Lagrangian Relaxation & Partial Cover

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Set Cover

$\mathcal{U}$ = ground set $\qquad c : \mathcal{S} \to \mathbb{R}^+$

$\mathcal{S} \subseteq \mathcal{P}(\mathcal{U})$

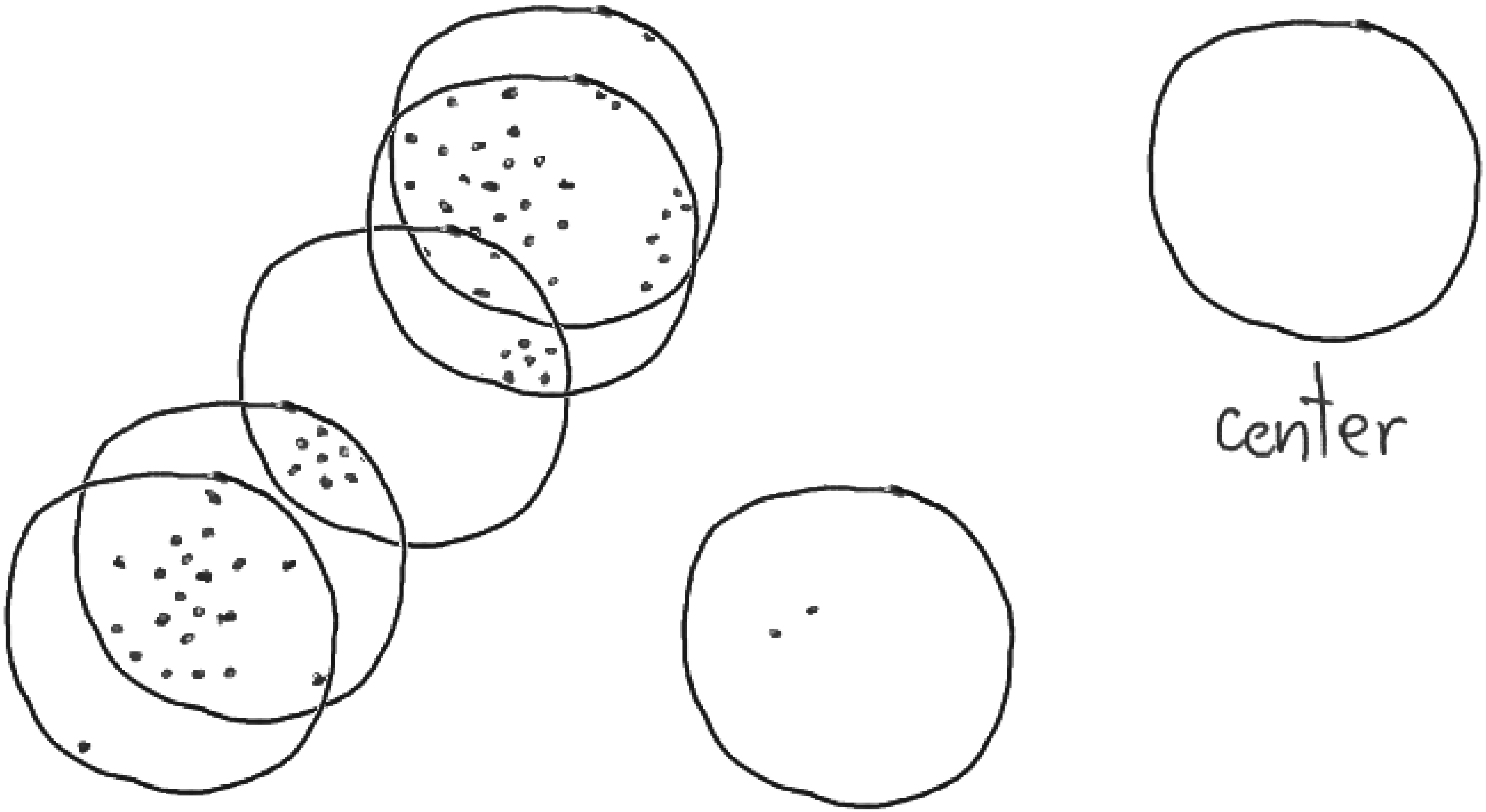
Find $\mathcal{C} \subseteq \mathcal{S}$ minimizing $c(\mathcal{C})$

such that $\bigcup_{S \in \mathcal{C}} S = \mathcal{U}$

Set Cover

- Greedy is a H_n-approximation
- Best possible unless...
- Easier/special cases
  - Balanced Cover
  - Vertex Cover

