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Flexural-Torsional Buckling of Columns with Oblique Eccentric Restraints

Research Report No R841

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Abstract:

This paper is concerned with the elastic flexural-torsional buckling of concentrically loaded columns with oblique eccentric restraints. Oblique flexural restraints may resist deflections as well as flexural rotations, while torsional restraints may resist twist rotations and warping. Flexural restraints may act at the shear centre or may be eccentric. Restraints may be concentrated at points along a column or distributed along portions of its length, and may be rigid or elastic.

Oblique flexural restraints cause coupling between the principal axis deflections and rotations, while eccentric flexural restraints cause coupling between the principal axis deflections and rotations and the twist rotations and warping displacements. The general buckling mode involves simultaneous bending about both principal axes and torsion.

This paper discusses the nature of oblique eccentric restraints, summarises their finite element analysis, presents examples of their effects on the elastic buckling of columns, and demonstrates the design of columns with oblique eccentric restraints.
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Keywords:

buckling, columns, design, elasticity, restraints, steel.

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1 INTRODUCTION

This paper is concerned with the elastic flexural-torsional buckling of concentrically loaded columns with oblique eccentric restraints. The restraints may resist deflections as well as rotations, and may be rigid or elastic. They may be concentrated at points along a column, or distributed along portions of its length. This subject may be considered to be the general case of the buckling of a compression member in a three-dimensional frame, which ignores any primary bending moments and inelastic behaviour.

The elastic flexural buckling of a doubly symmetric column with restraints which act in a principal plane has been thoroughly researched and is well understood. Such a column buckles in one of the two principal planes xz or yz shown in Fig. 1. In this case, the buckling analysis is simplified to the consideration of the independent planar buckling modes. The effects of unequal rotational end restraints (Trahair et al, 2001) on planar buckling have been analysed, and graphs (SA, 1998 and BSI, 2000) and nomograms (AISC, 1999b) for calculating the column effective length factors used to represent the buckling loads are available.

The analysis of the elastic flexural-torsional buckling of simply supported columns of general cross-section is presented in a number of textbooks (Timoshenko and Gere, 1961; Vlasov, 1961; and Trahair, 1993), while the buckling of columns of general cross-section with distributed or concentrated eccentric restraints which act in planes parallel to the principal planes is analysed in Trahair (1993).

However, there appear to have been no studies of the elastic flexural-torsional buckling of a column with oblique eccentric restraints which act in an oblique plane Xz inclined at an angle θ to the principal xz plane and eccentric from the column shear centre (S) axis, as shown in Fig. 2. In this case, the oblique restraints cause coupling between the principal axis deflections and rotations, while the restraint eccentricities cause coupling between the deflections and rotations and the twist rotations and warping displacements. The buckling mode involves simultaneous bending about both principal axes and torsion.

Practical examples of oblique restraints include angle and zed-section columns connected to restraints through the legs or webs, and columns whose principal axes are rotated relative to restraining beams or grids of beams. Depending on the stiffnesses of the restraining beam(s) relative to the stiffnesses of the column, the supporting beam(s) may be assumed to rigidly or elastically restrain deflections and rotations of the column in one or two oblique planes. Eccentric restraints frequently occur when restraints are connected to the column at points away from the shear centre.

Recently, the elastic flexural buckling of doubly symmetric columns with concentrated oblique translational and rotational elastic or rigid restraints has been analysed (Rasmussen and Trahair, 2004; Trahair and Rasmussen, 2005). These analyses have been used to study the effects of restraint type, angle, and stiffness on the flexural buckling of columns with end loads. However, any torsional buckling effects were eliminated from these studies by their restriction to doubly symmetric section columns with concentric restraints.

Trahair (1969) studied the elastic buckling of an unequal angle section column with oblique restraint lines (pinned about an X_U axis so that the moment M_{XU} = 0 and fixed in the X_Uz plane so that the rotation dU/dz = 0) at each end. The elastic buckling load varied with the angle θ_U between the restraint line and the column principal x axis in an approximately sinusoidal fashion. The magnitudes of the buckling loads were affected by torsional effects because of the asymmetry of the unequal angle section.

In this paper, only the effects of concentrated oblique eccentric restraints on the elastic flexural-torsional buckling of columns are considered, as the extension to the effects of continuous restraints is very straightforward. The following sections discuss the nature of oblique eccentric restraints, summarise their analysis, present examples of their effects on the elastic buckling of columns, and demonstrate the design of columns with oblique eccentric restraints.

2 ELASTIC RESTRAINTS

An elastic column of length L and axial compression N is shown in Fig. 1, in which x and y are the principal axes of the column cross-section and z is the distance along the column. The column is restrained against deflections and rotations at one or more points x_U, y_U along its length, as shown for example in Fig. 2, in which X_U is an oblique restraint axis through the point which is inclined at θ_U to the x principal axis. The restraints may exert a force F_{XU} which opposes deflection U in the X_U direction, a moment M_{YU} which opposes rotation $-dU/dz$ in the $X_U z$ plane, and a torque M_z and a bimoment B about the longitudinal axis through the shear centre x_0, y_0 which oppose a twist rotation ϕ and warping deflections proportional to $d\phi/dz$.

