 = v_p - v_1 \]
\[ \Delta v_2 = v_2 - v_a \]
2D Orbital Mechanics – Time Solution

- Finding radius and true anomaly as a function of time

\[
P = \frac{2\pi}{n} = \text{period} \\
\omega = \sqrt{\frac{\mu}{a^3}} = \text{mean motion} \\
M = n(t - \tau) = \text{mean anomaly}
\]

\[t = \text{specified time}\]
\[\tau = \text{time of perigee passage}\]

Solve for \(E\) \[M = E - e \sin E\]

\[
\tan \frac{\theta}{2} = \frac{1 + e}{1 - e} \sqrt{\frac{E}{2}}
\]

Mean Anomaly and Eccentric Anomaly:
2D Orbital Mechanics – Solving Kepler’s Equation

• Need to solve Kepler’s equation for $E$ -> this equation is transcendental !!!
• We will use an iterative Newton-Raphson method to solve for $E$

\[ M = E - e \sin E \]

\[ E - e \sin E - M = 0 = f(E) \]

Start with: \( E = M \)

Iterate until convergence:

\[ E = E - \frac{f(E)}{f'(E)} \quad , \quad f'(E) = 1 - e \cos E \]
3D Orbital Mechanics

- Three parameters are required to describe the orientation of an orbit in space

- **Inclination angle (i):** angle of inclination of the orbit with the Earth’s equator
- **Right ascension of the ascending node (Ω):** angle between vernal equinox and orbit’s south-north crossing
- **Argument of perigee (ω):** angle between north-south crossing and perigee, measured in orbital plane
3D Orbital Mechanics

- Transformation from orbital co-ordinates into ECI:

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{\text{orbit}} =
\begin{bmatrix}
  r \cos(\theta) \\
  r \sin(\theta) \\
  0
\end{bmatrix}
\]

\[
C_{\text{ECI}}^{\text{orbit}} =
\begin{bmatrix}
  \cos(\Omega)\cos(\omega) - \sin(\Omega)\sin(\omega)\cos(i) & -\cos(\Omega)\sin(\omega) - \sin(\Omega)\cos(\omega)\cos(i) & \sin(\Omega)\sin(i) \\
  \sin(\Omega)\cos(\omega) + \cos(\Omega)\sin(\omega)\cos(i) & -\sin(\Omega)\sin(\omega) + \cos(\Omega)\cos(\omega)\cos(i) & -\cos(\Omega)\sin(i) \\
  \sin(\omega)\sin(i) & \cos(\omega)\sin(i) & \cos(i)
\end{bmatrix}
\]
Orbital Parameters

• There are 6 orbital parameters in total that are used to define an orbit
• 3 parameters for orbit shape and timing \((a,e,M)\) \((M\) at some reference time \(\tau\))
• 3 parameters for orbit orientation \((i,\Omega,\omega)\)

• Orbital parameters are used to predict the motion of the satellite at a given time \(t\)
Summary of orbit motion prediction

- Orbital parameters \((a,e,i,\Omega,\omega,M) + \text{time } t\)

\[
P = \frac{2\pi}{n} = \text{period}
\]
\[
n = \sqrt{\frac{\mu}{a^3}} = \text{mean motion}
\]
\[
M = n(t - \tau) = \text{mean anomaly}
\]

\[
M = E - e \sin E
\]

\[
\tan \frac{\theta}{2} = \left[ \frac{1 + e}{1 - e} \right]^{1/2} \tan \frac{E}{2}
\]

Find radius and true anomaly

\[
r = \frac{p}{1 + e \cos \theta}
\]

Solve for \(E\)

 Orbit shape

\[
a = \text{semimajor axis} = \frac{r_a + r_p}{2}
\]
\[
b = \text{semiminor axis}
\]
\[
e = \text{eccentricity} = \frac{r_a - r_p}{r_a + r_p}
\]
\[
\theta = \text{true anomaly}
\]
\[
r_a = \text{apogee radius} = a(1 + e)
\]
\[
r_p = \text{perigee radius} = a(1 - e)
\]
\[
p = a(1 - e^2) = \frac{b^2}{a} = r_p(1 + e)
\]
\[
r_a(1 - e) = \text{semilatus rectum}
\]
\[
\gamma = \text{flight-path angle} = \pi/2 - \beta
\]

Transform from orbit co-ordinates to ECI, ECEF etc.

