Dual Systems of Finite Displacement Screws in Screw Triangle

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DUAL SYSTEMS OF FINITE DISPLACEMENT SCREWS
IN THE SCREW TRIANGLE

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Abstract. It has recently been shown that finite displacement screws of a particular form — referred to here as a sin-screw — enter into linear patterns of combination and display the familiar structures of the screw systems, when they are used to describe certain generic kinematic displacements. A particular kinematic situation of importance demonstrates the same simple linearity if normal usage is extended to admit dual coefficients of combination.

The screw triangle rule for composing the resultant displacement screw of two given finite displacement screws is examined in this paper. On regarding the lines of the given screws as fixed, and displacement movements about them as variable, it is found that all available sin-screw resultants occupy a structure which is the sum of two axially-orthogonal dual 2-systems. Each of these systems contains all linear combinations, using complementary dual sinusoidal coefficients, of two basis screws. The basis screws and nodal lines for the systems are found to lie on the triad of mutually orthogonal mirror-symmetry axes for the lines of the given screws.

1. INTRODUCTION

It is well-known that a sequence of finite displacement operations — when applied to a rigid body — may not generally be commuted without change to the resultant displacement; and a widespread perception derives from this that the composition of finite displacements is a complex and essentially non-linear matter. A series of recent investigations [1, 2, 3, 4, 5, 6, 7] has shown, however, that screws of a particular specification — suited to characterising the finite displacement of a rigid body — fall into linear patterns of
combination when they are used to describe certain generic kinematic displacements.

There is merit in recognising such linearity for the simplicity it brings to solution methods. Amongst the kinematic situations investigated, a number are found in which the coefficients of linear combination are real. These yield the spatial patterns of screws which have been made familiar in descriptions of the real linear screw systems \cite{8, 9, 10, 11, 12}, and whose support for simple solutions is already well established in other areas \cite{13, 14, 15}.

Simplicity of solution is not lost, however, if the linear combinations under consideration are extended to admit the use of coefficients in the form of the dual numbers of Clifford \cite{16}. Any real linear combination of screws which is to be solved for some property normally provides separate expressions for the direction and moment parts of the combination screw. Stated equivalently — if the screws themselves are formulated in terms of Clifford’s quasi-scalar $\varepsilon$ (see Section 2) — the real linear combination

$$\hat{S} = (\mathbf{S} + \varepsilon \mathbf{S}_p) = a_1 (\mathbf{S}_1 + \varepsilon \mathbf{S}_{p_1}) + a_2 (\mathbf{S}_2 + \varepsilon \mathbf{S}_{p_2}) + \cdots,$$

separates into real-part and dual-part expressions

$$\mathbf{S} = a_1 \mathbf{S}_1 + a_2 \mathbf{S}_2 + \cdots \quad \text{and} \quad \mathbf{S}_p = a_1 \mathbf{S}_{p_1} + a_2 \mathbf{S}_{p_2} + \cdots.$$

Now it is a characteristic of the defining property of that quasi-scalar, namely $\varepsilon^2 = 0$, that extension to the use of dual coefficients of the form $\hat{a} = a + \varepsilon \mathbf{d}$ in the dual linear combination

$$\hat{\mathbf{S}} = (\mathbf{S} + \varepsilon \mathbf{S}_p) = \hat{a}_1 (\mathbf{S}_1 + \varepsilon \mathbf{S}_{p_1}) + \hat{a}_2 (\mathbf{S}_2 + \varepsilon \mathbf{S}_{p_2}) + \cdots,$$

has no effect other than to add real multiples of the real-part components into the dual-part expression, thus

$$\mathbf{S} = a_1 \mathbf{S}_1 + a_2 \mathbf{S}_2 + \cdots \quad \text{and} \quad \mathbf{S}_p = a_1 \mathbf{S}_{p_1} + d_1 \mathbf{S}_1 + a_2 \mathbf{S}_{p_2} + d_2 \mathbf{S}_2 + \cdots,$$

and so does not make the expressions significantly more complex of solution.

Strong support for the recognition of linearity over dual coefficients comes from a recent investigation of those finite displacement screws which are effective in carrying a particular directed line into another directed line \cite{5}. In contrast with more complex descriptions given previously \cite{3, 4}, this set of screws is found to be completely and exactly specified as a set of linear combinations of just two screws.
when dual coefficients are used. Specifically, this set may be written

\[ \mathbf{\hat{S}} = \sin \hat{\psi} \mathbf{\hat{S}}_y + \cos \hat{\psi} \mathbf{\hat{S}}_x, \]  

(1.1)

where \( \mathbf{\hat{S}}_x \) and \( \mathbf{\hat{S}}_y \) are a certain pair of orthogonal basis screws and their coefficients, as shown, are complementary sinusoidal functions of a single dual angle parameter \( \hat{\psi} \). Since this form is the dual analogue of a commonly used representation of a \textit{(real)} 2-system, namely

\[ \mathbf{\hat{S}} = \sin \psi \mathbf{\hat{S}}_y + \cos \psi \mathbf{\hat{S}}_x, \]  

(1.2)

(in which the coefficients are real), it appears appropriate to describe the form (1.1) as a \textit{dual} 2-system. It can be shown [5] that any such dual 2-system contains infinitely many real 2-systems of both general and special kinds.