When the column buckles, it deflects u, v parallel to the x, y directions and undergoes twist rotations ϕ about the longitudinal axis through the shear centre, as shown in Fig. 2. The point x_U, y_U deflects U in the X_U direction and rotates $-dU/dz$ in the $X_U z$ plane. These deflections and rotations are related by

$$\{\delta_R\} = [T_R]\{\delta_r\} \tag{1}$$

in which

$$\{\delta_R\} = \{U_U \quad U_U' \quad \phi \quad \phi'\}^T \tag{2}$$

is a vector of restraint deformations, ‘ ≡ d/dz,

$$\{\delta_r\} = \{u \quad u' \quad v \quad v' \quad \phi \quad \phi'\}^T \tag{3}$$

is a vector of principal shear centre deformations, and the transformation matrix $[T_R]$ is given by

$$[T_R] = \begin{bmatrix} C_U & 0 & S_U & 0 & -y_{U0}C_U + x_{U0}S_U & 0 \\ 0 & C_U & 0 & S_U & 0 & -y_{U0}C_U + x_{U0}S_U \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

in which

$$y_{U0} = y_U - y_0 \quad (5a)$$

$$x_{U0} = x_U - x_0 \quad (5b)$$

$$C_U = \cos\theta_U \quad (5c)$$

$$S_U = \sin\theta_U \quad (5d)$$

If the restraints are elastic, then they may be expressed as

$$\{F_R\} = [\alpha_R]\{\delta_R\} \quad (6)$$

in which

$$\{F_R\} = \{F_{XU} \quad M_{YU} \quad M_z \quad B\}^T \quad (7)$$

is the vector of the restraint actions, and

$$\{\alpha_R\} = \begin{bmatrix} \alpha_{TU} & 0 & 0 & 0 \\ 0 & \alpha_{RU} & 0 & 0 \\ 0 & 0 & \alpha_z & 0 \\ 0 & 0 & 0 & \alpha_w \end{bmatrix} \quad (8)$$

is the matrix of restraint stiffnesses.

When the column buckles, these restraints store strain energy

$$U_R = \frac{1}{2}\{\delta_R\}^T[\alpha_R]\{\delta_R\} \quad (9)$$

which may be transformed into the principal axis system to

$$U_r = \frac{1}{2}\{\delta_r\}^T[\alpha_r]\{\delta_r\} \quad (10)$$

in which

$$[\alpha_r] = [T_R]^T[\alpha_R][T_R] \quad (11)$$

3 FINITE ELEMENT ANALYSIS OF ELASTIC BUCKLING

A finite element computer program FTBCOR has been prepared which can analyse the elastic buckling of columns with varying axial force and concentrated elastic or rigid oblique eccentric restraints against deflections and rotations which can act at a number of points along the column length. This program uses the principle of conservation of energy at elastic buckling, as expressed by the energy equation

$$\delta U + \delta V = 0 \tag{12}$$

in which

$$\delta U = \sum_e \frac{1}{2}\int_0^{L_e}\left(EI_{ye}u''^2 + EI_{xe}v''^2 + GJ_e\phi'^2 + EI_{we}\phi''^2\right)dz + \sum_r \frac{1}{2}\{\delta_r\}^T[\alpha_r]\{\delta_r\} \tag{13}$$

$$\delta V = -\sum_e \frac{\lambda}{2}\int_0^{L_e} N_e\left[u'^2 + v'^2 + \{(I_{xe}+I_{ye})/A_e + x_{0e}^2 + y_{0e}^2\}\phi'^2 - 2x_{0e}v'\phi' + 2y_{0e}u'\phi'\right]dz = 0 \tag{14}$$

in which E and G are the Young's and shear moduli of elasticity, A_e is an element's area, x_{0e}, y_{0e} are the shear centre coordinates, I_{xe}, I_{ye} are the second moments of area about the x, y principal axes, L_e is the length of and N_e is the initial axial force in the element, z is the distance along the element, and λ is the load factor at buckling.

In the finite element method, cubic fields (Trahair, 1993) are used to represent the displacements u, v and twist rotations ϕ in terms of the element nodal deformations $\{\delta_e\}$, and the components of Equations 13 and 14 are transformed to

$$\frac{1}{2}\int_0^{L_e}\left(EI_{ye}u''^2 + EI_{xe}v''^2 + GJ_e\phi'^2 + EI_{we}\phi''^2\right)dz = \frac{1}{2}\{\delta_e\}^T[k_e]\{\delta_e\} \tag{15a}$$

$$\frac{1}{2}\int_0^{L_e} N_e\left[u'^2 + v'^2 + \{(I_{xe}+I_{ye})/A_e + x_{0e}^2 + y_{0e}^2\}\phi'^2 - 2x_{0e}v'\phi' + 2y_{0e}u'\phi'\right]dz = \frac{1}{2}\{\delta_e\}^T[g_e]\{\delta_e\} \tag{15b}$$

in which $[k_e]$ and $[g_e]$ are the element stiffness and stability matrices (Trahair, 1993). These are then summed for all the elements to form the global matrices $[K]$ and $[G]$ and these are added to the restraint stiffnesses $[\alpha_r]$ to form

$$\frac{1}{2}\{\Delta\}^T\left([K]+[\alpha]-\lambda[G]\right)\{\Delta\} = 0 \tag{16}$$

Rigid restraints require some of the global deformations $\{\Delta\}$ to be condensed out of Equation 16, reducing it to

$$\frac{1}{2}\{\Delta_C\}^T\left([K_C]+[\alpha_C]-\lambda[G_C]\right)\{\Delta_C\} = 0 \tag{17}$$

A discussion of this condensation is given in the following section.

