Primal-Dual Algorithms for Combinatorial Optimization

Julián Mestre

Combinatorial Optimization

$$\left.\begin{array}{l}\text{Travelling Salesman}\\ \text{Facility Location}\\ \text{Edge Coloring}\end{array}\right\}\text{NP-hard}$$

(feasible solutions, cost function)

This dissertation

Approximation Algorithms for NP-hard optimization problems

Unifying theme: linear duality & primal-dual schema

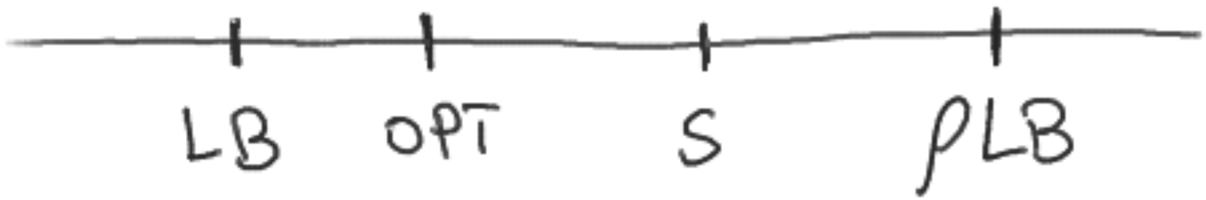
Talk Outline

* Background
* Summary of Results
* Adaptive Local Ratio

Approximation Algorithms

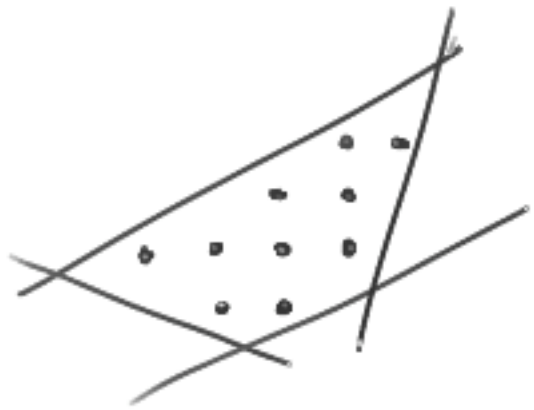
Let $(\mathcal{F}, c)$ be a minimization problem

Find $S \in \mathcal{F} : \underbrace{c(S) \leq \rho\, c(A)}_{c(S) \leq \rho\, LB \leq \rho\, c(A)} \quad \forall A \in \mathcal{F} \quad [\rho \geq 1]$



Linear Programming

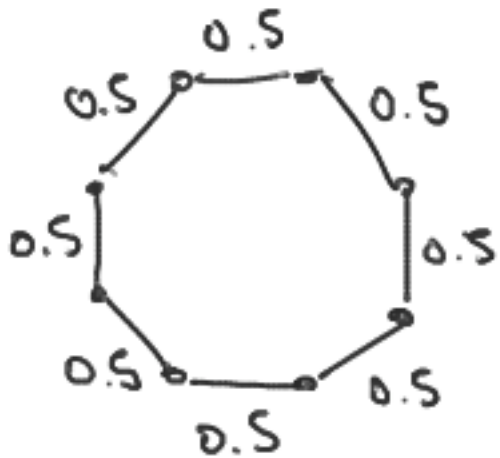
$$\left\{\begin{array}{l}\min c\cdot x\\ Ax\geq b\\ \quad x\in\mathbb{Z}^n_+\end{array}\right\} \geq \left\{\begin{array}{l}\min c\cdot x\\ Ax\geq b\\ \quad x\geq 0\end{array}\right\}$$