All spatial arrangements of screws of the kinds explored in [1, 2, 3, 4, 5, 6, 7] derive by specialisation from the general rule for composing finite displacements. This rule, which specifies the resultant finite displacement (screw) when a sequence of one finite displacement (screw), and then another, is applied to a rigid body, comprises the geometry of the \textit{screw triangle} [17]. Written as an expression for the resultant screw, this rule is examined here for the linear systems of finite displacement screws which it contains; in the light of the foregoing discussion, systems involving both real and dual coefficients of combination are admitted to consideration.

2. SPECIFICATION OF A SCREW

We specify a \textit{screw} in Plücker co-ordinates; that is, we use a six-tuple of real values \((L, M, N; P, Q, R)\) which we express alternatively in the form of a bivector \((\mathbf{S}; \mathbf{S}_p)\). The 3-vector \( \mathbf{S} = (L, M, N) \) is the \textit{direction component} of the screw, its length \( |\mathbf{S}| \) being the \textit{magnitude} of the screw, in which \( L, M, \) and \( N \) are the direction numbers of the \textit{line of the screw}.

The 3-vector \( \mathbf{S}_p = (P, Q, R) \) is the \textit{moment component} of the screw and can be expressed as \( \mathbf{S}_p = p \mathbf{S} + \mathbf{S}_0 \) where the real scalar \( p \) is the \textit{pitch} of the screw and the 3-vector \( \mathbf{S}_0 \), which is orthogonal to \( \mathbf{S} \) (so that \( \mathbf{S} \cdot \mathbf{S}_0 = 0 \)), is referred to as the \textit{moment component of the line of the screw}. The vector
\( S_0 \) is given variously by

\[
S_0 = R \times S, \quad R = \frac{1}{S^2} S \times S_0, \quad S_0 = V \times S.
\] (2.1)

where \( R \) is the perpendicular vector from the origin to the line of the screw and \( V \) is any point on that line. In these terms a screw is a line of direction \( S \) at radius \( R \) from the origin with an associated scalar pitch value \( p \); and a line, in its own right, is a screw of zero pitch.

For many purposes it is convenient to write the same screw as a dual 3-vector,

\[
\hat{S} = S + \varepsilon S_0 = (\hat{L}, \hat{M}, \hat{N}) = (L + \varepsilon P, M + \varepsilon Q, N + \varepsilon R),
\] (2.2)

in which the overwritten hat symbol \( \hat{} \) indicates the presence of dual quantities and, following Clifford [16], \( \varepsilon \) is a quasi-scalar with the property \( \varepsilon^2 = 0 \). The screw \( \hat{S} \) then has the familiar form of a 3-vector in which each element, such as \( \hat{L} = L + \varepsilon P \), is a dual number which has a real part \( L \) and a dual part \( P \). All standard 3-vector identities apply to screws expressed in this way. It follows that

\[
\hat{S} = (1 + \varepsilon p) S + \varepsilon S_0, \quad \hat{S} \cdot \hat{S} = (1 + \varepsilon p)^2 S^2,
\]

so that the process of (full- or dual-) normalisation of \( \hat{S} \) yields the screw

\[
\hat{s} = \frac{\hat{S}}{\sqrt{\hat{S} \cdot \hat{S}}} = \frac{1 - \varepsilon p}{|S|} \hat{S} = \frac{S}{|S|} + \varepsilon \frac{S_0}{|S|}
\] (2.3)

which is the unit line, of unit magnitude and zero pitch, of the screw \( \hat{S} \). We may then write

\[
\hat{S} = |S| (1 + \varepsilon p) \hat{s}.
\] (2.4)

Whenever, throughout this paper, we refer to the normalised instance of a screw \( \hat{S} \), we shall mean the unit line \( \hat{s} \) just defined (rather than the pitched screw \( \hat{S}/|S| \) which is obtained by real normalisation). We shall follow familiar usage in writing a lower case bold symbol, here \( \hat{s} \), for the normalised instance of the screw denoted by the corresponding upper case symbol, here \( \hat{S} \); and we shall write \( s \) for the unit direction 3-vector of that screw. Frequent appeal will be made to the following 3-vector properties: for any two screws
\[ \hat{S}_1 = |S_1| \left( 1 + \varepsilon \ p_1 \right) \hat{s}_1, \quad \hat{S}_2 = |S_2| \left( 1 + \varepsilon \ p_2 \right) \hat{s}_2. \] (2.5)

of respective pitches \( p_1 \) and \( p_2 \), their cross product screw \( \hat{S}_1 \times \hat{S}_2 \) is sited in the common perpendicular line of \( \hat{S}_1 \) and \( \hat{S}_2 \), and has pitch \( p_1 + p_2 + d \cot \theta \) where \( d \) is the distance and \( \theta \) is the angle from \( \hat{S}_1 \) to \( \hat{S}_2 \) as measured along and about that common perpendicular line. We shall say that two screws \( \hat{S}_1 \) and \( \hat{S}_2 \) are orthogonal if \( \hat{S}_1 \cdot \hat{S}_2 = 0 \), which implies that each is a perpendicular intersector of the other.