Outliers



Partial Cover

$\mathcal{U}$ = ground set $\quad c : \mathcal{S} \to \mathbb{R}^+$ $\quad P$

$\mathcal{S} \subseteq \mathcal{P}(\mathcal{U}) \quad p : \mathcal{U} \to \mathbb{R}^+$

Find $\mathcal{C} \subseteq \mathcal{S}$ minimizing $c(\mathcal{C})$

such that $p(\mathcal{C}) \geq P$

Lagrangian Relaxation

Find $\mathcal{B} \subseteq S$ minimizing $c(\mathcal{B})$: $\overline{p(\mathcal{B})} \geq P$

$$c(\mathcal{B}) + \rho \lambda \overline{p(\mathcal{B})} \leq \rho \left[c(\mathcal{B}') + \lambda \overline{p(\mathcal{B}')} \right] \quad \forall \mathcal{B}'$$

ρ-LMP property [JV 01]

Lagrangian Relaxation

$$\begin{array}{l} \mathcal{A}(\lambda^* - \varepsilon) \longrightarrow \mathcal{C}_1 \searrow \\ \qquad\qquad\qquad\qquad\qquad\quad \mathcal{C} \text{ (cheap feasible)} \\ \mathcal{A}(\lambda^* + \varepsilon) \longrightarrow \mathcal{C}_2 \nearrow \end{array}$$

ρ-LMP $\Rightarrow \frac{4}{3}\rho$-approx [KPS 06]

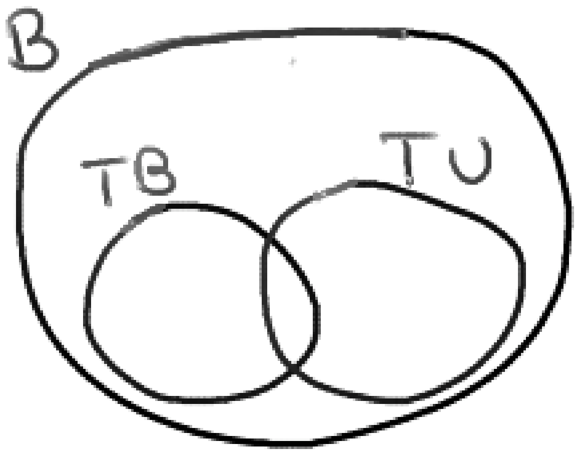
Our Contributions

<u>Thm</u>: Lower bound of $\frac{4}{3}\rho$ in general

<u>Thm</u>: Tight characterization of integrality gap for TB cover

<u>Coro</u>: $\exists\, \mathcal{C}_1$, Al $\mathcal{C}_2$ $\rightarrow$ dual solution $\rightarrow$ $\mathcal{C}$ tiny blocks