$$\min \sum w_e x_e$$

$$\sum_{e\in(S,\bar{S})} x_e \geq 1 \quad \forall S\subseteq V$$

$$x_e\geq 0$$



Linear Programming Duality

$$\min \quad 5x_1 + 6x_2$$

$$x_1 + 3x_2 \geqslant 3 \qquad (\times 2) \qquad (\times 1)$$

$$2x_1 + x_2 \geqslant 2 \qquad\qquad\qquad (\times 2)$$

$$x_1, x_2 \geqslant 0$$

$$5x_1 + 6x_2 \geqslant 2x_1 + 6x_2 \geqslant 6$$

$$5x_1 + 6x_2 \geqslant 5x_1 + 5x_2 \geqslant 7$$

Linear Programming Duality

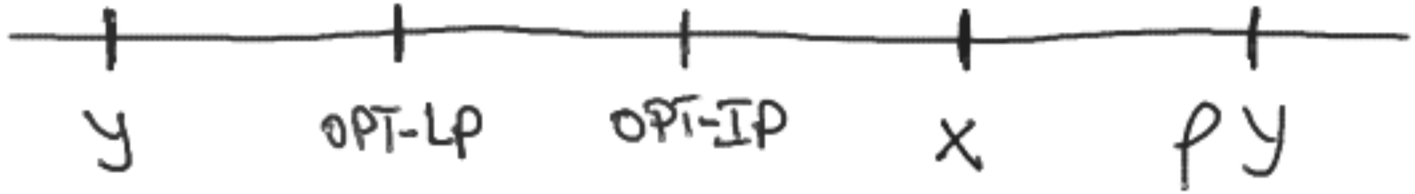
$$\left\{\begin{array}{c} \min\ c\cdot x \\ Ax \geq b \\ x \geq 0 \end{array}\right\} \overset{?}{=} \left\{\begin{array}{c} \max\ b\cdot y \\ A^T y \leq c \\ y \geq 0 \end{array}\right\}$$

strong
~~weak~~ duality $\cdot y$

Primal-Dual Schema [GW 91]

$$\left\{\begin{array}{c}\min c\cdot x\\ Ax\geq b\\ x\in\mathbb{Z}^n_+\end{array}\right\} \geq \left\{\begin{array}{c}\min c\cdot x\\ Ax\geq b\\ x\geq 0\end{array}\right\} = \left\{\begin{array}{c}\max b\cdot y\\ A^T y\leq c\\ y\geq 0\end{array}\right\}$$

Find feasible $x\in\mathbb{Z}^n_+$, $y\in\mathbb{R}^m_+$:



