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Abstract

This paper discusses a generalization of strongly chordal graphs. We consider characteristic elimination orderings for these graphs and prove the perfectness of these graphs.

1 Introduction

Strongly chordal graphs have become interesting to solve domination problems efficiently [5]. They were introduced by characteristic elimination orderings, the *strongly perfect elimination orderings.*

1. for $xy, xz \in E$, such that $x < y$ and $x < z$, also $yz \in E$,
2. for $x_1y_2, x_2y_1, x_1x_2 \in E$, such that $x_1 < y_1$ and $x_2 < y_2$, we have $y_1y_2 \in E$.

If an ordering on the vertices satisfies only the first condition then we call it a *perfect elimination ordering.* It is well known that a graph has a perfect elimination ordering if and only each cycle of length greater than three can has a shortcut by an edge joining two non consecutive vertices of the cycle (*chord*).

We even can define an analogous definition of chordality for bipartite graphs. A bipartite graph is called *chordal bipartite* if each cycle of length greater than four has a chord [7]. These graphs are just those bipartite graphs that have an ordering that satisfies the second condition of a strongly perfect elimination ordering.

In this paper we deal with the general problem to characterize graphs having an ordering satisfying the second condition of a perfect elimination ordering. We call these graphs *generalized strongly chordal graphs.* We call an ordering satisfying only the second condition of a strongly perfect elimination ordering a generalized strongly perfect elimination ordering.

In the second section, we introduce the basic notions that are necessary for this paper. In the section afterwards, we consider the problem to characterize generalized strongly chordal graphs by forbidden subgraphs and to compute a generalized strongly perfect elimination ordering by the double lexical ordering procedure of Lubiv [9]. In the fourth section, we discuss perfectness properties of generalized strongly chordal graphs. The fifth section deals with a minimum coloring algorithms for generalized strongly chordal graphs.

2 Preliminaries

A *graph* $G = (V, E)$ consists of a *vertex set* V and an *edge set* E. Multiple edges and loops are not allowed. The edge joining x and y is denoted by xy.

We say that x is a *neighbor* of y iff $xy \in E$. The *full neighborhood* of x is the set $\{x\} \cup \{y : xy \in E\}$ consisting of x and all neighbors of x and is denoted by $N(x)$.

A *path* is a sequence $(x_1 \dots x_k)$ of distinct vertices such that $x_i x_{i+1} \in E$.

A *cycle* is a closed path, that means a sequence $(x_0 \dots x_{k-1} x_0)$ such that $x_i x_{i+1 \pmod{k}} \in E$.

A *subgraph* of (V, E) is a graph (V', E') such that $V' \subset V$, $E' \subset E$.

An *induced subgraph* is an edge-preserving subgraph, that means (V', E') is an induced subgraph of (V, E) iff $V' \subset V$ and $E' = \{xy \in E : x, y \in V'\}$.

A graph is called *chordal* iff each cycle of greater length than three has two non consecutive vertices that are joined by an edge. We call such an edge a *chord*. A bipartite graph is called *chordal bipartite* if each cycle of length greater than four has a chord.

Proposition 1 *[6] A graph $G = (V, E)$ is chordal iff there is an ordering $<$ on V, such that with $xy \in E$, $xz \in E$, $x < y$, and $x < z$, $yz \in E$. We call such an ordering a* perfect elimination ordering.

An ordering $<$ on the vertex set V of the graph $G = (V, E)$ is called *strongly perfect elimination ordering* if it is a perfect elimination ordering and it satisfies the following *strong elimination property*:

for $x_1 y_2, x_2 y_1, x_1 x_2 \in E$, such that $x_1 < y_1$ and $x_2 < y_2$, we have $y_1 y_2 \in E$.

3 Characterization of Generalized Strongly Chordal Graphs by Forbidden Induced Subgraphs

We begin with a characterization of strongly chordal graphs by forbidden subgraphs [5].

Proposition 2 *A chordal graph is strongly chordal iff it has no* trampoline *as an induced subgraph, i.e. no even cycle of length greater than four alternating between a complete and an independent set as an induced subgraph.*

We can relax this proposition as follows [3].