3. CHASLES’S AXIS FOR A FINITE DISPLACEMENT

Consider a rigid body of general shape to undergo an arbitrary finite displacement between an initial and a final location in space. Then there is a directed line \( \hat{s} \) — Chasles’s axis for the displacement [18] — such that the displacement could have been achieved by means of a translation parallel to that line and by a rotation about it. The line \( \hat{s} \) is unique except in the special case that the displacement is comprised by a pure translation, in which situation any line parallel with the translation will serve. It is a matter of experience that the axial motions of translation and rotation about the line \( \hat{s} \) may be applied in arbitrary and piecemeal sequence to achieve the given displacement; and that to any rotation which (with some associated translation) achieves the displacement we may add a further rotation of \( 2\pi \) radians to achieve the same displacement.

We follow Hunt [19] in defining the cardinal motion for the given displacement as those axial motion components which achieve the final location from the initial location by means of translation through a distance \( 2\sigma \) in the direction of \( \hat{s} \) and by rotation through an angle \( 2\theta \), \( -\pi < 2\theta \leq \pi \), in a right-handed sense about that direction. The elements of this cardinal motion, namely the axis line \( \hat{s} \), the half-translation \( \sigma \), and the half-rotation \( \theta \), \( -\frac{\pi}{2} < \theta \leq \frac{\pi}{2} \), are used to define the specifying screws of the displacement which are described later.

4. SIN-SCREW SPECIFICATION OF A FINITE DISPLACEMENT

From the elements \( \hat{s} \), \( \sigma \) and \( \theta \), \( -\frac{\pi}{2} < \theta \leq \frac{\pi}{2} \), of the cardinal motion, we can readily construct a finite displacement screw \( \hat{S} \) which characterises the cardinal motion of the displacement. Adopting
the standard interpretation of a *dual angle*, namely

\[ \hat{\theta} \equiv \theta + \varepsilon \sigma , \quad \sin \hat{\theta} \equiv \sin \theta + \varepsilon \sigma \cos \theta , \quad \cos \hat{\theta} \equiv \cos \theta - \varepsilon \sigma \sin \theta , \quad (4.1) \]

we simply write

\[ \hat{S} \equiv \sin \hat{\theta} \hat{s} = \sin \theta \left( 1 + \varepsilon P_s \right) \hat{s} , \quad P_s = \frac{\sigma}{\tan \theta} , \quad (4.2) \]

which is a directed screw of amplitude \( \sin \theta \) and pitch \( P_s \) whose line is \( \hat{s} \). We shall refer to a such finite displacement screw \( \hat{S} \) as a *sin-screw*.

5. TAN-SCREW SPECIFICATION OF A FINITE DISPLACEMENT

With the same elements \( \hat{s} , \sigma \) and \( \theta , -\frac{\pi}{2} < \theta \leq \frac{\pi}{2} \), of the cardinal half-motion we may adopt the standard interpretation of the *dual tangent*, namely

\[ \tan \hat{\theta} \equiv \sin \hat{\theta} / \cos \hat{\theta} = \tan \theta + \varepsilon \sigma \left( 1 + \tan^2 \theta \right) , \quad (5.1) \]

and then, following Yang [20], write the finite displacement screw

\[
\hat{T} = \tan \hat{\theta} \hat{s} = \tan \theta \left( 1 + \varepsilon P_T \right) \hat{s} , \\
\]

\[ P_T = \sigma \left( \tan \theta + \frac{1}{\tan \theta} \right) = \frac{2\sigma}{\sin 2\theta} . \]

which is a directed screw of amplitude \( \tan \theta \) and pitch \( P_T \) whose line \( \hat{s} \) is Chasles's axis. We shall refer to such a \( \hat{T} \) as the *tan-screw* form of finite displacement screw.