Talk Outline

✓ Background

☆ Summary of Results

☆ Adaptive Local Ratio

Set Cover

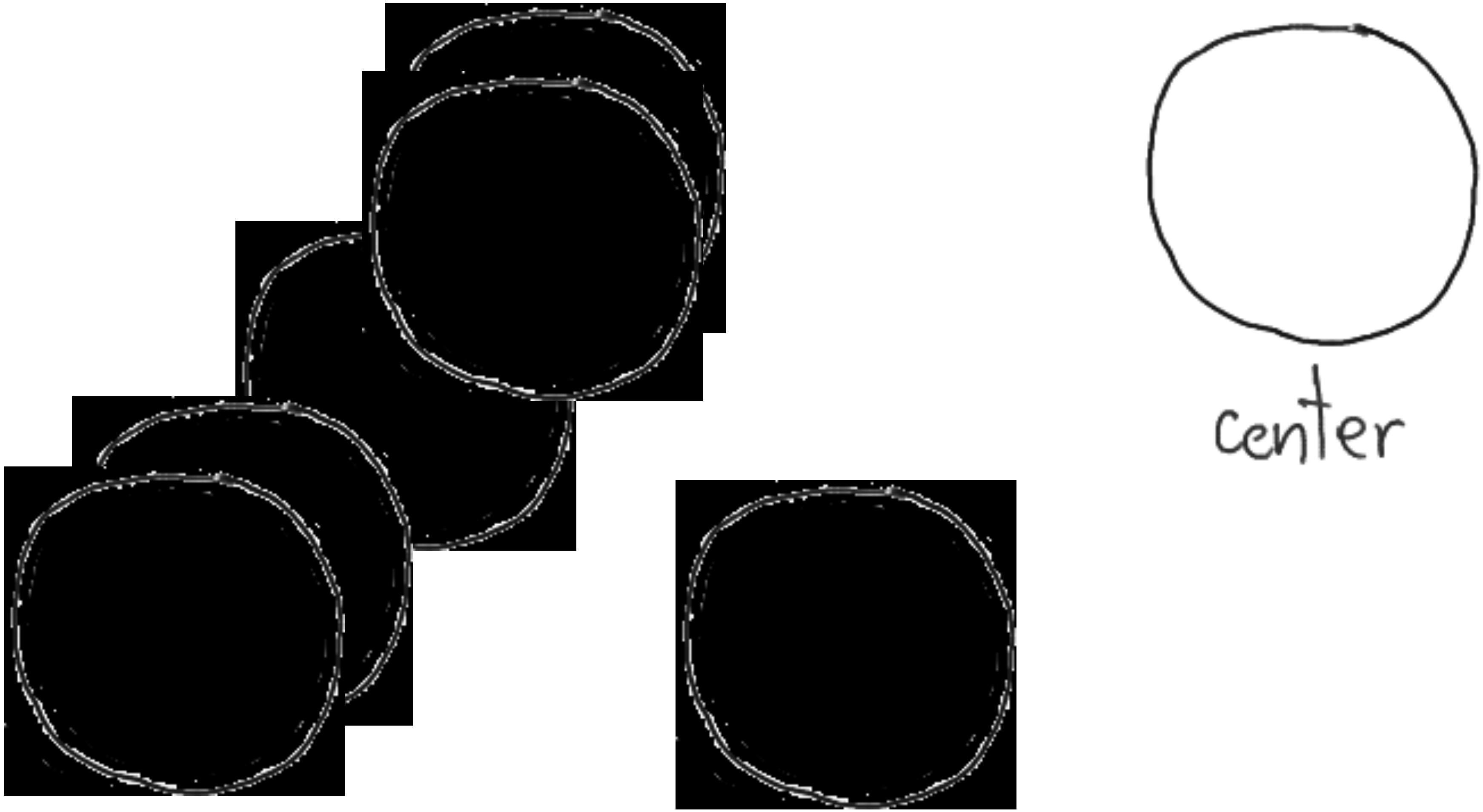
$\mathcal{U}$ = ground set $\quad c : \mathcal{S} \to \mathbb{R}^+$

$\mathcal{S} \subseteq \mathcal{P}(\mathcal{U})$

Find $\mathcal{C} \subseteq \mathcal{S}$ minimizing $c(\mathcal{C})$

such that $\bigcup_{S \in \mathcal{C}} S = \mathcal{U}$

Outliers



Partial Cover

$\mathcal{U}$ = ground set $\quad c : \mathcal{S} \to \mathbb{R}^+ \quad P$

$\mathcal{S} \subseteq \mathcal{P}(\mathcal{U}) \quad p : \mathcal{U} \to \mathbb{R}^+$

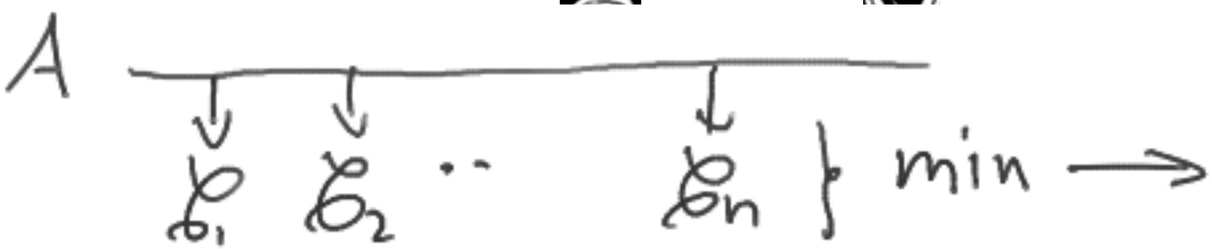
Find $\mathcal{C} \subseteq \mathcal{S}$ minimizing $c(\mathcal{C})$

such that $p(\mathcal{C}) \geqslant P$

Making Educated Guesses

<u>Thm</u>: $O(n\log n + m)$ time 2-approximation partial (capacitated) vertex cover

PD 2-approx for partial vertex cover [GKS 01]