Proposition 3 *A chordal graph is strongly chordal iff any cycle of even length has a chord that splits the cycle into two odd paths.*

Theorem 1 *In any generalized strongly chordal graph, each cycle of odd length greater than three has a chord and each cycle of even length greater than four has a chord that splits the cycle into two paths of odd length.*

Proof: First we suppose that $G = (V, E)$ is a generalized strongly chordal graph and $<$ is a generalized strongly perfect elimination ordering. Suppose $C = (v_0, \ldots, v_{k-1}, v_0)$ is is a cycle. Suppose $v_{i-2 \, mod \, k} < v_i$. If $v_{i-3 \, mod \, k} > v_{i-1 \, mod \, k}$ Then there is an edge $v_i v_{i-3}$ that splits C into two paths of odd length if C is of even length greater that four and is a chord if C is of odd length.

□

The converse of this theorem is not true. We consider the following example:

We define as the *house* a cycle of length five with exactly one chord. Note that the chord of the house splits the cycle of length five into a cycle of length three and a cycle of length four. We call those vertices belonging to the cycle of length three *triangle vertices.*

We consider the graph G consisting of two copies H_1 and H_2 of the house and one edge joining a non triangle vertex of H_1 with a non triangle vertex of H_2.

Obviously in G, any cycle of odd length has a chord and any cycle of even length is of length four, and therefore the conclusions of the theorem are trivially satisfied.

Let $(x_1, x_2, x_3, x_4, x_5, x_1)$ be the cycle of length five of the first copy of the house and x_3x_4 be the chord of the house.

Let $<$ be a generalized strongly perfect elimination ordering. If $x_3 < x_1$ then $x_4 < x_2$ and therefore $x_5 < x_3$ and $x_1 < x_4$. Therefore there exists at least one x_j with $x_1 < x_j$ such that x_1 and x_j are not adjacent, i.e. $j = 3, 4$.

Let $(y_1, y_2, y_3, y_4, y_5, y_1)$ be the cycle of length five of the second copy of the house, yx_3, y_4 be the chord of the second copy of the house. Moreover we assume that x_1y_1 forms an edge. Suppose $x_1 < x_j$, $j = 3$ or $j = 4$. Let $x_k = x_2$ if $x_j = x_3$ and $x_k = x_5$ if $x_j = x_4$. Then $y_1 < x_k$, since (x_j, x_k, x_1, y_1) is a path that cannot be shortcut by one edge. Even the path (x_k, x_1, y_1, y_2) and the path (x_k, x_1, y_1, y_5) cannot be shortcut by an edge. Therefore $y_2, y_5 < x_1$. By the same argument, $y_3, y_4 < y_1$. But, by the same argument as in the first copy of the house, $y_1 < y_3$ or $y_1 < y_4$. This is a contradiction.

Observe that in a house, y_1 cannot be the maximum of a generalized strongly perfect elimination ordering.

We improve the definition of a generalized strongly chordal graph. A graph is called a *good generalized strongly chordal graph* if each vertex is the maximum of a strongly perfect elimination ordering.

It might be senseful to characterize good generalized strongly chordal graphs by forbidden induced subgraphs.

We can generalize the fact that a non triangle vertex of a house cannot be the maximum of a generalized strongly perfect elimination ordering.

Lemma 1 *Suppose v is a vertex of a cycle C of odd length greater that three in a generalized strongly chordal graph such that v does not belong to a chord of C and each chord of C splits C into two paths such that v belongs to the path of odd length. Then v cannot be the maximum of a generalized strongly perfect elimination ordering.*

Proof: We can restrict our considerations to the vertices of C. Suppose $(v = x_1, \ldots, x_{2k-1}, x_{2k} = v)$ is the canonical enumeration of C, i.e. x_i and x_{i+1} are consecutive vertices of C. Note that the only neighbors of v are x_2 and x_{2k-1}. Therefore there is no chord x_1x_4 and no chord x_1x_{2k-3}. Moreover for any i, $2 \leq i < 2k-3$, there is no chord x_ix_{i+3}. Otherwise we had a chord that splits C into two paths such that v belongs to the path of even length.

Assume $<$ is a generalized strongly perfect elimination ordering with v as its maximum. Then $x_3 < x_1 = v$. But since there is no edge x_1x_4, we get $x_4 < x_2$, and by induction on i, we get by the same argument $x_{i+2} < x_i$ and therefore also $v = x_{2k} < x_{2k-2}$. This is a contradiction.