6. SIN-SCREW AND TAN-SCREW FORMULATIONS OF THE SCREW TRIANGLE

When a sequence of finite displacements, represented in succession by their sin-screws \( \hat{S}_1 , \hat{S}_2 , \hat{S}_3 , \ldots \), is applied to a rigid body, we will write both the sequence of these actions and the *resultant* screw which describes the aggregate displacement achieved, in the programmatic form

\[ \hat{S} = \left( \hat{S}_1 ; \hat{S}_2 ; \hat{S}_3 ; \ldots \right) . \]
The operations of such a sequence can not, in general, be \textit{commuted} without causing change to the resultant; but they may be freely \textit{associated} in pairs. As shown in [5], the sin-screw \( \hat{S} \) which is the resultant of the sequence of applying such a pair of sin-screws, first \( \hat{S}_1 \equiv \sin \hat{\theta}_1 \hat{s}_1 \) and then \( \hat{S}_2 \equiv \sin \hat{\theta}_2 \hat{s}_2 \), is given by the expression

\[
\hat{S} \equiv \{ \hat{S}_1 ; \hat{S}_2 \} = \cos \hat{\theta}_2 \hat{s}_1 + \cos \hat{\theta}_1 \hat{s}_2 - \hat{S}_1 \times \hat{S}_2 .
\]  

This is the sin-screw formulation of the \textit{screw triangle rule} [17] for composition of finite displacements. Division by \( \cos \hat{\theta}_1 \cos \hat{\theta}_2 \) followed by minor manipulation of geometric quantities yields the tan-screw formulation of the screw triangle as given by Yang [20], namely

\[
\hat{T} = \frac{\hat{T}_1 + \hat{T}_2 - \hat{T}_1 \times \hat{T}_2}{1 - \hat{T}_1 \cdot \hat{T}_2} .
\]  

\[ \text{(6.2)} \]

\[ \text{7. LOCATION GEOMETRY OF THE GIVEN SCREWS} \]

For two finite displacement operations applied in sequence, as represented in succession by their sin-screws \( \hat{S}_1 \equiv \sin \hat{\theta}_1 \hat{s}_1 \) and \( \hat{S}_2 \equiv \sin \hat{\theta}_2 \hat{s}_2 \), or by the corresponding tan-screws \( \hat{T}_1 \) and \( \hat{T}_2 \), we now proceed to examine the spatial distribution of resultant screws described by equations (6.1) or (6.2). Given the large degree of variability available in this problem, we shall consider the lines of the given screws, \( \hat{s}_1 \) and \( \hat{s}_2 \), to be substantially fixed while variations are considered in the translations and rotations made about them, as comprised in the dual angle parameters \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \).

It will be convenient to recognise certain symmetries which derive from the location geometry of the given lines \( \hat{s}_1 \) and \( \hat{s}_2 \). Let \( \hat{\phi}_{12} = \phi_{12} + \epsilon \frac{d_{12}}{2} \) be half of the dual angle measured from the line \( \hat{s}_1 \) to the line \( \hat{s}_2 \). Then we may write

\[
\hat{s}_1 + \hat{s}_2 = 2 \cos \hat{\phi}_{12} \hat{s}_x , \quad \hat{s}_1 - \hat{s}_2 = 2 \sin \hat{\phi}_{12} \hat{s}_y , \quad \hat{s}_x \times \hat{s}_y = \hat{s}_z .
\]  

\[ \text{(7.1)} \]

as definitions of the mutually orthogonal lines \( \hat{s}_x , \hat{s}_y \) and \( \hat{s}_z \) of the screws

\[
\begin{align*}
\hat{S}_x &= \cos \hat{\phi}_{12} \hat{s}_x = \frac{\hat{s}_1 + \hat{s}_2}{2}, \\
\hat{S}_y &= \sin \hat{\phi}_{12} \hat{s}_y = \frac{\hat{s}_1 - \hat{s}_2}{2}. 
\end{align*}
\]  

\[ \text{(7.2)} \]
which are axes of mirror-symmetry for the given lines.

We will refer certain later results to the reference frame whose \( xyz \)-axes are the mutually orthogonal lines \( \hat{s}_x \), \( \hat{s}_y \) and \( \hat{s}_z \) respectively. These lines intersect in an origin point at the mid-point between the given screws \( \hat{S}_1 \) and \( \hat{S}_2 \) on their common perpendicular line \( \hat{s}_z \). We note the useful derived identity

\[
- \hat{s}_1 \times \hat{s}_2 = \frac{(\hat{s}_1 + \hat{s}_2) \times (\hat{s}_1 - \hat{s}_2)}{2} = 2 \sin \phi_{12} \cos \phi_{12} \hat{s}_z = 2 \hat{s}_z.
\]

(7.3)

8. A DUAL 3-SYSTEM

On writing

\[
\begin{align*}
\hat{S}_1 &= \sin \theta_1 \hat{s}_1, \quad T_1 = \tan \theta_1 \hat{s}_1 = \tan \theta_1 (1 + \varepsilon \, P_{T_1}) \hat{s}_1, \\
\hat{S}_2 &= \sin \theta_2 \hat{s}_2, \quad T_2 = \tan \theta_2 \hat{s}_2 = \tan \theta_2 (1 + \varepsilon \, P_{T_2}) \hat{s}_2,
\end{align*}
\]

(8.1)

in which

\[
P_{T_1} = \frac{2 \sigma_1}{\sin 2 \theta_1} \quad \text{and} \quad P_{T_2} = \frac{2 \sigma_2}{\sin 2 \theta_2},
\]