Totally Balanced Cover



Kolen's Algorithm

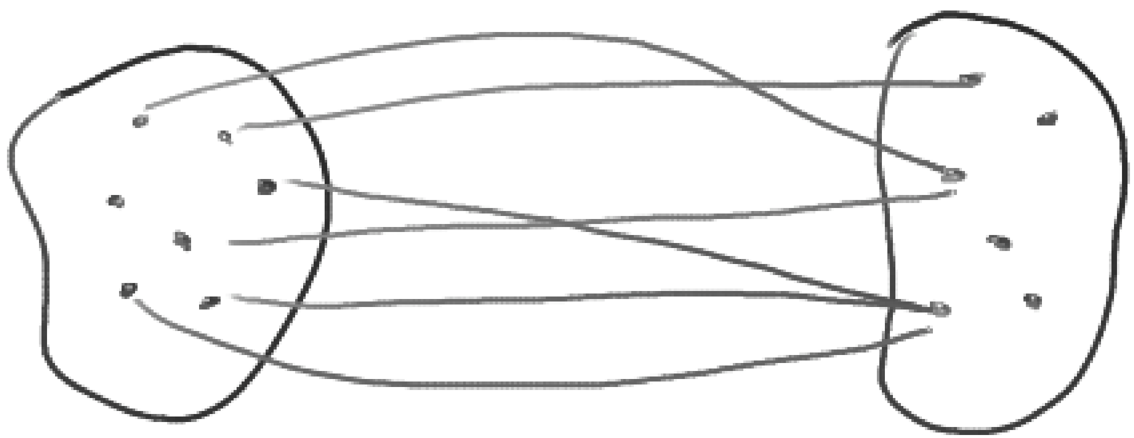
$$\lambda p_u \geqslant y_u \geqslant 0 \quad \text{for } u \in \mathcal{U}$$

$$\sum_{u \in S} y_u \leqslant c(S) \quad \text{for } S \in \mathcal{S}$$

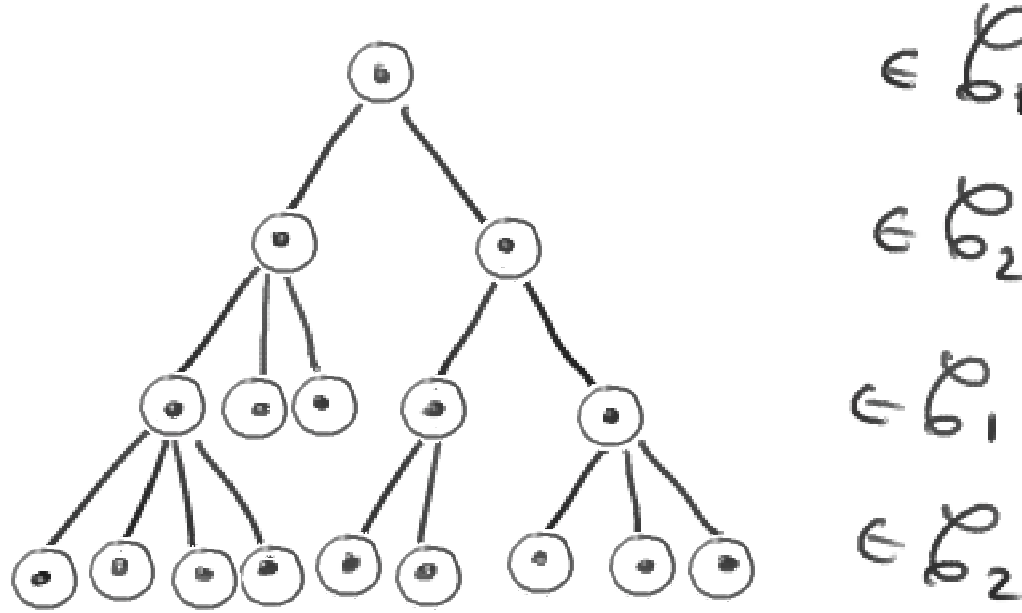
$$LB = \sum y_u$$

Lagrangian Relaxation

$\mathcal{C}_1$ (cheap unfeasible) $\qquad$ $\mathcal{C}_2$ (expensive feasible)



$A \bullet$ —— $\bullet B$ iff $u \in A \cap B$, $y_u > 0$



- tree property
- every edge must be covered
- if path is long, OPT is high

What do we get?

$$IP \leq \left(1 + \frac{1}{3^{k-1}}\right) LP + k\, c_{max}$$

$$IP > \left(1 + \frac{1}{3^{k-1}}\right) LP + \frac{k}{2}\, c_{max}$$

Open Problems

- LR for other problems
- Stronger notions than ρ-LMP
- Partial Totally unimodular Cover

Thanks for
your attention!