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Bending and Buckling of Tapered Steel Beam Structures
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ABSTRACT

This paper describes an efficient finite element method of analysing the elastic in-plane bending and out-of-plane buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I cross-section. The structure’s loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints.

Tapered finite element formulations are developed by numerical integration instead of the closed forms often used for uniform elements. Difficulties in specifying the load positions for tapered mono-symmetric members caused by the variations of the centroidal and shear centre axes are avoided by using an arbitrary axis system based on the web mid-line. Account is taken of additional Wagner torque terms arising from the inclination of the shear centre axis.

A computer program based on this method is used to analyse a number of examples of the elastic in-plane bending of tapered cantilevers and built-in beams, and very close agreement is found between its predictions and closed form solutions.

The program’s predictions of the elastic out-of-plane flexural-torsional buckling of a large number of uniform and tapered doubly and mono-symmetric beams and cantilevers under various loading and restraint conditions are generally in close agreement with existing predictions and test results. The common approximation in which tapered elements are replaced by uniform elements is shown to converge slowly, and to lead to incorrect predictions for tapered mono-symmetric beams.

KEYWORDS

Beam-columns, Bending, Buckling, Mono-symmetry, Steel, Structures, Taper, Torsion
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1 INTRODUCTION

The distributions of axial forces and bending moments in steel beam structures caused by the applied loads need to be determined before their elastic flexural-torsional buckling resistances can be analysed [1]. While these may be easily found for indeterminate structures with uniform members and for determinate structures with tapered members, they are less easily found for indeterminate tapered structures. It is noteworthy that most if not all of the early and later studies of the elastic buckling of tapered structures cited in [2, 3] and more recently in [4, 5] are for statically determinate beams or cantilevers.

The finite element stiffness matrices used to analyse the elastic in-plane bending and out-of-plane buckling of a tapered beam structure depend on the variations of the section properties along each element of which a structure is composed, and in general require numerical integrations to be made, in place of the use of the common closed form solutions for uniform elements.

When the tapered elements are of mono-symmetric cross-section (Fig.1a), the centroidal and shear centre axes wander [6] along the element length (Fig. 1b). If the problem is simplified by using an arbitrary axis system [7], then modifications need to be made for the eccentricity of the centroid from the chosen axis to allow for the moments induced by eccentric longitudinal loads or restraints.

This paper describes a finite element method of analysing the elastic in-plane bending and out-of-plane buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I-section. The structure’s loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints. The method is superior to the commonly used approximate method of replacing a tapered element by a large number of uniform elements, with a significant reduction in the number of elements required to obtain an accurate solution.

While this method gives solutions for bending and out-of-plane buckling, it also allows the analysis of the inelastic buckling of uniform steel beams for which non-uniform yielding along the beam causes both the in-plane and out-of-plane properties to vary, so that the member is effectively tapered [8]. It may be noted that the combination of the anti-symmetrical stresses due to the applied loading with the symmetrical residual stresses in a doubly symmetric I-section beam causes the effective section to become mono-symmetric [9].

A computer program FTBTM based on the method described in this paper is used to analyse a number of examples of the bending and out-of-plane buckling of tapered beam structures, and its predictions are compared with previous theoretical and experimental values.

2 TAPERED MONOSYMMETRIC MEMBERS

The member cross-section shown in Fig. 1a has top and bottom flange and web widths \(b_t\), \(b_b\), and \(b_w\) and thicknesses \(t_t\), \(t_b\), and \(t_w\), respectively. The element shown in Fig.1b may be linearly tapered in all cross-section dimensions, and while thickness tapering is very rare in practice, depth tapering is comparatively common. The member has an arbitrary but convenient [7] longitudinal axis \(Oz\) which is the locus of the web mid-heights, and an \(Oy\) axis which coincides with the web centreline. The section properties required for an in-plane analysis are

\[
I_0 = A = A = \int_A dA = b_t t_t + b_b t_b + b_w t_w
\]

(1)

\[
I_1 = y_c A = \int_A y dA = -b_t t_t b_w / 2 + b_b t_b b_w / 2
\]