(8.2)

the screw triangle rule of equations (6.1) and (6.2) may be re-expressed as

\[
\frac{\hat{S}}{\sin \theta_1 \sin \theta_2} = \frac{1 - T_1 \cdot T_2}{\tan \theta_1 \tan \theta_2} \hat{T}
\]

\[
= \cot \theta_2 (1 - \varepsilon \, P_{T_2}) \hat{s}_1 + \cot \theta_1 (1 - \varepsilon \, P_{T_1}) \hat{s}_2 - \hat{s}_1 \times \hat{s}_2.
\]

(8.3)

If the lines \( \hat{s}_1 \) and \( \hat{s}_2 \) are linearly independent of one another, so that both are necessarily linearly independent of the screw \( \hat{s}_1 \times \hat{s}_2 \), we may regard the right-hand side of this equation as describing a dual 3-system in those three screws as basis, all member screws of that system being generated under independent full-range variation of the dual coefficients \( \cot \theta_1 (1 - \varepsilon \, P_{T_1}) \) and \( \cot \theta_2 (1 - \varepsilon \, P_{T_2}) \).
9. CONTAINED REAL 3-SYSTEMS

If, under variation of the rotation parameters $\theta_1$ and $\theta_2$, each of the pitch quantities $P_{T_1}$ and $P_{T_2}$ is held at some constant value by continuous adjustment of the translation parameters $\sigma_1$ and $\sigma_2$, the right-hand side of equation (8.3) describes a real 3-system of general form, with real coefficients $\cot \theta_2$ and $\cot \theta_1$ in the fixed basis screws

\[
(1 - \epsilon \ P_{T_2}) \ \hat{s}_1, \quad (1 - \epsilon \ P_{T_1}) \ \hat{s}_2 \quad \text{and} \quad \hat{s}_1 \times \hat{s}_2. \tag{9.1}
\]

For fixed $P_{T_1}$ and $P_{T_2}$, independent full-range variation $-\infty \leq \cot \theta_2 \leq \infty$ and $-\infty \leq \cot \theta_1 \leq \infty$ of the coefficients $\cot \theta_2$ and $\cot \theta_1$ generates every screw of the real 3-system. Throughout those variations, corresponding in range to the respective angular variations $-\frac{\pi}{2} \leq \theta_2 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2}$, the distance quantities $\sigma_1$ and $\sigma_2$ vary continuously within the respective ranges $-P_{T_2}/2 \leq \sigma_2 \leq P_{T_2}/2$ and $-P_{T_1}/2 \leq \sigma_1 \leq P_{T_1}/2$.

Independent full-range variation of the quantities $\sigma_1$ and $\sigma_2$, taken together with the independent full-range variation of the coefficients $\cot \theta_1$ and $\cot \theta_2$, clearly exhausts all possible variations of the parameters of the dual 3-system.

When (zero-valued) invariance of the pitch quantities $P_{T_1}$ and $P_{T_2}$ is achieved by setting the distance parameters $\sigma_1$ and $\sigma_2$ to zero — which corresponds to the practically important situation of operating revolute joints on the axes $\hat{s}_1$ and $\hat{s}_2$ — the real 3-system described by equation (8.3) is that discovered by Huang [7].

10. THE GEOMETRY OF VARIATION FOR CONSTANT $P_T$

For the constant values specified in Section 9 for the pitch quantities $P_{T_1}$ and $P_{T_2}$, either of the distance parameters $\sigma_1$ and $\sigma_2$ has the form of variation given by

\[
\sigma = \frac{P_T}{2} \sin 2 \theta, \quad P_T = \text{constant}, \tag{10.1}
\]
in which \( \theta \) stands for the corresponding angle parameter \( \theta_1 \) or \( \theta_2 \).

In order to visualise this variation, imagine a variable line which is constrained to move in such a way that it remains at all times a perpendicular intersector of the corresponding axis \( \hat{s}_1 \) or \( \hat{s}_2 \). Let any one of its locations, arbitrarily chosen, be taken as the origin of measurement for both of the quantities \( \theta \) and \( \sigma \) about the axis. Then under variation of these quantities — regarded as coordinates of that variable line — the form (10.1) specifies that the reference line generates the ruled surface of a cylinder whose nodal line lies on the corresponding axis. This cylinder, which contains the origin reference line as one of its two central generators, has a nodal-line length between its extreme generators which is equal to the pitch value \( P_T \).