The lowest buckling load factor λ and the corresponding buckling mode defined by $\{\Delta_C\}$ may be extracted from Equation 17 by using standard eigenvalue procedures. The computer program FTBCOR has been written using the MATLAB (Mathworks, 1995) language and functions.

4 RIGID RESTRAINTS

The condensations for rigid restraints referred to above are quite straightforward when the rigid restraints prevent one or more of the principal shear centre deformations of Equation 3 since these correspond directly to some of the global deformations $\{\Delta\}$. When the rigid restraints act in oblique planes or are eccentric, it is simpler to transform the corresponding global deformations to the oblique planes, as described below.

It is assumed that there are two rigid restraint points, at (x_U, y_U) as in Fig. 2, and at (x_V, y_V). Restraint planes $X_U z$ and $X_V z$ through these two points are inclined at different angles θ_U, θ_V to the x axis. Rigid restraints at these two points may prevent translations U_U, U_V in the X_U, X_V directions and rotations U_U', U_V' in the $X_U z$, $X_V z$ planes.

Using Equations 1-4 for both points,

$$\begin{Bmatrix} U_U \\ U_V \end{Bmatrix} = \begin{bmatrix} C_U & S_U & -y_{U0}C_U + x_{U0}S_U \\ C_V & S_V & -y_{V0}C_V + x_{V0}S_V \end{bmatrix} \begin{Bmatrix} u \\ v \\ \phi \end{Bmatrix} \tag{18}$$

whence

$$\{u \quad v \quad \phi\}^T = [T_{RR}]\{U_U \quad U_V \quad \phi\}^T \tag{19}$$

in which

$$[T_{RR}] = \begin{bmatrix} \dfrac{S_V}{S_V C_U - S_U C_V} & \dfrac{-S_U}{S_V C_U - S_U C_V} & \dfrac{-(x_{U0} - x_{V0})S_U S_V + (y_{U0} S_V C_U - y_{V0} S_U C_V)}{S_V C_U - S_U C_V} \\ \dfrac{-C_V}{S_V C_U - S_U C_V} & \dfrac{C_U}{S_V C_U - S_U C_V} & \dfrac{(y_{U0} - y_{V0})C_U C_V - (x_{U0} S_U C_V - x_{V0} S_V C_U)}{S_V C_U - S_U C_V} \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

Similarly,

$$\{u' \quad v' \quad \phi'\}^T = [T_{RR}]\{U'_U \quad U'_V \quad \phi'\}^T \tag{21}$$

Equations 19 - 21 may be used to transform the appropriate principal shear centre deformations of the global deformations $\{\Delta\}$ into the corresponding oblique eccentric deformations, following which the appropriate oblique eccentric deformations corresponding to the rigid restraints can be condensed out of the transformed Equation 17.

5 APPLICATIONS

5.1 Oblique Translational Restraints

The finite element computer program has been used to analyse the elastic flexural-torsional buckling of a uniform tee section (Fig. 3a) column whose end deflections and twist rotations are prevented but whose end flexural rotations and warping are free. The properties of the tee section were obtained by using the computer program THIN-WALL (Papangelis and Hancock, 1997) and are shown in Table 1. When the column has no central restraints, then the buckling load N_0 is equal to the flexural-torsional buckling load N_{yz} obtained from (Timoshenko and Gere, 1961 and Trahair, 1993)

$$N_{yz} = \frac{(N_y + N_z) - \sqrt{\{(N_y + N_z)^2 - 4N_y N_z r_0^2 /(r_0^2 + y_0^2)\}}}{2r_0^2 /(r_0^2 + y_0^2)} \tag{22}$$

in which

$$r_0^2 = (I_x + I_y) / A \tag{23}$$

$$N_y = \pi^2 EI_y / L^2 \tag{24}$$

is the y axis flexural buckling load, and

$$N_z = (GJ + \pi^2 EI_w / L^2)/(r_0^2 + x_0^2 + y_0^2) \tag{25}$$

is the torsional buckling load.

The solutions of Equation 22 for the tee section column (with x_0 = 0) are shown in Fig. 4 as the variations of the dimensionless buckling load N_0/N_z with the dimensionless length L/L_{yz}, in which L_{yz} is the column length at which N_y = N_z. It can be seen that N_0 = N_{yz} decreases from N_z towards N_y as the dimensionless length L/L_{yz} increases from zero. Also shown in Fig. 4 are the corresponding finite element solutions (obtained using eight equal elements), which are in very close agreement.

The finite element predictions for columns with rigid oblique restraints against central translation (at z = L/2) in a plane Xz through the shear centre (S) (and therefore concentric) at θ to the xz principal plane are also shown in Fig. 4. When θ = 90°, the central restraint has no effect on the y axis flexural buckling load, and the column buckles at N_0 = N_{yz}. When θ = 0°, the central restraint increases the y axis flexural buckling load from N_y to a value greater than the x axis flexural buckling load

$$N_x = \pi^2 EI_x / L^2 \tag{26}$$

and the flexural-torsional buckling load to a value greater than N_{yz}, so that the column buckles at the lesser of N_x and the increased flexural-torsional buckling load. For intermediate values of θ, the buckling load N_0 generally increases from N_{yz} towards N_x as θ decreases from 90° towards 0°, except at low values of L/L_{yz}.