Tracking

Transform from ECEF to ground station co-ordinates
Orbit Types

- **Low Earth Orbit (LEO)** – Typically ~300-1500km altitude
- **Medium Earth Orbit (MEO)** – ~1500-35800km altitude
- **Geostationary/Geosynchronous Orbits (GEO)** – 35800km altitude, circular (orbital period matched with earth 24hrs)
- **Highly Elliptical Orbit (HEO)** – Typically <1000km perigee, 15000-40000km apogee
Fig. 2.9. Sample ground track of the circular low-Earth orbit of Echo 1 ($a = 7978$ km, $i = 47.2^\circ$, $T = 118^m$3)
Polar orbits

- Polar orbit ground trace, greater coverage particularly over polar regions

- High inclination angle (>80°)
Geostationary orbits

- Generally ground trace should be a single point on the ground, however some figure-8 motion is observed due to disturbances to inclination angle, eccentricity etc.
Molniya orbits

- Most time spent in Northern hemisphere

Fig. 2.11. 24th ground track of a sample Molniya type orbit ($a = 26555 \text{ km}, e = 0.7222, i = 63.4^\circ$, $\omega = 270^\circ$, $T = 12^h0$)
Orbital Perturbations

- Until now we have only considered the two body problem with no other disturbing forces
- Disturbances to the Keplerian orbital model come from several sources:
  - Earth oblateness, gravity harmonics
  - Solar/Lunar gravity forces
  - Aerodynamic drag
  - Solar radiation
Orbital Perturbations
Earth Oblateness Effects

- Non-spherical Earth shape causes anomalies in gravity field as satellite moves around the Earth
- Most pronounced effect is from J2 zonal harmonic (primary bulge at equator due to Earth rotation)

\[
\begin{align*}
\dot{\Omega} &= -\frac{3}{2} \frac{J_2 R^2}{p^2} \bar{n} \cos i \\
\dot{\omega} &= \frac{3}{2} \frac{J_2 R^2}{p^2} \bar{n} \left(2 - \frac{5}{2} \sin^2 i \right) \\
\bar{n} &= \sqrt{\frac{\mu}{a_0^3}} \left[1 + \frac{3}{2} \frac{J_2 R^2}{p^2} \left(1 - \frac{3}{2} \sin^2 i \right) \left(1 - \epsilon^2 \right)^{\frac{3}{2}} \right]
\end{align*}
\]

\(\dot{\Omega}\) = rate of ascending node  \\
\(\dot{\omega}\) = argument of perigee rate  \\
\(\bar{n}\) = orbit mean motion with J2 correction  \\
\(J_2\) = 0.00108263  \\
\(R\) = Earth equatorial radius  \\
\(i\) = orbit inclination  \\
\(\mu\) = gravitational constant  \\
\(a_0\) = semimajor axis at epoch  \\
\(e\) = eccentricity  \\
\(p\) = \(a_0(1 - \epsilon^2)\)
Sun-Synchronous Orbits

- The sun-synchronous orbit takes advantage of the nodal precession such that $\Omega$ moves 360° in a year.
- The orbital plane maintains a constant angle with the sun vector resulting in predictable sun/shade time ratios and passes over the same point on the Earth always at the same time every day.
Sun-Synchronous Orbits

• Typically 600-800km altitude, 98° inclination
Critical Inclination

- Critical inclination angles are those for which apsidal rotation becomes zero
- Angle is independent of altitude, eccentricity or even J2 value, only dependent on inclination
- Typically used by Molniya orbits, long overhead time is maintained in northern hemisphere

\[ \omega = \frac{\Delta \text{APGEE} + \Delta \text{PERIGEE}}{2} \]

\[ h_a - h_p = \frac{e (h_a + h_p + 2R)}{h_a + h_p} \]

\[ R = 3464 \text{ nmi} \]

\[ h = \begin{cases} 0 & \text{for } i = 0 \\ 100 \text{ nmi} & \text{for } i = 30 \\ 200 \text{ nmi} & \text{for } i = 60 \\ 300 \text{ nmi} & \text{for } i = 90 \end{cases} \]

\[ i = 63.43^\circ \text{ or } 116.57^\circ \]
Atmospheric Drag

• Atmospheric drag acts to reduce the altitude of the orbit over time

• More pronounced effect at perigee, tends to circularise the orbit

\[
\frac{da}{dt} = -na^2 \left( \frac{\rho g_0}{B} \right)
\]

- \( n \) = mean motion
- \( \rho \) = atmosphere density at that altitude
- \( g_0 \) = gravitational acceleration at sea level
- \( B \) = ballistic coefficient = \( W/C_d A \)
Satellite Orbital Models

- SGP4/SDP4 (Simplified General Perturbations/Simplified Deepspace Perturbations) orbital models
- Takes into account main perturbations due to gravity harmonics and atmospheric drag
- Model parameters can be updated on the order of several times a day to once or twice a week, depending on accuracy requirements, altitude etc.
Two Line Elements (TLEs)

**Standard Format**

Line 1

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<th>Column</th>
<th>Description</th>
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<td>Line Number of Element Data</td>
</tr>
<tr>
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<tr>
<td>1.3</td>
<td>08</td>
<td>Classification</td>
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<tr>
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<td>International Designator (Last two digits of launch year)</td>
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<td>International Designator (Launch number of the year)</td>
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<td>International Designator (Piece of the launch)</td>
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<td>Epoch (Day of the year and fractional portion of the day)</td>
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<td>Second Time Derivative of Mean Motion (decimal point assumed)</td>
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<td>BSTAR drag term (decimal point assumed)</td>
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- Used with SPG4 model for satellite orbit prediction
- TLEs are updated by NORAD via ranging measurements for model correction