□

Theorem 2 *A graph is a good generalized strongly chordal graph iff each cycle of even length greater than four has a chord splitting it into two paths of odd length and for each vertex v of a cycle C of odd length greater than three there is a chord such that v belongs to or it splits C into two paths such that v belongs to the path of even length.*

Proof: The second statement follows from the first statement by previous lemma. To prove the other way round, we introduce the notion of a lexical ordering.

We call an ordering of the vertices of G a *lexical ordering* [9] if the following requirement is satisfied:

P0 If $a < b, ac \in E, bc \notin E$ then there is a vertex b' such that $bb' \in E, ab' \notin E, c < b'$.

It is well known that any graph has a lexical ordering [9]. Note that not each vertex may be the maximum of a lexical ordering. But for any vertex v, we can consider the graph G_v consisting of two copies of G such that the copies of v are identified. If the maximum of the lexical ordering of G_v is in the one copy of G then v is the maximum in the other copy of G in G_v. Therefore we can make any vertex v a maximum of a lexical ordering by extending the graph G to a graph G_v such that

1. G is an induced subgraph of G_v,
2. v is the maximum of a lexical ordering on G_v in G.

Now it is sufficient to prove that any lexical ordering is a generalized strongly perfect elimination ordering.

Assume the lexical ordering $<$ is not a generalized strongly perfect elimination ordering. Then there are x_0, y_0, x_1, y_1, such that $x_0 < x_1$, $y_0 < x_1$, x_0, y_0, x_0y_1, $y_0x_1 \in E$ and $x_1y_1 \notin E$. Since $<$ is a lexical ordering, there must be an x_2 and a y_2 such that $x_1y_2 \in E$, $y_1x_2 \in E$, $x_0y_2 \notin E$, and $y_0x_2 \notin E$. We choose x_2 and y_2 to be maximal, i.e. for all $y > y_2$, $x_1y \in E$ iff $x_0y \in E$ and for all $x > x_2$, $y_1x \in E$ iff $y_0x \in E$.

If $x_2 = y_2$ then we have a cycle of odd length five such that all chords are only of the form x_ix_j or y_iy_j. Therefore all chords partition this cycle into two paths such that the odd path contains $x_2 = y_2$. Therefore we can exclude the case $x_2 = y_2$.

If $x_2y_2 \in E$ then we have a cycle of even length and all chord x_ix_j or y_iy_j partition into two paths of even length. Therefore we even can exclude the case that $x_2y_2 \in E$.

We have a sequence $(x_0, \ldots, x_k)$ and a sequence $(y_0, \ldots, y_k)$ with the following properties.

1. $x_i < x_{i+1}$, $y_i < y_{i+1}$,
2. $x_iy_j \in E$ iff $|i - j| = 1$ or $i = j = 0$,

3. For each i, and all $y > y_{i+2}$ and all $x > x_{i+2}$, $x_i y \in E$ iff $x_{i+1} y \in E$ and $y_i x \in E$ iff $y_{i+1} x \in E$.

We extend this sequence by vertices x_{k+1} and y_{k+1} as follows. Since $x_k y_k \notin E$, $x_{k-1} y_k \in E$, there is a y_{k+1} such that $x_k y_{k+1} \in E$ and $x_{k-1} y_{k+1} \notin E$. Again we choose a largest y_{k+1}. Therefore since $<$ is a lexical ordering, for all $y > y_{k+1}$, $x_{k-1} y \in E$ iff $x_k y \in E$. Applying condition 3 for each $i < k$, y_{k+1} is not adjacent to any x_i, $i < k$.

x_{k+1} is constructed analogously.

By the same arguments as in the construction of x_2 and y_2, x_{k+1} and y_{k+1} can neither be equal nor adjacent.

Since we deal with finite graphs, we get a contradiction.

□

4 Perfectness Properties

One of the most important problems in graph theory is to label each vertex of any graph with a color such that adjacent vertices are colored differently. The number of colors should be minimal. We call the minimum number of colors needed to color a graph in such a way the *chromatic number* of a graph. Clearly the maximum size of a clique of a graph is bounded by the chromatic number. We call a graph *perfect* if for each induced subgraph, the chromatic number and the maximum clique size coincide. It is an open problem to recognize perfect graphs in polynomial time. It is even an open problem to characterize perfect graphs by forbidden induced subgraphs.