5.2 Eccentric Translational Restraints

The finite element computer program has been used to analyse the elastic flexural-torsional buckling of a uniform I-section (Fig. 3b) column whose end deflections and twist rotations are prevented but whose end flexural rotations and warping are free. The properties of the I-section were obtained by using the computer program THIN-WALL (Papangelis and Hancock, 1997) and are shown in Table 1. When the column has no central restraints, then the buckling load N_0 is equal to the y axis flexural buckling load N_y, which is lower than the x axis flexural buckling load N_x and the torsional buckling load N_z. The variations of the dimensionless buckling load $N_0/N_{x=z}$ with the dimensionless length L/L_{xz} (in which L_{xz} is the column length at which $N_x = N_z = N_{x=z}$) are shown in Fig. 5.

The finite element predictions for I-section columns with rigid restraints against central translation (at $z = L/2$) in a plane Xz parallel to the xz plane but at an eccentricity e from the xz principal plane are also shown in Fig. 5. When $2e/h = 0$ (in which h is the distance between flange centroids), the central restraint increases the y axis flexural buckling load from N_y to $4N_y > N_x$, so that the column buckles at the lower of the torsional buckling load N_z (at low values of L/L_{xz}) and the x axis flexural buckling load N_x (at high values of L/L_{xz}). As the dimensionless eccentricity $2e/h$ increases towards ∞, the buckling load N_0 decreases towards N_y.

If it is assumed that the central restraint causes the column to buckle with an enforced centre of rotation at an axis through the restraint point (Trahair, 1993), then the buckling load can be determined from

$$N_0 = \frac{N_y e^2 + N_z r_0^2}{r_0^2 + e^2} \tag{27}$$

Solutions obtained from Equation 27 are also shown in Fig. 5. They are very close to the finite element solutions.

It is commonly assumed by the designers of steel structures that a rigid central translational restraint is sufficient to increase the elastic buckling load from N_y to the lesser of 4 N_y and N_x. The making of this assumption is encouraged by the absence of a full consideration of the possibility of torsional or flexural-torsional buckling in many design codes (SA, 1998; BSI, 2000; and AISC, 1999a). However, Fig. 5 shows that this assumption is unwarranted, even for concentric restraints, and becomes increasingly dangerous as the eccentricity increases. This danger may be avoided by providing restraints which prevent twist rotation as well as x axis translation.

5.3 Oblique Eccentric Restraints

5.3.1 Equal angle section columns

The finite element computer program has been used to analyse the elastic flexural-torsional buckling of a uniform equal angle section (Fig. 3c) column whose end deflections and twist rotations are prevented but whose end flexural rotations and warping are free. The properties of the angle section were obtained by using the computer program THIN-WALL (Papangelis and Hancock, 1997) and are shown in Table 1. The buckling load N_0 is the lower of (Timoshenko and Gere, 1961 and Trahair, 1993) the y axis flexural buckling load N_y and the flexural-torsional buckling load

$$N_{xz} = \frac{(N_x + N_z) - \sqrt{\{(N_x + N_z)^2 - 4N_x N_z r_0^2 /(r_0^2 + x_0^2)\}}}{2r_0^2 /(r_0^2 + x_0^2)} \tag{28}$$

The solutions of Equations 24 and 28 for the equal angle section column (with y_0 = 0) are shown in Fig. 6 as the variations of the dimensionless buckling load N_0/N_z with the dimensionless length L/L_{yz}. At low values of L/L_{yz}, $N_y > N_{xz}$ and the column buckles at $N_0 = N_{xz}$ in a flexural-torsional mode by deflecting v in the y direction and twisting ϕ. However, at higher values of L/L_{yz}, $N_y < N_{xz}$ and the column buckles in a flexural mode at $N_0 = N_y$ by deflecting u in the x direction. Also shown in Fig. 6 are the corresponding finite element, which are in very close agreement.

The finite element predictions for equal angle section columns with rigid oblique restraints against end rotations in a plane Xz through the centroid (and therefore eccentric to the shear centre) at θ to the xz principal plane are also shown in Fig. 6. When $\theta = 0^o$, the end restraints increase the y axis flexural buckling load from N_y to $4N_y = N_x$, which is always greater than the flexural-torsional buckling load N_{xz}, so that the column always buckles at $N_0 = N_{xz}$. When $\theta = 90^o$, the end restraints have no effect on the y axis flexural buckling load, and so the column generally buckles at N_y , except at low values of L/L_{yz}. For intermediate values of θ, the buckling load generally increases from N_y towards N_{xz} as θ decreases from 90^o towards 0^o, except at low values of L/L_{yz}.

5.3.2 Unequal angle section column

The finite element computer program has been used to analyse the elastic flexural-torsional buckling of a uniform unequal angle section (Fig. 3d) column of length 1409.7mm whose end deflections and twist rotations are prevented. The properties of the angle section were obtained by using the computer program THIN-WALL (Papangelis and Hancock, 1997) and are shown in Table 1.

The finite element predictions of the variations of the dimensionless buckling load N_0/N_y with θ for columns with rigid oblique centroidal restraints against end rotations in a plane Xz at θ to the xz principal plane are shown in Fig. 7. Also shown in Fig. 7 are the corresponding predictions obtained from Trahair (1969). It can be seen that these are generally in close agreement.