(2)

\[
I_2 = I_1 - y_c^2 A = \int_A y^2 dA = b_t t_t b_w^2 / 4 + b_b t_b b_w^2 / 4 + b_w t_w / 12 + b_w t_w y_c^2
\]

(3)

in the coordinate \(y_c\) of the centroid is given by

\[
y_c = \frac{b_w b_b t_b - b_t t_t}{2 A}
\]

(4)
The coordinate of the shear centre S (Fig. 1a) is defined by

\[ y_s = y_c + y_b = \frac{b_w b_t^3 t_b - b_t^3 t_w}{2 b_t^3 t_b + b_t^3 t_w} \]  \hspace{1cm} (5)

Concentrated transverse and longitudinal loads \( Q \) (at \( y_Q \)), \( N \) (at \( y_N \)), and moments \( M \) may act at a point \( z \) along the axis \( Oz \) as shown in Fig. 2a, as well as rigid or elastic restraints (at \( y_R \)). Distributed transverse and longitudinal loads \( q \) (at \( y_q \)) and \( n \) (at \( y_n \)) may act along a length \( L \) of the element as shown in Fig. 2b, as well as elastic restraints (at \( y_e \)).

### 3 IN-PLANE ANALYSIS AND BEHAVIOUR

#### 3.1 FINITE ELEMENT FORMULATION

A finite element method of carrying out the in-plane (pre-flexural-torsional buckling) linear elastic analysis of uniform beam-columns is detailed in [1]. In this, the equilibrium equations are represented by

\[ K_i \Delta_i = Q_{in} + Q_{ie} \]  \hspace{1cm} (6)

in which \( K_i \) is the in-plane global stiffness matrix, \( \Delta_i \) are the in-plane global nodal deflections and rotations, \( Q_{in} \) are the nodal loads, and \( Q_{ie} \) are the nodal loads equivalent to the loads distributed along the elements. The global stiffness matrix is given by

\[ K_i = \sum_k T_{ie}^T k_{ie} T_{ie} \]  \hspace{1cm} (7)

in which \( k_{ie} \) is an element stiffness matrix and \( T_{ie} \) is a matrix which transforms the element in-plane end deflections and rotations \( \delta_i \) into the corresponding global deflections and rotations \( \Delta_i \) according to

\[ \delta_{ie} = T_{ie} \Delta_i \]  \hspace{1cm} (8)

The element stiffness matrix is given by

\[ k_{ie} = C_i^{-T} \int_0^L B_i^T D_i B_i dz C_i^{-1} \]  \hspace{1cm} (9)

in which \( C_i \) is a matrix related the cubic and linear functions used to approximate the element deflections \( v \) and \( w \), \( B_i \) is a matrix related to the length \( L \) of the element and the distance \( z \) along it, and \( D_i \) is given by

\[ D_i = \begin{bmatrix} EI_x & 0 \\ 0 & EA \end{bmatrix} \]  \hspace{1cm} (10)

in which \( E \) is the Young’s modulus of elasticity.

When the typical element is of mono-symmetric cross-section it is convenient [7] to use the arbitrary axis system of Fig. 1. Thus all deflections \( v \) and \( w \) are referred to these axes, as are the nodal loads and moments \( N, Q, \) and \( M \) and distributed loads \( n \) and \( q \). When \( N \) or \( n \) act eccentrically to the \( Oz \) axis, then they are replaced by their static equivalents referred to the \( Oz \) axis.

The \( D \) matrix then needs to be replaced by

\[ D = \begin{bmatrix} EI_2 & EI_1 \\ EI_1 & EA \end{bmatrix} \]  \hspace{1cm} (11)

in which the values of \( A, I_1, \) and \( I_2 \) are given in Equations 1-3.

When the element is tapered so that \( A, I_1, \) and \( I_2 \) vary with \( z \), it is difficult to develop exact formulations for the integrals required in Equation 9. Instead, it is much simpler to use the approximate but highly accurate 4-point Gaussian integration [10].
The effects of rigid nodal restraints which prevent the corresponding nodal deflections or rotations are allowed for by making appropriate modifications to the equilibrium Equations 6. When a rigid restraint against deflection parallel to the $Oz$ axis acts eccentrically, then it is convenient to transform the global $\Delta_i$ so that they include the deflection at the eccentric restraint, as in [11]. The effects of nodal elastic restraints against the in-plane deflections and rotations may be allowed for by augmenting the global stiffness matrix $K_i$ by the stiffnesses of the elastic restraints, as in [1]. The effects of uniformly distributed element restraints may be allowed for by augmenting the element stiffness matrix $k_{ei}$ by the stiffnesses of the elastic restraints, as in [1].