Lagrangian Relaxation

Find $\mathcal{C} \subseteq S$ minimizing $c(\mathcal{C}) + \lambda\, \overline{p(\mathcal{C})}$

$$c(\mathcal{C}) + \rho \lambda\, \overline{p(\mathcal{C})} \leq \rho \left[c(\mathcal{C}') + \lambda p(\mathcal{C}') \right] \quad \forall \mathcal{C}'$$

ρ-LMP property [JV 01]

Lagrangian Relaxation

$$\begin{tikzcd}
\mathcal{A}(\lambda-\varepsilon) \arrow[r] & \mathcal{C}_1 \arrow[dr] & \\
& & \mathcal{C} \ \left(\begin{array}{c}\text{cheap}\\ \text{feasible}\end{array}\right) \\
\mathcal{A}(\lambda+\varepsilon) \arrow[r] & \mathcal{C}_2 \arrow[ur] &
\end{tikzcd}$$

ρ-LMP $\Rightarrow$ $\frac{4}{3}\rho$-approx [KPS 06]

Lagrangian Relaxation

<u>Thm</u>: Lower bound of $\frac{4}{3}\rho$ in general

<u>Thm</u>: Tight characterization of integrality gap for TB cover

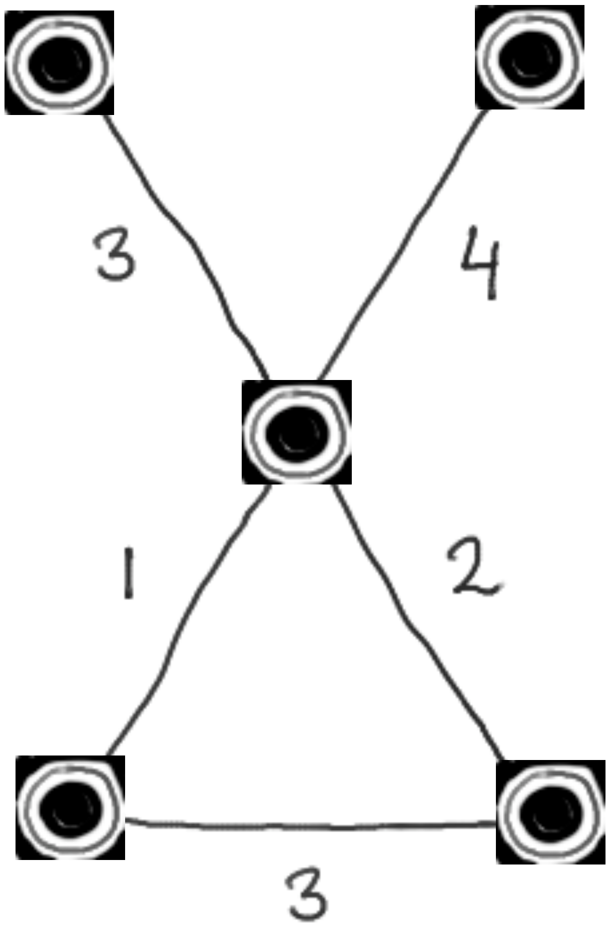
<u>Coro</u>: $\exists$ Al $\mathcal{C}_1$, $\mathcal{C}_2$ $\to$ dual solution $\to$ $\mathcal{C}$ tiny 2-blocks

Data Migration

Transfer graph $G=(V,E)$ $\begin{cases} V: \text{disks} \\ E: \text{transfers} \end{cases}$

Want to schedule the transfers $\equiv$ Partition E into matchings $M_1, \dots, M_k$

$$\min \sum_v C_v \;/\; \min \sum_e C_e$$



$$\sum C_v = 17$$

$$\sum C_e = 13$$

Beyond Complementary Slackness

Ty More sophisticated method for $\min \sum C_{\sigma:i} \}$

<u>Thm</u>: 3-approx for unit-length

<u>Thm</u>: 5.83-approx for arb-length

Local Ratio

Alternative approch, doesn't require LP

- Decompose $w = w_1 + w_2 + \ldots + w_n$
- Construct a solution S
- Argue $w_k(S) \leq \rho\, w_k(A) \quad \forall A$