6 DESIGN OF COLUMNS WITH OBLIQUE RESTRAINTS

6.1 Design by Buckling Analysis

The simplest method of designing steel columns with oblique eccentric restraints which is compatible with the widely used methods of designing columns with principal axis concentric restraints is to use the method of design by buckling analysis, which is incorporated either directly or indirectly in design codes such as the AISC Specification (AISC, 1999a), the Australian Standard AS4100 (SA, 1998), or the British Standard BS5950 (BSI, 2000). In this method, the results of an analysis for the elastic buckling load N_0 of the obliquely and eccentrically restrained column is used with the squash load

$$N_Y = Af_y \tag{29}$$

to calculate a modified slenderness

$$\lambda_c = \sqrt{(N_Y / N_0)} \tag{30}$$

which is then used to determine the nominal design capacity N_n.

For example, the AISC Specification and the Australian/New Zealand Standard (SA, 1996) for cold-formed steel structures both use

$$\frac{N_n}{N_Y} = 0.658\hat{}(\lambda_c^{\,2}) \qquad (\lambda_c \le 1.5) \tag{31a}$$

$$\frac{N_n}{N_Y} = 0.877/(\lambda_c^{\,2}) \qquad (\lambda_c > 1.5) \tag{31b}$$

as shown in Fig. 8.

6.2 Worked Example

A 1409.7 mm long steel unequal angle section (UA) column has the dimensions shown in Fig. 3d, the section properties shown in Table 1, and a yield stress of 282.7 MPa (Trahair, 1969). The column is prevented from deflecting and twisting at both ends, and is prevented from rotating in the centroidal plane parallel to the short leg but is free to rotate in the centroidal plane parallel to the long leg at both ends. The design compression capacity may be determined as follows.

(Fig. 3d)	θ = 180-23.87 = 156.13^o
(Fig. 7)	N_0/N_y = 3.2
(Equation 24)	$N_y = \pi^2$ x 1.9306E5 x 9.052E4 / 1409.7^2 = 86.8E3 N.
	N_0 = 3.2 x 86.8E3 = 277.7E3 N.
(Equation 29)	N_Y = 766.1 x 282.7 = 216.5E3 N.
(Equation 30)	λ_c = √(216.5E3 / 277.7E3) = 0.883 < 1.5
(Equation 31a)	N_n = 216.5E3 x (0.658 ^ (0.883^2)) = 156.2E3 N.

If the capacity factor is 0.85, the design capacity is N_d = 0.85 x 156.2E3 = 132.8E3 N.

7 CONCLUSIONS

Column flexural restraints which act in planes oblique to the principal planes cause coupling between the principal axis deflections and rotations, while eccentric (from the shear centre) flexural restraints cause coupling between the principal axis deflections and rotations and the twist rotations and warping displacements. The general buckling mode involves simultaneous bending about both principal axes and torsion.

This paper describes a finite element program for analyzing the elastic flexural-torsional buckling of columns under varying axial compression and with oblique, eccentric, concentrated restraints which may be rigid or elastic. The program was validated by comparisons with well-known results for columns with principal axis rotational and translational restraints, and with recently obtained solutions for the flexural buckling of columns with oblique restraints.

The program was used to analyse the flexural-torsional buckling of tee section columns with oblique concentric (through the shear centre) central translational restraints. Unrestrained columns buckle in a flexural-torsional mode at loads N_0 which tend to the torsional buckling load N_z as the slenderness decreases and to the flexural buckling load N_y as the slenderness increases. Oblique central translational restraints increase the effective flexural buckling loads and so there are consequent increases in the buckling load N_0. These increase as the angle θ between the restraint plane and the xz plane decreases from 90^o to 0^o.

The program was also used to analyse the flexural-torsional buckling of I-section columns with eccentric central translational restraints. Unrestrained columns buckle in a flexural mode at loads N_0 equal to the flexural buckling load N_y. Eccentric central translational restraints increase the effective flexural buckling load and so there are consequent increases in the buckling load N_0. These increase as the eccentricity e decreases towards 0, until the lower of the torsional buckling load N_z and the flexural buckling load N_x is reached. It was concluded that central restraint may not be as effective as is commonly assumed. It is recommended that such restraints be required to prevent twist rotation as well as deflection.

The program was also used to analyse the flexural-torsional buckling of equal angle section columns with oblique centroidal (and therefore eccentric from the shear centre) rotational end restraints. Unrestrained columns of low slenderness buckle in a flexural-torsional mode at loads N_0 which tend to the torsional buckling load N_z as the slenderness decreases, and to the flexural buckling load N_x as the slenderness increases. At higher slendernesses the buckling load changes to the flexural buckling load N_y. Oblique rotational end restraints increase the effective flexural buckling loads and so there are consequent increases in the buckling load N_0. These increase as the angle θ between the restraint plane and the xz plane decreases from 90^o to 0^o.

The program was also used to analyse the flexural-torsional buckling of an unequal angle section column with oblique centroidal (eccentric) rotational end restraints. It was found that the buckling load increases as the angle θ between the restraint plane and the xz plane increases from 90^o approximately (when the restraint plane coincides with the stiffer xz principal plane) to 170 ^{o} approximately (which is close to the minor yz principal plane), and then decreases.

A worked example is presented which demonstrates, by using the method of design by buckling analysis, that restrained steel columns with oblique eccentric restraints can be designed in a way which is consistent with the traditional methods of designing columns with concentric principal axis restraints.