A MATLAB [12] computer program FTBTM for the elastic flexural-torsional buckling of tapered members has been written which includes the above formulation for the in-plane (pre-flexural-torsional buckling) analysis. Examples of the application of this program are discussed in the following sub-sections.

3.2 TAPERED CANTILEVERS

A tapered cantilever of length $L$ shown in Fig. 3a has end actions $M$ or $Q$ or uniformly distributed load $q$. The second moment of area $I_z$ varies linearly along its length according to $I_z = I_{z0}(1 + \alpha z / L)$ (12) in which $\alpha$ is a constant which defines the degree of taper. The in-plane deformations of the cantilever are analysed in Appendix A. The variations of the dimensionless in-plane end ($z = 0$) deflections $v_0 / v_{0u}$ (in which $v_{0u}$ is the value of $v_0$ for a uniform beams with $\alpha = 0$) with $\alpha$ are shown in Fig. 4. These are in very close agreement with the values determined by the computer program FTBTM, as demonstrated in Fig. 4.

3.3 BUILT-IN BEAMS

A built-in tapered beam of length $L$ with a uniformly distributed load $q$ is shown in Fig. 3b. The second moment of area $I_z$ varies along its length according to Equation 12. The redundant end actions $M_0$ and $Q_0$ are analysed in Appendix B. The variations of the dimensionless redundant actions $12M_0 / q L^2$ and $2Q_0 / q L$ with $\alpha$ are shown in Fig. 5. These are in very close agreement with the values determined by the computer program FTBTM, as demonstrated in Fig. 5.

4 FLEXURAL-TORSIONAL BUCKLING

4.1 BUCKLING ANALYSIS

4.1.1 Uniform members

A finite element method of carrying out the out-of-plane flexural-torsional buckling analysis of uniform beam-columns is detailed in [1]. In this, the energy equation at elastic buckling is represented by

$$\frac{1}{2} \Delta_o^T (K_o + \lambda G_o) \Delta_o = 0$$

(13)

in which $K_o$ and $G_o$ are the out-of-plane global stiffness and stability matrices, $\Delta_o$ are the global out-of-plane nodal deflections and rotations, and $\lambda$ is the buckling load factor.

The global stiffness matrix is given by

$$K_o = \sum_k T_{oe}^T k_{oe} T_{oe}$$

(14)

in which $k_{oe}$ is an element stiffness matrix and $T_{oe}$ is a matrix which transforms the element end deflections and rotations $\delta_{oe}$ into the corresponding global deflections and rotations $\Delta_o$ according to

$$\delta_{oe} = T_{oe} \Delta_o$$

(15)
The element stiffness matrix \( k_{oe} \) may be obtained from the element strain energy increase during buckling

\[
\frac{1}{2} U_e = \frac{1}{2} \int_0^L e_u^T D_u e_u \, dz
\]

in which

\[
e_u = \begin{bmatrix} u_x, v_x, \phi, \phi' \end{bmatrix}^T
\]

and

\[
D_u = \begin{bmatrix} EI_v & 0 & 0 \\
0 & GJ & 0 \\
0 & 0 & EI_w \end{bmatrix}
\]

in which \( u_x \) is the shear centre displacement, \( \phi \) is the twist rotation, \( G \) is the shear modulus of elasticity, \( I_z \) is the second moment of area about the \( y \) axis of symmetry, and \( J \) and \( I_w \) are the uniform torsion and warping section constants. The element stiffness matrix can be expressed as

\[
k_{oe} = C_u^{-T} \int_0^L B_v^T D_v B_v \, dz \, C_u^{-1}
\]

in which \( C_u \) is a matrix related the cubic functions used to approximate the element out-of-plane deflections and rotations and \( B_v \) is a matrix related to the length \( L \) of the element and the distance \( z \) along it.