$\Rightarrow S$ is ρ-approximate

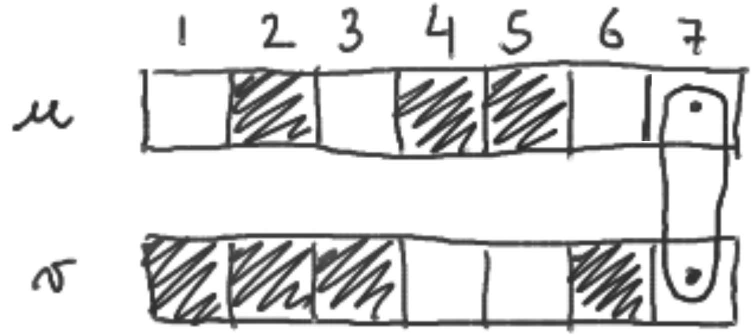
Primal-dual & Local-Ratio

- Equivalent paradigms
- Hundreds of papers
- Usually models are "simple"

<u>Thm</u>: $1+\phi$ approximation for min $\sum_v c_v$

$$\min \sum C_e$$

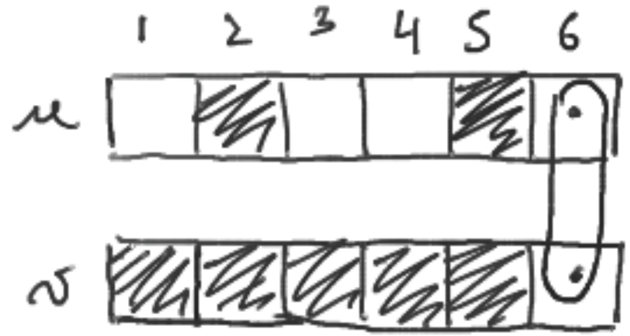
A schedule $M_1, \ldots, M_k$ is minimal if:



Minimal schedules are 2-approx [BBHST 98]

Dual Fitting

A schedule $M_1, \ldots, M_\Delta$ is strongly minimal if



<u>Thm</u>: Strongly minimal schedules are $\sqrt{2}$-approx

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- ✓ Summary of Results
- ☆ Adaptive Local Ratio

Local Ratio

Alternative approach, doesn't require LP

- Decompose $w = w_1 + w_2 + \ldots + w_n$
- Construct a solution S
- Argue $w_k(S) \leq \rho\, w_k(A) \quad \forall A$