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APPENDIX 2 NOTATION

A	area of cross-section
A_e	area of element cross-section
B	restraint bimoment
$C_{U, V}$	cos $\theta_{U, V}$
e	eccentricity of translational restraint
E	Young's modulus of elasticity
$\{F_R\}$	vector of restraint actions
F_{XU}	restraint force
f_y	yield stress
$[G]$, $[K]$	global stiffness and stability matrices
$[G_C]$, $[K_C]$	condensed global stiffness and stability matrices
$[g_e]$, $[k_e]$	element stiffness and stability matrices
h	distance between flange centroids
I_x, I_y	column second moments of area about the x, y principal axes
I_{xe}, I_{ye}	element second moments of area about the x, y principal axes
L	column length
L_e	element length
L_{xz}	length of a column for which $N_x = N_z$
L_{yz}	length of a column for which $N_y = N_z$
M_{XU}, M_{YU}	restraint moments
M_z	restraint torque
N	concentric axial load
N_d	design compression capacity
N_e	element axial compression
N_n	nominal compression capacity
N_x, N_y	Euler flexural buckling loads
N_{xz}, N_{yz}	flexural- torsional buckling loads
$N_{x=z}$	N_0 for a column of length L_{xz}
N_z	torsional buckling load
N_Y	squash load
N_0	elastic buckling load
r_0	radius of gyration
$S_{U, V}$	sin $\theta_{U, V}$
$[T_R]$, $[T_{RR}]$	transformation matrices
u, v	shear centre deflections parallel to the x, y principal axes
U_R, U_r	strain energy of the restraints
U_U, U_V	deflections in restraint planes
x, y	principal axes
x_0, y_0	coordinates of shear centre
x_{0e}, y_{0e}	element coordinates of shear centre
X	oblique direction of restraint plane
x_U, x_V	x coordinates of restraint points
x_{U0}, y_{V0}	x, y distances from restraint point to shear centre
y_U, y_V	y coordinates of restraint points
z	distance along element

$[\alpha]$	restraint stiffness matrix
$[\alpha_C]$	condensed restraint stiffness matrix
$[\alpha_R]$	oblique restraint stiffness matrix
$[\alpha_r]$	principal axis restraint stiffness matrix
α_{RU}	stiffness of rotational restraint
α_{TU}	stiffness of translational restraint
α_w	stiffness of warping restraint
α_z	stiffness of torsional restraint
$\{\delta_R\}$	vector of oblique plane deformations at a restraint point
$\{\delta_r\}$	vector of principal axis deformations at a restraint point
δU	increase in strain energy
δV	increase in potential energy
$\{\Delta\}$	vector of global nodal deformations
$\{\Delta_C\}$	condensed vector of global nodal deformations
θ_U, θ_V	inclinations of restraint planes to the xz principal plane
ϕ	angle of twist rotation
λ	buckling load factor
λ_c	modified slenderness

Quantity	T	I	EA	UA
A (mm^2)	8800	2E4	8000	766.1
I_x (mm^4)	3.515E7	6.933E8	5.333E7	5.205E5
I_y (mm^4)	2.304E7	2.133E8	1.333E7	9.052E4
J (mm^4)	1.173E6	2.267E6	1.067E6	1.030E4
I_w (mm^6)	0	8.533E12	0	0
x_0 (mm)	0	0	- 70.71	- 17.54
y_0 (mm)	- 45.46	0	0	- 16.40
E (MPa)	2E5	2E5	2E5	1.9306E5
G (MPa)	8E4	8E4	8E4	7.7224E4

Table 1 Section Properties

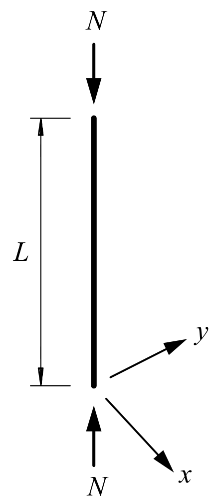


Fig.1 Elastic Column

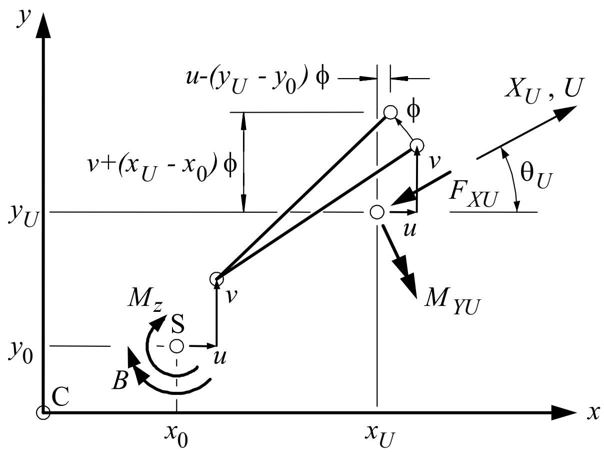
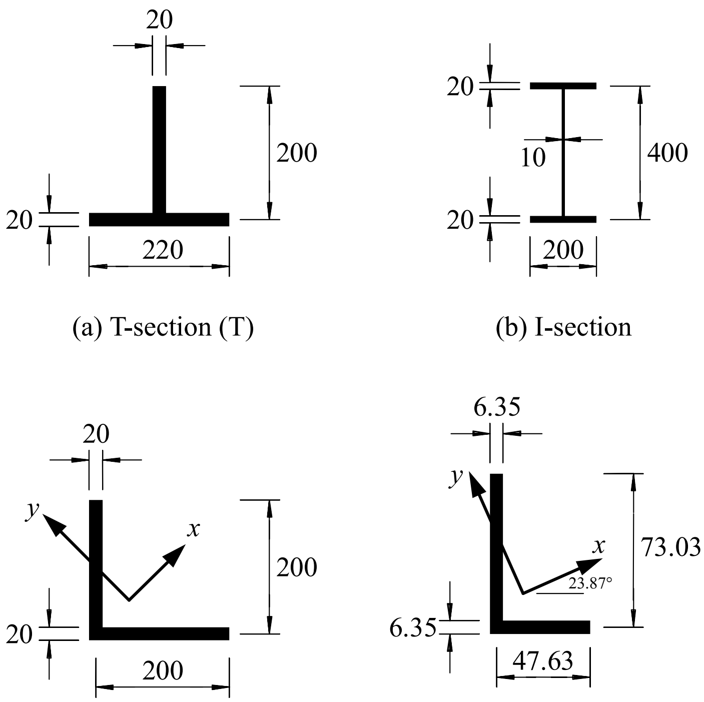