The out-of-plane global stability matrix is given by

\[
G_o = \frac{1}{2} \sum \left[ \phi_n^2 + \int e_v^T D_v e_v \, dz \right] + \sum T_{oe}^T g_{oe} T_{oe}
\]

in which the first term represents the increase in potential energy during buckling of the initial nodal loads \( Q \) which result from their distance \( (y_Q - y_s) \) below the shear centre, and the second term contains the element stability matrix \( g_{oe} \). It may be obtained from the increase in the element potential energy

\[
\frac{1}{2} \lambda V_e = \frac{1}{2} \int_0^L e_v^T D_v e_v \, dz
\]

in which

\[
e_v = \begin{bmatrix} u_x', v_x, \phi', \phi'' \end{bmatrix}^T
\]

and

\[
D_v = \begin{bmatrix} N_z & 0 & 0 & N_z y_0 & 0 \\
0 & 0 & -M_x & 0 & 0 \\
0 & -M_x & q (y_Q - y_s) & 0 & 0 \\
N_z y_0 & 0 & 0 & N_z (I_p / A + y_0^2) - M_x \beta_x & 0 \\
0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

in which \( N_z \) and \( M_x \) are the initial internal axial compression and moment, \( q \) is the distributed vertical load, \( (y_Q - y_s) \) is the distance below the shear centre at which \( q \) acts, \( I_p \) is the polar moment of area, and \( \beta_x \) is the monosymmetry constant [1]. The element stability matrix can be expressed as

\[
g_{oe} = C_v^{-T} \int_0^L B_v^T D_v B_v \, dz \, C_v^{-1}
\]

in which \( C_v \) is a matrix related the cubic functions used to approximate the element out-of-plane deflections and rotations, and \( B_v \) is a matrix related to the length \( L \) of the element and the distance \( z \) along it.
4.1.2 Tapered members

When the element is tapered, it is convenient to replace the shear centre deflection \( u \) and its derivatives by the web mid-height values

\[
\begin{align*}
u &= u_s + y_s \phi \\
u' &= u_s' + y_s' \phi + y_s \phi' \\
u'' &= u_s'' + 2y_s' \phi' + y_s \phi''
\end{align*}
\]

When these are used, Equation 18 becomes

\[
D_{ul} = \begin{bmatrix}
(EI_t + EI_b) & (EI_t - EI_b)b_w' & (EI_t - EI_b)b_w / 2 \\
(EI_t - EI_b)b_w' & GJ + (EI_t + EI_b)b_w'^2 & (EI_t + EI_b)b_w b_w'^2 / 2 \\
(EI_t - EI_b)b_w / 2 & (EI_t + EI_b)b_w b_w'^2 / 2 & (EI_t - EI_b)b_w^3 / 4
\end{bmatrix}
\]

in which \( I_t \) and \( I_b \) are the top and bottom flange second moments of area about the \( y \) axis, and Equation 23 becomes

\[
D_{el} = \begin{bmatrix}
N_z & 0 & -N_z y_s' & -N_z y_s \\
0 & 0 & -M_s & 0 \\
-N_z y_s' & -M_s & N_z y_s'^2 - q(y_q - y_s) & N_z y_s'(y_s + y_c) + 2.5M_s y_s^' & M_s y_s \\
-N_z y_c & 0 & N_z y_s'(y_s' + y_s) + 2.5M_s y_s^' & N_z(Y_p / A + y_s'^2) - M_s \phi_i & 0 \\
0 & 0 & M_s y_s & 0 & 0
\end{bmatrix}
\]

This contains additional terms which arise from the additional Wagner [6, 13] torque components \((-N_z y_0 y_s' \phi)\) and \((-M_s y_s' \phi)\) in

\[
T_w = -N_z \{(I_p / A + y_s'^2) \phi_i + y_0 y_s' \phi\} + M_s \{\phi_i y_s' - y_s \phi\}
\]

cau##ed by the slope \( y_s'\) of the shear centre axis. The derivation of the Wagner torque \( T_w \) is given in Appendix C.

