Weighted Popular Matchings
(2006; Mestre)

Julián Mestre, University of Maryland, http://www.cs.umd.edu/~jmestre/

1 Problem definition

Consider the problem of matching a set of individuals X to a set of items Y where each individual has a weight and a personal preferences over the items. The objective is to construct a matching M that is stable in the sense that there is no matching M' such that the weighted majority vote will choose M' over M.

More formally, we are given a bipartite graph (X, Y, E), a weight $w(x) \in R^+$ for each individual $x \in X$, and a rank function $r : E \to \{1, \ldots, |Y|\}$ encoding the individual preferences. For every applicant x and items $y_1, y_2 \in Y$ we say applicant x prefers y_1 over y_2 if $r(x, y_1) < r(x, y_2)$, and x is indifferent between y_1 and y_2 if $r(x, y_1) = r(x, y_2)$. The preference lists are said to be strictly ordered if applicants are never indifferent between two items, otherwise the preference lists are said to contain ties.

Let M and M' be two matchings. An applicant x prefers M over M' if x prefers the item he/she gets in M over the item he/she gets in M'. A matching M is *more popular than* M' if the applicants that prefer M over M' outweigh those that prefer M' over M. Finally, a matching M is *weighted popular* if there is no matching M' more popular than M.

In the *weighted popular matching problem* we must determine if a given instance admits a popular matching, and if so, to produce one. In the *maximum weighted popular matching problem* we must find a popular matching of maximum cardinality, provided one exists.

Abraham *et al.* [2] gave the first polynomial time algorithms for the special case of these problems where the weights are uniform. Later, Mestre [8] introduced the weighted variant and developed polynomial time algorithms for it.

2 Key results

Theorem 1. *The weighted popular matching and maximum weighted popular matching problems on instances with strictly-ordered preferences can be solved in $O(|X| + |E|)$ time.*

Theorem 2. *The weighted popular matching and maximum weighted popular matching problems on instances with arbitrary preferences can be solved $O(\min\{k\sqrt{|X|}, |X|\}|E|)$ time.*

Both results rely on an alternative easy-to-compute characterization of weighted popular matchings called *well-formed* matchings. It can be shown that every popular matching is well-formed. While in unweighted instances every well-formed matching is popular [2], in weighted instances there may be well-formed matchings that are not popular. These non-popular well-formed matchings can be weeded out by pruning certain bad edges that cannot be part of any popular matching. In other words, the instance can be pruned so that a matching is popular if and only if it is well-formed and is contained in the pruned instance [8].

3 Applications

Many real-life problems can be modeled using one-sided preferences. For example, the assignment of graduates to training positions [5], families to government-subsidized housing [10], students to projects [9] and Internet rental markets [1] such as Netflix where subscribers are assigned DVDs.

Furthermore, the weighted framework allows us to model the naturally-occurring situation in which some subset of users has priority over the rest. For example, an Internet rental site may offer a "premium" subscription plan and promise priority over "regular" subscribers.

4 Cross References

Ranked Matching, Stable Marriage, Maximum Cardinality Bipartite Matching.

5 Recommended reading

[1] D. J. Abraham, N. Chen, V. Kumar, and V. Mirrokni. Assignment problems in rental markets. In *Proceedings of the 2nd Workshop on Internet and Network Economics*, 2006.

[2] D. J. Abraham, R. W. Irving, T. Kavitha, and K. Mehlhorn. Popular matchings. In *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 424–432, 2005.

[3] D. J. Abraham and T. Kavitha. Dynamic matching markets and voting paths. In *Proceedings of the 10th Scandinavian Workshop on Algorithm Theory (SWAT)*, pages 65–76, 2006.

[4] P. Gardenfors. Match making: assignments based on bilateral preferences. *Behavioural Sciences*, 20:166–173, 1975.

[5] A. Hylland and R. Zeeckhauser. The efficent allocation of individuals to positions. *Journal of Political Economy*, 87(2):293–314, 1979.

[6] M. Mahdian. Random popular matchings. In *Proceedings of the 7th ACM Conference on Electronic Commerce (EC)*, pages 238–242, 2006.

[7] D. Manlove and C. Sng. Popular matchings in the capacitated house allocation problem. In *Proceedings of the 14th Annual European Symposium on Algorithms (ESA)*, pages 492–503, 2006.

[8] J. Mestre. Weighted popular matchings. In *Proceedings of the 16th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 715–726, 2006.

[9] L. G. Proll. A simple method of assigning projects to students. *Operational Research Quarterly*, 23(23):195–201, 1972.

[10] Y. Yuan. Residence exchange wanted: a stable residence exchange problem. *European Journal of Operational Research*, 90:536–546, 1996.