Conjecture [1] (see also [2]) A graph is perfect iff it has no cycle of odd length greater three and not the complement of a cycle of odd length greater three as an induced subgraph.

We call a graph *Berge perfect* if it has not a cycle and not the complement of a cycle of odd length greater three as an induced subgraph.

In context to generalized strongly chordal graphs, we can prove the following.

Theorem 3 *Let $G = (V, E)$ be a graph.*

1. *If each cycle C of G of even length greater than four has a chord that splits C into two paths of odd length and each cycle of G of odd length greater than three has a chord then G is Berge-perfect.*
2. *If G is a generalized strongly chordal graph then G is perfect.*

Proof: First we prove the following.

Lemma 2 *The complement of a path of length six has a cycle of even length with no chord splitting into two odd length paths.*

Proof of Lemma: Consider vertices $x_1, \ldots, x_6$ such that $x_i x_j \in E$ iff $|i - j| \neq 1$. The cycle $(x_1, x_3, x_5, x_2, x_4, x_6, x_1)$ has no chord that splits into paths of odd length.

□(Lemma)

From this lemma, it follows immediately that any graph with the property that each even cycle of length greater than four has a chord that splits into two paths of odd length has no complement of any cycle of length greater than six as an induced subgraph. Since the complement of the cycle of length five is the cycle of length five, a graph satisfying the requirements of part 1 is Berge-perfect.

We continue with a proof of part 2. A graph is called *weakly triangulated* if the graph itself and its complement have the property that any cycle of length greater than four has a chord [8].

Lemma 3 *[8] Any weakly triangulated graph is perfect.*

We prove the following.

Lemma 4 *Any generalized strongly chordal graph is weakly triangulated.*

Proof of Lemma: Since any generalized strongly chordal graph satisfies the requirements of part 1, we only have to prove that a generalized strongly chordal graph does not contain the complement of a cycle of length six as an induced subgraph.

We consider the vertex set $\{x_0, \ldots, x_5\}$. $x_i x_j \in E$ iff neither $i-j$ nor $j-1$ is $1 \, mod \, 6$. We assume there is a generalized strongly perfect elimination ordering $<$. Suppose $x_1 < x_0$. The path (x_0, x_3, x_1, x_5) has the property that the end vertices are not adjacent. Therefore $x_5 < x_3$. Even the path (x_3, x_1, x_5, x_2) has the property that the end vertices are not adjacent. Therefore $x_2 < x_1$. By the same argument, one gets $x_i < x_{i-1 \, mod \, 6}$, for any $i = 0, \ldots 5$. This is a contradiction. If $x_0 < x_1$ then we get the same contradiction. □(Lemma)

Herewith even part 2 has been proved.

□ (Theorem)

5 A Minimum Coloring Algorithm for General Strongly Chordal Graphs

Suppose a generalized strongly perfect elimination ordering of $G = (V, E)$ is given. One might mention that the right minimum strategy is backward greedy coloring. There we find the following counterexample. Consider the path (x_1, x_2, x_3, x_4) that has no chords. Suppose $x_2 < x_1 < x_3 < x_4$. Then backward greedy coloring needs three colors. But a coloring with two colors such that adjacent vertices have different colors is possible.

Before we state a minimum coloring algorithm, we consider the structure of greater neighbors of any vertex.

Lemma 5 *Suppose $<$ is a generalized strongly perfect elimination ordering on $G = (V, E)$ and $v \in V$. Then there is a neighbor $x > v$ of v such that all neighbors y with $v < y \leq x$ form an independent set and all neighbors z of v with $x \leq z$ induce a complete subgraph.*

Proof: Let $N = \{y > v | yv \in E\}$.

Fact 1 *If $x, y \in N$, $x < y < z$, and $xz \in E$ then $yz \in E$.*

Proof of Fact: Note that vx, vy, $yz \in E$. Then the fact follows directly from the assumption that $<$ is a generalized strongly perfect elimination ordering.

□(Fact)

Fact 2 *If $x, y, z \in N$, $xy \in E$, and $x < y < z$ then $yz \in E$.*

Proof of Fact: Note that xy, xv, $vz \in E$. Then the fact follows immediately from the assumption that $<$ is a generalized strongly perfect elimination ordering.