When a rigid restraint against deflection parallel to the \( Ox \) axis or rotation about the \( Oy \) axis acts eccentrically, then it is convenient to transform the global \( \Delta_o \) so that they include the deflection or rotation at the eccentric restraint, as in [11]. The effects of nodal elastic restraints against the out-of-plane deflections and rotations may be allowed for by augmenting the global stiffness matrix \( K_o \) by the stiffnesses of the elastic restraints, as in [1]. The effects of uniformly distributed element restraints may be allowed for by augmenting the element stiffness matrix \( k_{eo} \) by the stiffnesses of the elastic restraints, as in [1].

Examples of the application of a MATLAB [12] computer program FTBTM for the elastic flexural-torsional buckling of tapered members are discussed in the following sub-sections.

4.2 MONO-SYMMETRIC UNIFORM BEAMS

Flexural-torsional buckling predictions obtained using the computer program FTBTM have demonstrated good agreement with a wide range of independent theoretical [1, 14] and test results. An example [15] is shown in Fig. 6 for a mono-symmetric uniform cantilever with an end load acting at different distances \((y_Q - y_s)\) below the shear centre (it may be noted that although the theory presented in Section 4.1 above was based on a mid-web axis system, the distance below the shear centre at which transverse loads \( Q \), \( q \) act is used in Equations 20 and 27). There is close agreement with previous theoretical and test results.
4.3 DOUBLY SYMMETRIC TAPERED BEAMS

Flexural-torsional buckling predictions obtained using FTBTM for doubly symmetric tapered beams [2] are shown in Fig. 7. There is good agreement with previous theoretical and test results.

4.4 MONO-SYMMETRIC TAPERED BEAMS AND CANTILEVERS

Flexural-torsional buckling predictions obtained using FTBTM for mono-symmetric width-tapered beams [6] are shown in Fig. 8. There is good agreement with previous theoretical and test results. The convergence of the dimensionless buckling loads \( \frac{Q}{Q_{\text{test}}} \) with the number of elements is demonstrated in Fig. 9. It can be seen that convergence within about 2% is obtained using 4 elements, and that the converged values are within about 3% of the test values [6].

Also shown in Fig. 9 are the corresponding values for approximations for which the tapered elements are replaced by uniform elements. Not only is the convergence much slower, but the converged values are significantly different from the test values. It is probable that this occurs because the use of uniform elements neglects the effects of the slope of the shear centre axis, which is responsible for the additional terms in the Wagner torque of Equation 28.

Buckling predictions obtained using FTBTM for mono-symmetric tapered cantilevers are shown in Fig. 10, and are a little lower than previous theoretical results [3].

5 CONCLUSIONS

This paper describes a finite element method of analysing the elastic in-plane bending and out-of-plane flexural-torsional buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I cross-section. The structure's loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints.

Tapered finite element formulations developed by numerical integration replaced the closed forms often used for uniform elements. Difficulties in specifying the load positions for tapered mono-symmetric members caused by the variations of the centroidal and shear centre axes were avoided by using an arbitrary axis system based on the web mid-line. Account was taken of additional Wagner torque terms arising from the inclination of the shear centre axis.

A computer program based on this method was used to analyse a number of examples of the elastic in-plane bending of tapered cantilevers and built-in beams, and very close agreement was found between its predictions and closed form solutions.

The program's predictions of the elastic out-of-plane flexural-torsional buckling of a large number of uniform beams and cantilevers under various loading and restraint conditions were in close agreement with those of an existing finite element program [14]. The predictions for cantilevers with concentrated end loads were also in close agreement with test results.

The program was used to analyse the flexural-torsional buckling of a number of tapered doubly symmetric beams, and again good agreement was found with existing theoretical predictions and test results. Close agreement was also found for tapered mono-symmetric beams with central concentrated loads. Predictions for tapered mono-symmetric cantilevers with end loads were a little lower than previous theoretical predictions.