11. THE CENTRE OF THE REAL 3-SYSTEM

Let us locate the central or principal screws of the real 3-system which is specified by equation (8.3) for a particular pair of fixed pitches \( P_{T_1} \) and \( P_{T_2} \). We observe that under independent variation of the real coefficients \( \cot \theta_2 \) and \( \cot \theta_1 \), all linear combinations of the basis screws

\[
(1 - \epsilon P_{T_2}) \hat{s}_1 \quad \text{and} \quad (1 - \epsilon P_{T_1}) \hat{s}_2 \tag{11.1}
\]

are members of the real 3-system and generate a real 2-system whose nodal line lies on the screw \( \hat{s}_1 \times \hat{s}_2 \). The two orthogonal screws — let us call them \( \hat{S}_x \) and \( \hat{S}_y \) — which are the principal screws of that 2-system at its centre, taken together with their perpendicular intersector screw \( \hat{s}_1 \times \hat{s}_2 \), constitute a triad of mutually orthogonal screws which are members of the real 3-system of interest. These can immediately be recognised as being the three principal screws of that 3-system.

We shall determine the distance \( d_{xy} \) and the angle \( \phi_{xy} \) — as measured along and about \( \hat{s}_1 \times \hat{s}_2 \) — from the symmetry-centre screws \( \hat{S}_x \) and \( \hat{S}_y \) of Section 7 to these principal screws \( \hat{S}_x \) and \( \hat{S}_y \). Let us denote the pitches of \( \hat{S}_x \) and \( \hat{S}_y \) by \( p_x \) and \( p_y \) respectively. Then it is a matter of standard analysis [8, 9, 12] that the typical 2-system screw \( \hat{S} \) which lies at angle \( \psi \) from \( \hat{S}_x \), has a pitch \( p \) and intersects the nodal line perpendicularly at a distance \( z \) from \( \hat{S}_x \), which are given respectively by

\[
p = \frac{p_x + p_y}{2} - \frac{p_y - p_x}{2} \cos 2 \psi \quad \text{and} \quad z = \frac{p_y - p_x}{2} \sin 2 \psi \tag{11.2}
\]
Now the basis screws $(1 - \varepsilon P_{T_2}) \hat{s}_1$ and $(1 - \varepsilon P_{T_1}) \hat{s}_2$ of equation (11.1) are particular screws of the 2-system. Making reference to Section 7 we see that the former, of pitch $-P_{T_2}$, lies at angle $-\phi_{12} - \phi_{xy}$ and at distance $-d_{12} - d_{xy}$ from the screw $\hat{s}_2$; the latter, of pitch $-P_{T_1}$, lies at angle $\phi_{12} - \phi_{xy}$ and at distance $d_{12} - d_{xy}$ from $\hat{s}_1$. So, on substituting these values into the pitch-equation (11.2), we obtain

$$
\begin{align*}
-P_{T_2} &= \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2} \cos 2(\phi_{12} + \phi_{xy}) , \\
-P_{T_1} &= \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2} \cos 2(\phi_{12} - \phi_{xy}) ,
\end{align*}
$$

from which we learn

$$P_{T_1} - P_{T_2} = (p_x - p_y) \sin 2 \phi_{12} \sin 2 \phi_{xy} . \tag{11.3}$$

Similarly, substitution into the distance-equation (11.2) yields

$$
\begin{align*}
d_{12} + d_{xy} &= \frac{p_x - p_y}{2} \sin 2(\phi_{12} + \phi_{xy}) , \\
d_{12} - d_{xy} &= \frac{p_x - p_y}{2} \sin 2(\phi_{12} - \phi_{xy}) ,
\end{align*}
$$

from which, by use of equation (11.3), we derive

$$
\begin{align*}
2d_{12} &= (p_x - p_y) \sin 2 \phi_{12} \cos 2 \phi_{xy} = (P_{T_1} - P_{T_2}) \cot 2 \phi_{xy} , \\
2d_{xy} &= (p_x - p_y) \cos 2 \phi_{12} \sin 2 \phi_{xy} = (P_{T_1} - P_{T_2}) \cot 2 \phi_{12} .
\end{align*}
$$

It follows that

$$P_{T_1} - P_{T_2} = \frac{2d_{12}}{\cot 2 \phi_{xy}} = \frac{2d_{xy}}{\cot 2 \phi_{12}} , \quad \frac{2d_{xy}}{\tan 2 \phi_{xy}} = \frac{2d_{12}}{\tan 2 \phi_{12}} , \tag{11.4}$$

the latter ratio being an invariant, independent of the value of $P_{T_1} - P_{T_2}$. These relationships allow the coordinates $\phi_{xy}$ and $d_{xy}$ of the screws $\hat{s}_1$ and $\hat{s}_2$ to be determined from the known quantities $\phi_{12}$, $d_{12}$ and $P_{T_1} - P_{T_2}$. 
We observe in particular that when the basis screws \((1 - \varepsilon^2 P_{T_2}) \hat{s}_1\) and \((1 - \varepsilon P_{T_1}) \hat{s}_2\) are of equal pitch, so that the pitch difference \(P_{T_1} - P_{T_2}\) vanishes, the principal screws of the real 3-system lie along the mutually orthogonal lines \(\hat{s}_x\), \(\hat{s}_y\) and \(\hat{s}_z\) which are axes of mirror-symmetry for those basis screws. Specifically, the real 3-system identified by Huang [7] for the situation in which those pitches have zero values is so sited.