$\Rightarrow$ S is ρ-approximate

Algorithmic Framework

Find $\hat{w}$ so that $\hat{w}(S) \leq \rho\, \hat{w}(A)$ $\forall A$

Bounding the Local Ratio

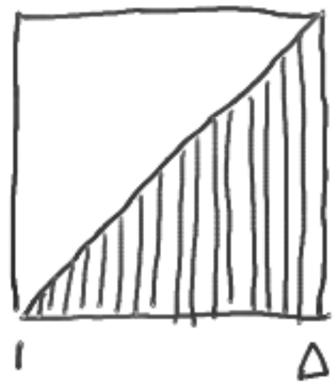
Lemma: $c_i \leq \Delta + d_i - 1$

$$\hat{w}(S) \leq \Delta^2 + \sum d_i$$

$\hat{w}_i = 1$

$\nearrow$ $\hat{w}(A) \geq \sum d_i$

$\searrow$ $\hat{w}(A) \geq \frac{1}{2}\Delta(\Delta+1)$



Using the best model

Given $d = (d_1, d_2, \ldots, d_\Delta)$

find $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_\Delta)$

minimizing $\rho(d) = \dfrac{UB(d, \hat{\omega})}{LB(d, \hat{\omega})}$

LP formulation

$$\rho(d) = \min \sum_i \hat{\omega}_i (d_i + \Delta - 1)$$

$$\text{s.t.} \quad \sum (y_i - z_i) \geq 1$$

$$y_i - z_j \leq \hat{\omega}_i \max(d_i, j)$$

$$\hat{\omega}_i, y_i, z_i \geq 0$$

Bounding the Local Ratio

Def: $\rho = \sup_d \rho(d)$

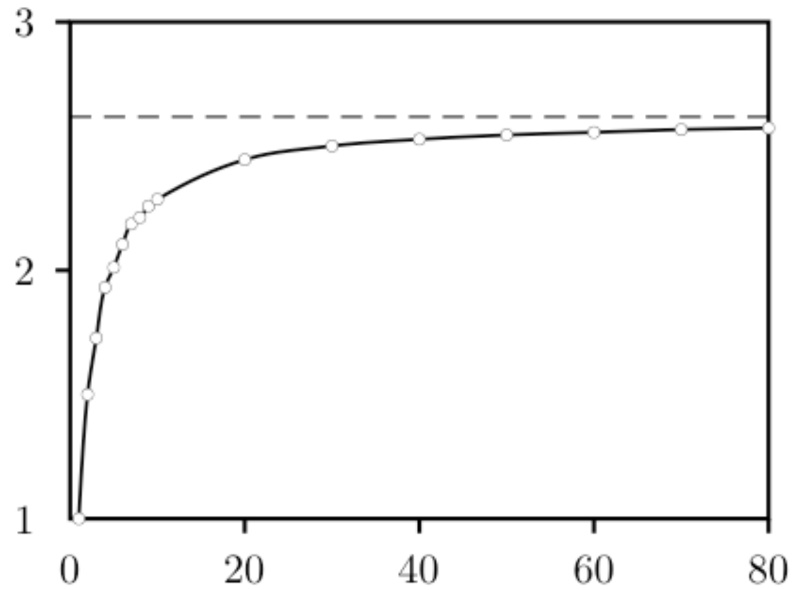
Thm: S is ρ-approximate and this is tight

Now we only need to bound ρ

Experimental Evaluation

Exhaustive search for small values of Δ

$$\rho_\Delta = \max_{d : |d| = \Delta} \rho(d)$$



$$\rho = \sup_{d} \rho(d)$$

$$\rho_\Delta = \max \alpha$$

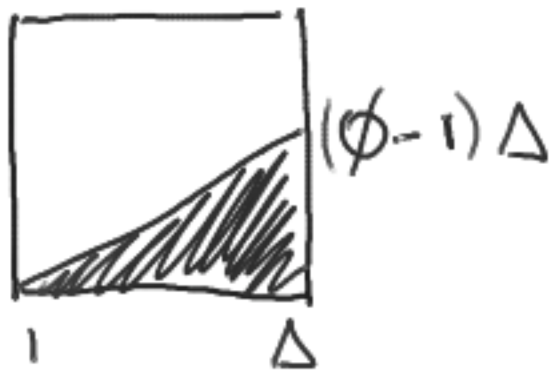
$$\sum_i x_{ij} \geq \alpha \qquad \forall j$$

$$\sum_j x_{ij} \leq \alpha \qquad \forall i$$

$$\sum_j x_{ij} \max(d_i, j) \leq d_i + \Delta - 1 \qquad \forall i$$

$$d_i, x_{ij}, \alpha \geq 0$$

Thm: $\rho = 1 + \phi \approx 2.61$



Intuition:

- weights exploit irregularities of d
- if d is "flat", only two models are useful