It may be concluded that the computer program developed from the finite element analysis of this paper provides an efficient method of predicting the in-plane bending and out-of-plane buckling of indeterminate beam structures composed of tapered mono-symmetric members under a wide range of loading and restraint conditions.
6 REFERENCES


7 PRINCIPAL NOTATION

A Area of cross-section
b_{bt,wb} Bottom and top flange widths and web depth
D_{D,i,u,v,ut} Generalised elasticity matrices
E Young’s modulus of elasticity
f Stress
G Shear modulus of elasticity
G_o Global stability matrix
g_{oe} Element stability matrix
I_{xy} Second moments of area about x, y axes
I_{bt} Second moments of area of bottom and top flanges
I_0 Value of I, at z = 0
I_p = I_k + I_t
I_w Warping section constant
J Uniform torsion constant
K_{i,o} Global in-plane and out-of-plane stiffness matrices
k_{oe} Element in-plane and out-of-plane stiffness matrices
L Length
M External moment
M_i Internal bending moment
N, n Concentrated and distributed longitudinal loads
N_c Internal compression force
Q, q Concentrated and distributed transverse loads
Q_{oe} Generalised equivalent distributed and concentrated nodal loads
Q_o Value of Q for a uniform member
r_{ls} Distance of point (x, y) from shear centre (0, y_s)
T_{oe} In-plane and out-of-plane element to global transformation matrices
Tw Wagner torque
(t_{bt,wb} Bottom and top flange and web thicknesses
U_e Element strain energy increase
u, v, w Displacements in x, y, z directions
us Shear centre displacement parallel to x axis
V_e Potential energy increase
x, y Cross-section coordinates of arbitrary axis system
y_{cs} Centroid and shear centre coordinates
y_0 Distance of shear centre from centroid
y_{N,n} Positions of N, n loads
y_{Q,q} Positions of Q, q loads
y_{R,r} Positions of concentrated and distributed restraints
z Distance along member
\alpha_{dx,dy} Taper constants
\beta = (1 + \alpha z/L)
\beta_t Mono-symmetry section constant
\Delta_{i,o} Global in-plane and out-of-plane nodal deflections and rotations
\Delta_{oe} Element in-plane and out-of-plane nodal deflections and rotations
\varepsilon_{uv} Generalised strain vectors
\lambda Buckling load factor
\phi Twist rotation
APPENDIX A  IN-PLANE DEFORMATIONS OF TAPERED CANTILEVERS

A tapered cantilever of length $L$ is shown in Fig. 3a. The second moment of area $I_x$ varies linearly along its length according to

$$I_x = I_0 (1 + \alpha z / L)$$  \hspace{1cm} (A1)

The in-plane deformations of the cantilever caused by the loading $M$, $Q$, $q$ shown in Fig. 3a may be determined from the differential equation of bending

$$-EI_y v'' = -M - Qz - qz^2 / 2$$  \hspace{1cm} (A2)

The general solution of this equation is

$$EI_0 v = \begin{bmatrix} \beta \ln \beta - \beta \\ \beta \ln \beta - \beta^2 / 2 \\ \beta \ln \beta - \beta^2 / 2 - \beta^3 / 6 \end{bmatrix}^T \begin{bmatrix} ML^2 / \alpha^2 \\ QL^3 / \alpha^3 \\ qL^4 / 2\alpha^4 \end{bmatrix} + A_1 \beta + A_2$$  \hspace{1cm} (A3)

in which

$$\beta = (1 + \alpha z / L)$$  \hspace{1cm} (A4)

and $A_1$ and $A_2$ are constants of integration. These may be determined by using the boundary conditions

$$v_L = 0 \; ; \; v_L' = 0$$  \hspace{1cm} (A5)

as

$$A_1 = EI_0 \begin{bmatrix} - \ln(1 + \alpha) \\ \ln(1 + \alpha) - (1 + \alpha) \\ - \ln(1 + \alpha) + (1 + \alpha) - (1 + \alpha)^2 / 2 \end{bmatrix}^T \begin{bmatrix} ML^2 / \alpha^2 \\ QL^3 / \alpha^3 \\ qL^4 / 2\alpha^4 \end{bmatrix}$$  \hspace{1cm} (A6)

and

$$A_2 = \begin{bmatrix} (1 + \alpha) \\ -(1 + \alpha) + (1 + \alpha)^2 / 2 \\ (1 + \alpha) - (1 + \alpha)^2 + (1 + \alpha)^3 / 3 \end{bmatrix}^T \begin{bmatrix} ML^2 / \alpha^2 \\ QL^3 / \alpha^3 \\ qL^4 / 2\alpha^4 \end{bmatrix}$$  \hspace{1cm} (A7)