12. A SUM OF DUAL 2-SYSTEMS

Examination of the left-hand side(s) of equation (8.3) reveals that the member screws of the dual 3-system (on the right) are neither sin-screws nor tan-screws which have relevance to the problem context. In distinction from the screw forms \(\hat{S}\) and \(\hat{T}\) which have immediate relevance (as the resultant screw), the items on the left are significantly modified in both magnitude and pitch by dual multiplying factors.

For comparison with results obtained in other kinematic situations [2, 4, 5, 6] (which, in effect, were derived in terms of sin-screws), and for the wider purpose of achieving more straightforward applicability, we now derive an alternative to equation (8.3) which expresses the unmodified sin-screw \(\hat{S}\) in terms of basis screws of the same (sin-screw) type.

Writing \(\hat{S}_1 = \sin \hat{\theta}_1 \hat{s}_1\) and \(\hat{S}_2 = \sin \hat{\theta}_2 \hat{s}_2\), we may re-express the screw triangle rule of equation (6.1) as

\[
\hat{S} = \sin \hat{\theta}_1 \cos \hat{\theta}_2 \hat{s}_1 + \sin \hat{\theta}_2 \cos \hat{\theta}_1 \hat{s}_2 - \sin \hat{\theta}_1 \sin \hat{\theta}_2 \hat{s}_1 \times \hat{s}_2
\]

\[
= \sin (\hat{\theta}_1 + \hat{\theta}_2) \frac{\hat{s}_1 + \hat{s}_2}{2} + \sin (\hat{\theta}_1 - \hat{\theta}_2) \frac{\hat{s}_1 - \hat{s}_2}{2} - \sin \hat{\theta}_1 \sin \hat{\theta}_2 \hat{s}_1 \times \hat{s}_2
\]

which, through the definitions at equations (7.2) and (7.3),

\[
= \sin (\hat{\theta}_1 + \hat{\theta}_2) \hat{S}_X + \sin (\hat{\theta}_1 - \hat{\theta}_2) \hat{S}_Y + 2 \sin \hat{\theta}_1 \sin \hat{\theta}_2 \hat{S}_Z \tag{12.1}
\]

when expressed in terms of the mutually orthogonal screws \(\hat{S}_X\), \(\hat{S}_Y\) and \(\hat{S}_Z\). An equation which is similar to equation (8.3) in describing a dual 3-system, but which is expressed in terms of these mirror-symmetry axes, is obtained by dividing throughout by \(\sin \hat{\theta}_1 \sin \hat{\theta}_2\); thus
\[ \hat{S} \sin \theta_1 \sin \theta_2 = (\cot \hat{\theta}_1 + \cot \hat{\theta}_2) \hat{S}_x + (\cot \hat{\theta}_1 - \cot \hat{\theta}_2) \hat{S}_y + 2 \hat{S}_z. \] (12.2)

However, a more informative analysis of structure is obtained by defining new independent parameters

\[ \hat{\psi}_s = \hat{\theta}_1 + \hat{\theta}_2, \quad \hat{\psi}_c = \hat{\theta}_1 - \hat{\theta}_2. \] (12.3)

Re-writing equation (12.1), we then find that

\[ \hat{S} = \sin \hat{\psi}_s \hat{S}_x + \sin \hat{\psi}_c \hat{S}_y - (\cos \hat{\psi}_s - \cos \hat{\psi}_c) \hat{S}_z \] (12.4)

which, on re-grouping terms,

\[ = (\sin \hat{\psi}_s \hat{S}_x + \cos \hat{\psi}_s \hat{S}_z) + (\sin \hat{\psi}_c \hat{S}_x - \cos \hat{\psi}_c \hat{S}_z), \] (12.5)

which, in form, is the sum of two dual 2-systems, each combined by complementary dual sinusoids, whose structure has been studied in [5]. These dual 2-systems in the independent parameters \( \hat{\psi}_s \) and \( \hat{\psi}_c \), have orthogonal nodal lines on the axes \( \hat{S}_X \) and \( \hat{S}_Y \) respectively; and each finds its orthogonal basis in the remainder of the set \( \hat{S}_X, \hat{S}_Y \) and \( \hat{S}_Z \).