□(Fact)

Fact 3 *If $x, y, z \in N$, $x < y < z$, and $xy \in E$ then $xz \in E$.*

Proof of Fact: Note that xy, yv, $vz \in E$. Then the assumption that $<$ is a generalized strongly perfect elimination ordering forces an edge xz.

□(Fact)

From these facts, we can conclude immediately:

Fact 4 *If $x, y \in N$, y is the immediate successor of x with respect to $<$ in N, and $xy \in E$ then $\{y \geq x | y \in N\}$ induces a complete subgraph.*

Now let x be the smallest vertex in N such that $\{y \geq x | y \in N\}$ induces a complete subgraph.

We have to prove that $\{y \in N | y \leq x\}$ is independent, i.e. no vertices in this set are adjacent.

Suppose $u, w \in N$, $u < w \leq x$. If $uw \in N$ and w' is the immediate predecessor of w in N with respect to $<$ then $w'w \in E$ and therefore all vertices $\leq w'$ in N induce a complete subgraph. This is a contradiction to the minimality of x.

A minimum coloring of a generalized strongly chordal graph can be determined by the following procedure SCOLOR.

We assume that a generalized strongly perfect elimination ordering is given.

1. We give the maximum vertex the color with the minimum number.
2. Suppose all vertices $w > v$ have a color $c(w)$. To color v, we first determine the smallest $x > v$ such that
   - $vx \in E$ and
   - $xy \in E$, for all y with $vy \in E$ and $y > x$.

   The color of v is the smallest color that does not coincide with the color $c(y)$ of some $y \geq x$ with $yv \in E$. Finally we update the colors $c(y)$ with $y < x$ and $yv \in E$. These y get the same color as x.

Proposition 4 *SCOLOR computes a minimum coloring.*

Proof: First we prove that adjacent vertices have different colors. We took care that at each application of the second step, v and its neighbors have different colors. We only have to consider the neighbors y of v with $v < y < x$. Since $<$ is a generalized strongly perfect elimination ordering, all neighbors z of such y that are greater than v are also neighbors of x. Since these vertices y

form an independent set together with x, we can color these vertices y with the same color as x.

To prove that SCOLOR computes a minimum coloring, we prove that SCOLOR computes a coloring with a number of colors that coincides with the cardinality of a maximum clique.

Let k be the number of colors that are used in SCOLOR and v be the vertex with color k that is maximal with respect to $<$. Let x be defined as in SCOLOR. Let C be the set of all neighbors $y \geq x$ of v. Then all colors $< k$ appear in C and C is a complete set. Therefore C is of cardinality $k-1$. Note that $\{v\} \cup C$ is a complete set and of cardinality k.

Box

It remains to check the complexity. We denote the number of vertices by n and the number of edges by m.

Proposition 5 *SCOLOR can be implemented in time $O(n+m)\log n$.*

Proof: We assume that $<$ is known. For each vertex v, we can find the smallest neighbor $x > v$, such that all neighbors $y \geq x$ of v induce a complete subgraph, as follows:

1. Sort all neighbors of v with respect to $<$ to a sequence $(x_1, \ldots, x_l)$.
2. Fix x to be the minimum x_i such that $x_i x_{i+1} \in E$.

For all v, the first step can be done in $O(n+m)\log n$ time. The second step consists of the substeps to find all i such that $x_i x_{i+1}$ are adjacent and to find the smallest such x_i. For one v, the first step can be done in $O(l \log l)$ time where l is the degree of v. The second step can be done in $O(l)$ time. Therefore for all v, we get a time bound of $O(n+m)\log n$.

The step to find a minimum available color can be done in $O(n+m)$ time. Note that a color is updated at most so often as it has neighbors. Therefore the color update steps have an overall time bound of $O(m)$.

□

Combining all these results, we get the following.

Theorem 4 *Given a generalized strongly chordal graph with a generalized strongly perfect elimination ordering, a minimum coloring can be computed in* $O(n+m)\log n$ *time.*

6 Conclusions

With generalized strongly chordal graphs we defined a new class of perfect graphs that covers strongly chordal graphs and chordal bipartite graphs in a quite natural way. We developed a variation of greedy coloring that computes a minimum coloring for this graph class. We should mention that the greedy maximum matching algorithm for strongly chordal graphs and the NC-algorithm for maximum matching of [4] even work for generalized strongly chordal graphs.

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