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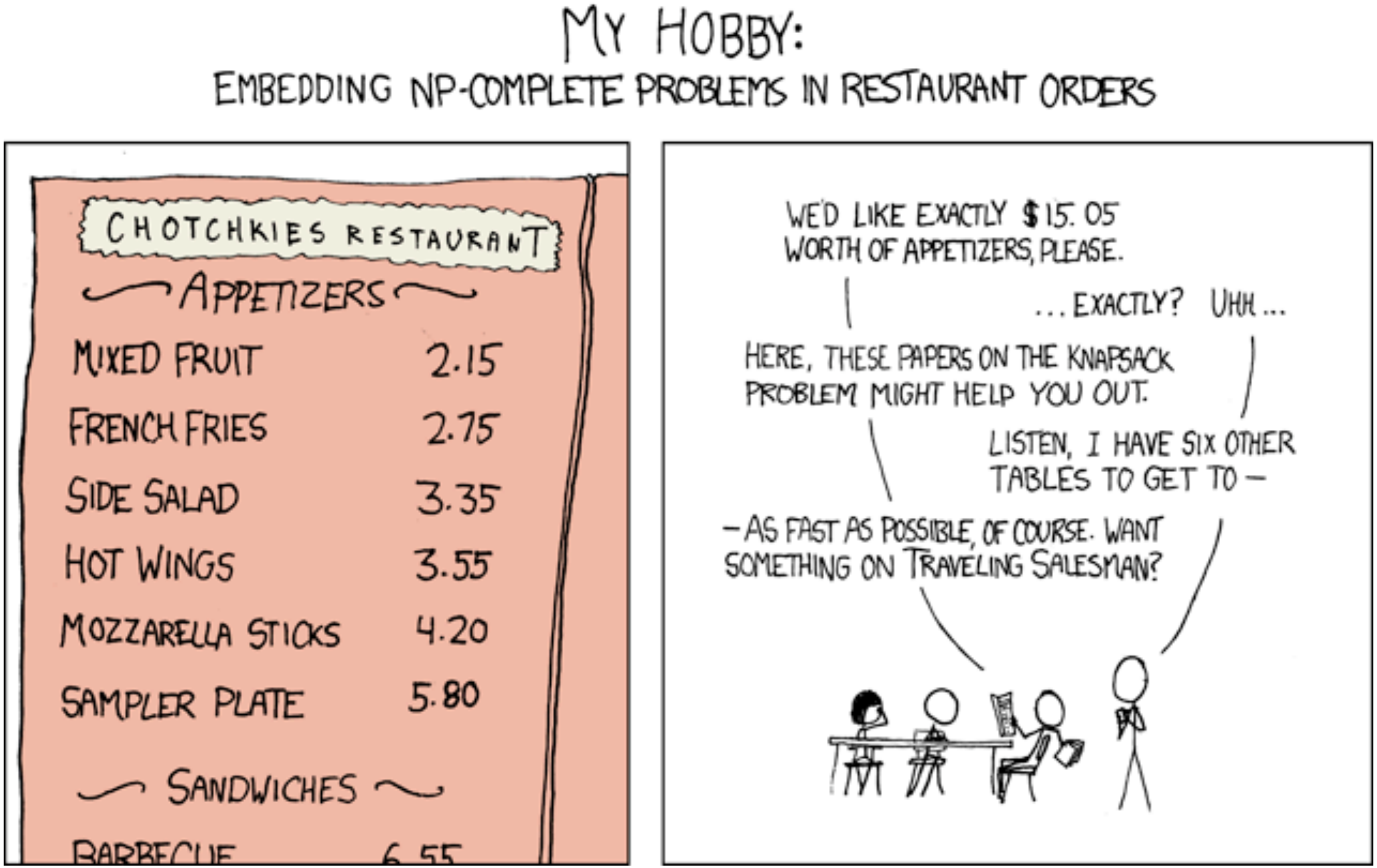
Conclusions

Primal-dual schema:

Powerful algorithmic technique

There is room for variations

Thanks for
your attention !



MY HOBBY:
EMBEDDING NP-COMPLETE PROBLEMS IN RESTAURANT ORDERS
CHOTCHKIES RESTAURANT
APPETIZERS
MIXED FRUIT 2.15
FRENCH FRIES 2.75
SIDE SALAD 3.35
HOT WINGS 3.55
MOZZARELLA STICKS 4.20
SAMPLER PLATE 5.80
SANDWICHES
WE'D LIKE EXACTLY $15.05 WORTH OF APPETIZERS, PLEASE.
...EXACTLY? UHH...
HERE, THESE PAPERS ON THE KNAPSACK PROBLEM MIGHT HELP YOU OUT.
LISTEN, I HAVE SIX OTHER TABLES TO GET TO –
– AS FAST AS POSSIBLE, OF COURSE. WANT SOMETHING ON TRAVELING SALESMAN?

xkcd