Fig.2 Restraints



(a) T-section (T) (b) I-section

(c) Equal angle (EA) (d) Unequal angle (UA)

Fig.3 Section Dimensions (mm)

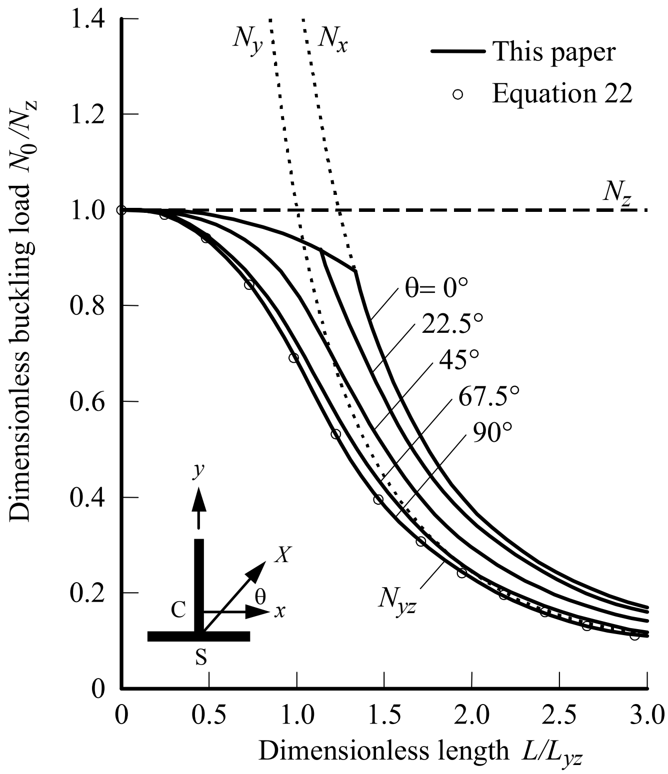


Fig.4 Buckling Loads of Tee Section Columns with Oblique Central Translational Restraints

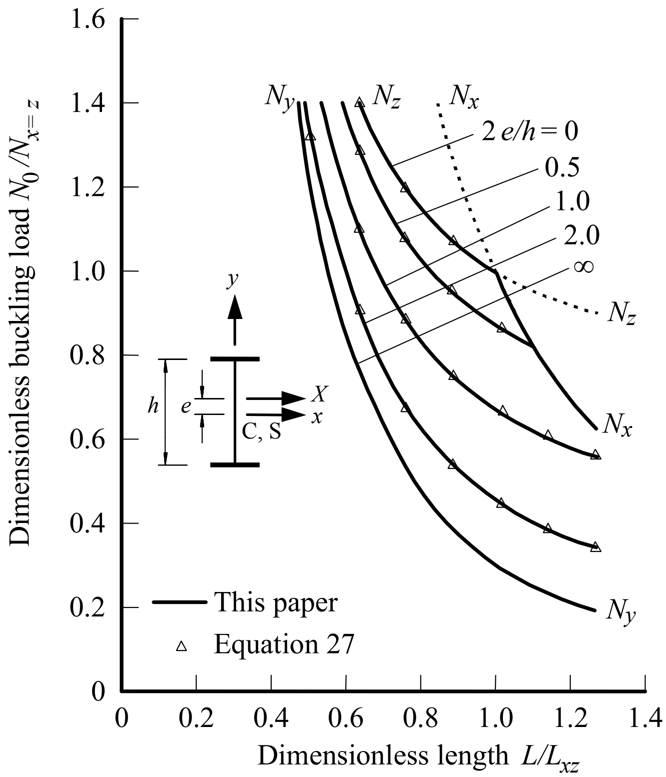


Fig.5 Buckling Loads of I-Section Columns with Eccentric Central Translational Restraints