It is of practical interest that the observed alternation of the internal signs of the dual 2-system components is reversed if the sequence of application of the screws \( \hat{S}_1 \) and \( \hat{S}_2 \) is reversed. That is

\[
\begin{align*}
\{ \hat{S}_1 ; \hat{S}_2 \} &= (\sin \hat{\psi}_s \hat{S}_x + \cos \hat{\psi}_s \hat{S}_z) + (\sin \hat{\psi}_c \hat{S}_x - \cos \hat{\psi}_c \hat{S}_z), \\
\{ \hat{S}_2 ; \hat{S}_1 \} &= (\sin \hat{\psi}_s \hat{S}_x - \cos \hat{\psi}_s \hat{S}_z) + (\sin \hat{\psi}_c \hat{S}_x + \cos \hat{\psi}_c \hat{S}_z). 
\end{align*}
\] (12.6)

13. DISCUSSION

Being a fixed sum which lacks variable coupling between the two dual 2-systems identified, the expression (12.5) cannot reasonably be referred to as a dual 3-system. This assertion is borne out by the finding, now to be demonstrated, that the expression (12.5) does not contain general real 3-systems analogous to those found in expression (8.3) and discussed in Sections 9, 10, and 11.
We may establish this finding in a simple way by considering the special case treated in [7] for which no translation takes place in the given displacements about the lines \( \hat{s}_1 \) and \( \hat{s}_2 \). As discussed in Section 9, the dual quantities \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) then adopt purely real values \( \theta_1 \) and \( \theta_2 \). Through the definitions (12.3) we see that the parameters \( \hat{\psi}_x \) and \( \hat{\psi}_y \) do likewise and, adopting purely real values which we may write as \( \psi_x \) and \( \psi_y \), provide in place of equation (12.5) the expression

\[
\hat{S} = (\sin \psi_x \hat{S}_x + \cos \psi_x \hat{S}_x) + (\sin \psi_y \hat{S}_y - \cos \psi_y \hat{S}_y).
\]  

(13.1)

This describes the fixed sum of two real 2-systems whose generator screws are fixed in amplitude. While the sharing of orthogonal basis screws between the 2-systems ensures that this sum constitutes a subset of some real 3-system (which we do not investigate), the sum does not itself describe a general real 3-system.

In examining this matter more closely, we might suppose for the moment that equation (13.1) does describe a real 3-system. Then, in view of the observed symmetry of parameterisation between the basis screws \( \hat{S}_x \) and \( \hat{S}_y \), we would expect to find within it a contained 2-system, infinite in cardinality, of screws which lie parallel with the plane of \( \hat{S}_x \) and \( \hat{S}_y \). These screws would occupy the same lines as the screws which — chosen for the present situation of pure rotations about the lines \( \hat{s}_1 \) and \( \hat{s}_2 \) — were discussed in the treatment of Section 11, and were there denoted \( S \) (and \( \hat{S}_1 \) and \( \hat{S}_2 \)).

In fact, when equation (13.1) is examined for those screws which lie parallel with the plane of \( \hat{S}_x \) and \( \hat{S}_y \) — that is, for which the direction component in the direction of \( \hat{S}_z \) is zero-valued — only two solutions are found, namely

\[
\cos \psi_x - \cos \psi_y = 0, \text{ i.e. } \psi_x = \pm \psi_y.
\]

Reference to the definitions (12.3) shows that these are equivalent to the alternative solutions \( \theta_1 = 0 \) or \( \theta_2 = 0 \). For the present situation in which only pure rotations are considered, these amount to the total absence of one or other of the given contributing displacements.

Thus, in review, we see that equation (13.1) specifies a system of resultants in which the only screws lying parallel with the plane of \( \hat{S}_x \) and \( \hat{S}_y \) are the screws \( \hat{S}_1 \) and \( \hat{S}_2 \) themselves; each of these occurs in the situation that it acts alone, as its own resultant. Infinite numbers of other screws which, in a general 3-system, would be expected to lie parallel with the plane of \( \hat{S}_x \) and \( \hat{S}_y \), are simply absent and do not exist.
as effective resultants. While this characteristic has been established in terms of screws of a special directional specification, similar but more protracted arguments may be made to establish the absence of 3-system screws lying in general directions.

14. CONCLUSION

An expression has been derived, in the form of the sum of two dual 2-systems at equation (12.5), which encompasses all of finite displacement screws which can arise by composition of two such screws when they act sequentially on the generally disposed lines \( \hat{s}_1 \) and \( \hat{s}_2 \). The relationship between these lines and the screws \( \hat{S}_x \), \( \hat{S}_y \) and \( \hat{S}_z \) which appear as shared bases and nodal lines of the dual 2-systems, has been specified in Section 7, and the effects of reversing the sequence of application of the contributing screws are summarised in equations (12.6).

Little more has been performed than to achieve a separation of variables as between the two-(dual) parameter system which first introduced the screw triangle at equation (6.1), and the two-(dual) parameter system of dual 2-systems which has been derived at equation (12.5). But this separation, and the associated transfer to orthogonal bases, has made the structure of the system more obvious. It makes clear, in particular, that the dual 3-system of modified screws which is apparent in equation (8.3) does not re-appear as a complete dual 3-system when expressed in sin-screws.

Both of the equations (6.1) and (12.5) provide compact expressions of the screw triangle rule [17] for the resultant sin-screw of successively applied displacements. As previous work [5, 6] has effectively shown, the latter equation (12.5) provides a useful starting point for studies of the sub-sets of resultants which are available when various kinematic constraints are applied.

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