The tip deformations may then be determined from

$$EI_0 v_0 = \begin{bmatrix} -1 \\ 3 / 2 \\ -11 / 6 \end{bmatrix}^T \begin{bmatrix} ML^2 / \alpha^2 \\ QL^3 / \alpha^3 \\ qL^4 / 2\alpha^4 \end{bmatrix} + A_1 + A_2$$  \hspace{1cm} (A8)

$$EI_0 v_0' = \begin{bmatrix} 0 \\ 1 \\ -3 / 2 \end{bmatrix}^T \begin{bmatrix} ML^2 / \alpha^2 \\ QL^3 / \alpha^3 \\ qL^4 / 2\alpha^4 \end{bmatrix} + A_1$$  \hspace{1cm} (A9)
APPENDIX B  FIXED END ACTIONS OF TAPERED BEAMS

A fixed ended tapered beam of length $L$ with a uniformly distributed load $q$ is shown in Fig. 3b. The second moment of area $I_x$ varies along its length according to Equation A1. The redundant end actions $M_0$ and $Q_0$ may be determined by applying the boundary conditions $v_0 = v'_0 = 0$ to Equations A8 and A9, which can then be rearranged in the form of

$$
\begin{bmatrix}
[F]
\end{bmatrix}
\begin{bmatrix}
M_0 \\
-Q_0 L / \alpha
\end{bmatrix}
= \frac{q}{2L^2 / \alpha^2} \{d\}
$$

whence

$$
\begin{bmatrix}
M_0 \\
-Q_0 L / \alpha
\end{bmatrix}
= \frac{q}{2L^2 \alpha^2} \left[F\right]^{-1} \{d\}
$$

(B2)
APPENDIX C  ADDITIONAL WAGNER TORQUE COMPONENTS

When a member twists, tensile longitudinal stresses $f$ exert restoring torque components, often described as Wagner [13] torques. The origin of these is demonstrated in Fig. 11, which shows that twist rotations $\phi$ about the shear centre $S(0, y_s)$ cause a longitudinal fibre through a point $(x, y)$ to rotate $(r, \phi)'$ about the line $rs$ joining the point to the shear centre. Thus the force $fdA$ along the fibre develops a transverse force component $fdA(r, \phi)'$, which exerts a torque component $fdA(r, \phi)' r_s$ about the shear centre. The total torque is

$$ T_w = \int_A fr_s (r, \phi)' dA $$

(C1)

The tensile stress is given by

$$ f = -N_z / A + M_s (y - y_s) / I_x $$

(C2)

and the distance $r_s$ by

$$ r_s = \sqrt{x^2 + (y - y_s)^2} $$

(C3)

It can be shown that

$$ \int_A r_s^2 dA = I_p + A y_0^2 $$

$$ \int_A (y - y_c) r_s^2 dA = I_s \beta_s $$

$$ \int_A r_s r_s' dA = y_0 y_s' A $$

$$ \int_A (y - y_c) r_s r_s' dA = -y_s' I_x $$

(C4)

so that

$$ T_w = -N_z \left\{ \left[ I_p / A + A y_0^2 \right] \phi' + y_0 y_s' \phi \right\} + M_s \left\{ \beta_s \phi' - y_s' \phi \right\} $$

(C5)
Figures

(a) Section

(b) Element

Fig. 1 Geometry of a Tapered Element
Fig. 2  Nodal Actions and  Element Loads

(a) Nodal Actions                                         (b)  Element Loads

Fig. 2  Nodal Actions and  Element Loads
Fig. 3  Tapered Members

(a) Cantilever

(b) Built-in beam
Fig. 4 In-Plane Deflections of Tapered Cantilevers
Fig. 5  Fixed End Actions of Tapered Beams
Fig. 6  Buckling of Uniform Mono-Symmetric Cantilevers
Fig. 7 Buckling of Doubly Symmetric Tapered Beams
Fig. 8 Buckling of Mono-Symmetric Tapered Beams
Fig. 9  Convergence
Fig. 10 Buckling of Mono-Symmetric Tapered Cantilevers
Fig. 11 Wagner Torque