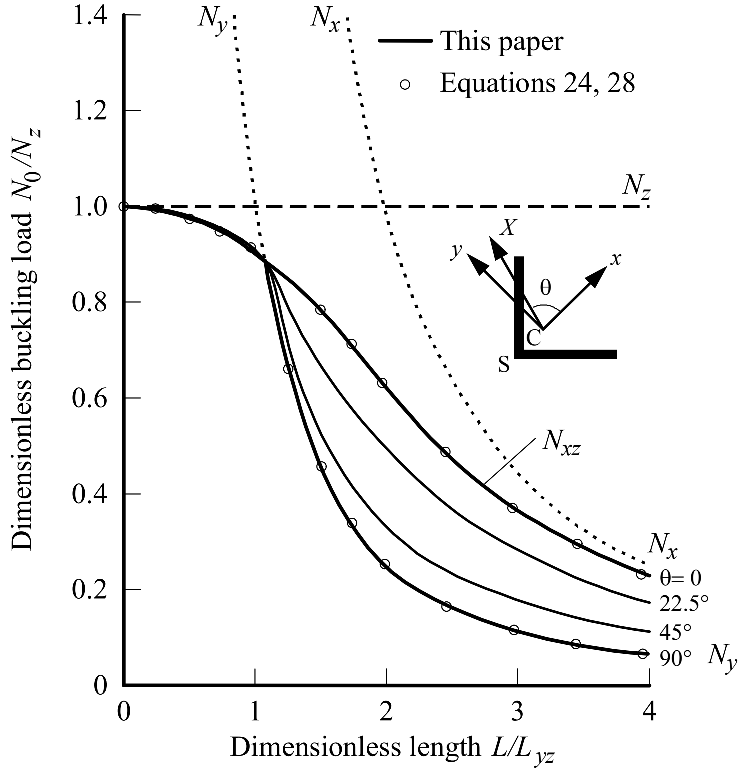


Fig.6 Buckling Loads of Equal Angle Columns with Oblique End Rotational Restraints

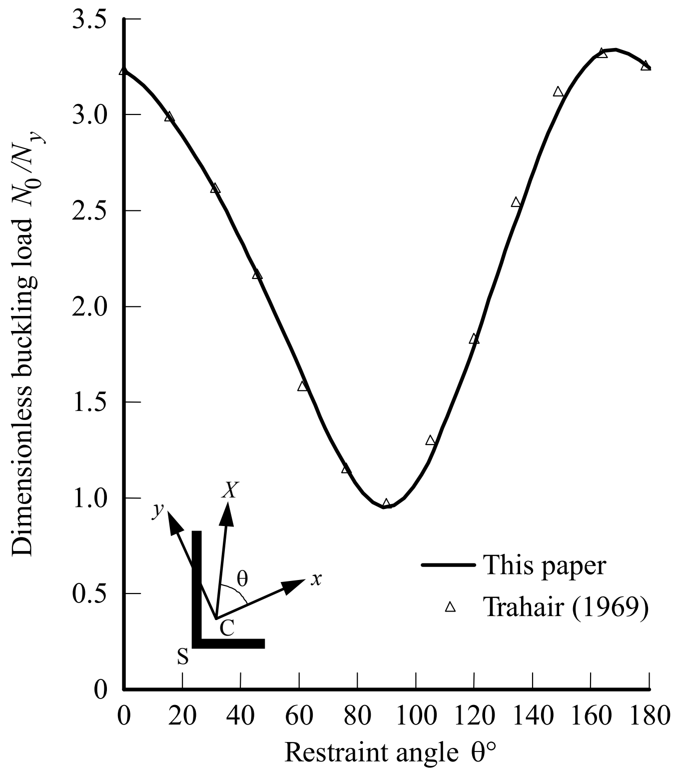


Fig.7 Buckling Loads of an Unequal Angle with Oblique End Rotational Restraints

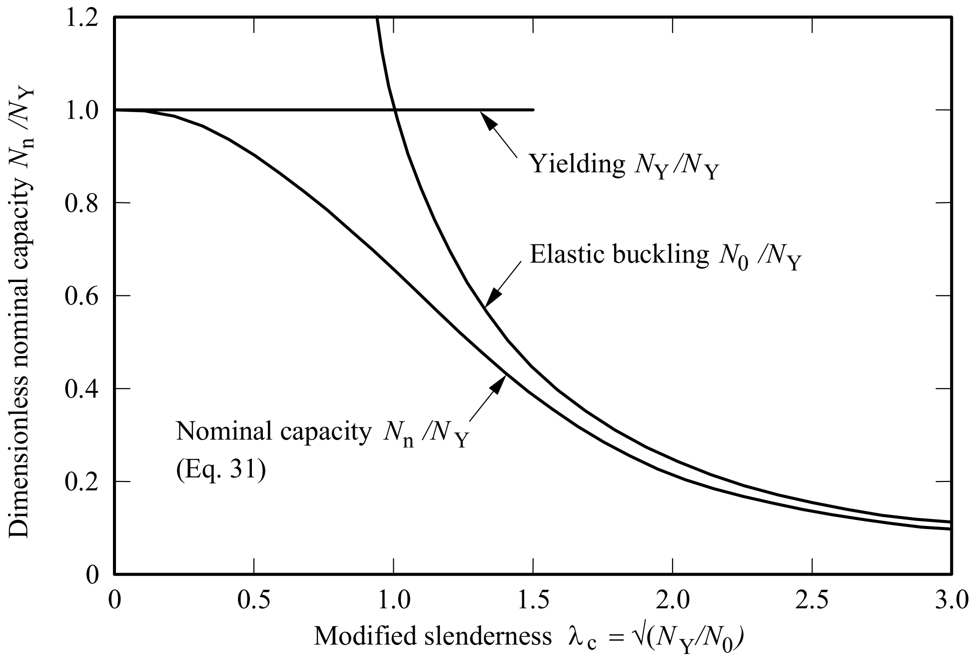


Fig.8 Design by Buckling Analysis