THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, EDUCATION, ENGINEERING AND SCIENCE

COSC 1001 – COMPUTATIONAL SCIENCE IN MATLAB
COSC 1901 – COMPUTATIONAL SCIENCE IN MATLAB (ADVANCED)

NOVEMBER 2009

TIME ALLOWED: 90 MINUTES

ALL QUESTIONS HAVE THE VALUE SHOWN

INSTRUCTIONS:

- Students in COSC 1001 should attempt questions 1, 2, and 3.
- Students in COSC 1901 should attempt questions 1, 2, and 4.

- This exam is not open-book.
- Non-programmable calculators are permitted.
1. A short-sighted student wanders down a street with six intersections. His home is at intersection 1, and his favourite internet cafe is at intersection 6. At each intersection other than his home or the cafe, he moves in the direction of the cafe with probability 4/5, and in the direction of his home with probability 1/5. He never wanders down a side-street. If he reaches his home or the cafe, he disappears into them, never to re-appear.

The following MATLAB code simulates the journey of a student who starts his wanderings at intersection 2.

```matlab
clear
expt=1e4;
for i=1:expt
    x=2;
    while x>1 && x<6
        if rand>1/5
            x=x+1;
        else
            x=x-1;
        end
    end
    count(i)=(x==1);
end
mean(count)
```

(a) If the output of the program is

```
an =
    0.2476
```

what is the probability (as a percentage) of reaching the cafe having started the journey from intersection 2?

(b) Given that the standard error in the mean for the calculation above is 0.0043, what might be some typical values of the quantity count if only 100 experiments were performed? Would it be reasonable to expect answers as low as 0.16?

(c) Consider some fixed quantity a, where 0 ≤ a ≤ 0.5. Explain why the expression `round(a+rand)` returns the numbers 0 and 1 in the ratio 0.5–a:0.5+a.

(d) Determine the value of a for which the expression in part (c) returns a zero one-fifth of the time, and thus write down a single-line expression which replaces lines 6–10 in the code above.

(10 marks)
2. The Leslie matrix for a three-band population model in which both adults and retirees have children has the form

\[ A = \begin{pmatrix} 0 & r_2 & r_3 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix}. \]

(a) Briefly explain why \( \alpha \) and \( \beta \) must both be less than unity.

(b) Write a line of MATLAB code that constructs the matrix \( A \) for a model where \( r_2 = 1, r_3 = 0.5, \alpha = 0.8 \) and \( \beta = 0.5 \).

(c) The MATLAB command that gives the eigenvectors and eigenvalues of this matrix, with output, is:

\[
>> [V,D] = eig(A)
\]

\[
V =
\begin{pmatrix}
0.7454 & -0.5970 & 0.1649 \\
0.5963 & 0.6600 & -0.4772 \\
0.2981 & -0.4561 & 0.8632
\end{pmatrix}
\]

\[
D =
\begin{pmatrix}
1.0000 & 0 & 0 \\
0 & -0.7236 & 0 \\
0 & 0 & -0.2764
\end{pmatrix}
\]

From an arbitrary starting point, what would you expect the ratio of children:adults:retirees to be after many generations? Will the total population decrease, increase or remain stable over time? Explain this with reference to the eigenvectors and eigenvalues in less than two paragraphs.

(d) Show that if the long-term population is to be constant over time then the following identity must hold.

\[ \alpha r_2 + \alpha \beta r_3 = 1. \]  

\textit{(12 marks)}
3. This question is for COSC 1001 students only

The value of $\pi$ can be estimated with a numerical integration using random numbers (known as Monte Carlo integration) of the area of a circle within a square. For the purposes of this question, consider a square defined by the range $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, in which you can draw a circle with a radius of 1 centred at the origin.

(a) Briefly outline how you would a random number generator to undertake the above Monte Carlo integration and estimate the value of $\pi$.

(b) The intrinsic MATLAB function `rand` returns a random number drawn from a uniform distribution between 0 and 1. Write out the MATLAB commands to calculate a vector of random numbers distributed uniformly between $-1$ and 1 using the `rand` function.

(c) Write a MATLAB function, called `estimatepi` to which you pass an integer $n$ and return an estimate of $\pi$, using the Monte Carlo approach you outlined in part (a) and the random number sequence you derived in part (b).

(d) Write down the output of the following MATLAB commands.

(i) $a=-1:2:8;$
   \[ \text{find}(a>5) \]

(ii) clear
    \for i=1:3:8
    \hspace{1cm} x(i)=i^4;
    \end
    \text{length(x)}
    \text{disp(i)}

(iii) $b=[2\ 3\ 8\ 6\ 8\ 9]$;
     \[ y=[-2\ 0\ 7\ -1\ -4\ 6] \]
     \[ b(y>0) \]

\(15 \text{ marks}\)
4. This question is for COSC 1901 students only
When modelling the interior of stars, it is important to calculate the fraction of Hydrogen and Helium in neutral and ionized states. This is described by the Saha equation, which we will write as:

\[ n_\text{H} = f_\text{H}(T)n_e^2 n_{\text{H}^+} \]  
\[ n_\text{He} = f_\text{He}(T)n_e^2 n_{\text{He}^+} \]  

(1)  
(2)
Here \( n_e \) is the number density of electrons (number of electrons per cubic meter), \( n_\text{H} \)
 is the number density of Hydrogen atoms, and so on. In these equations, \( T \) is the gas temperature, and the functions \( f_\text{H}(T) \) and \( f_\text{He}(T) \) depend on the ionization energies of H and He. Assume that these functions are given by MATLAB functions \( f_\text{H}(T) \) and \( f_\text{He}(T) \).

The total charge in the gas is zero, which means that the number density of electrons equals the sum of the number densities of H and He. The ratio of Hydrogen nuclei to Helium nuclei is 10:1.

(a) Formulate the problem of finding \( f_\text{H}, f_\text{He}, f_{\text{H}^+}, \) and \( F_{\text{He}^+} \) as a matrix equation.

(b) Write a MATLAB function that takes as input the electron number density and the gas temperature, and outputs \( n_\text{H}, n_{\text{H}^+}, n_\text{He}, \) and \( n_{\text{He}^+} \).

Random numbers can be used to integrate a function, \( f(x, y, z) \), over a volume; this is known as Monte Carlo integration. For this, we select random points within the volume over which we wish to integrate the function \( f(x, y, z) \). We then evaluate \( f(x, y, z) \) at these points and calculate its average value within the volume. Multiplying this average value by the volume gives an estimate of the integral. For example, if we want to integrate the function \( f(x, y, z) \) over a spherical volume centred at the origin, then

\[
\iiint f(x, y, z) \, dx \, dy \, dz \sim \langle f(x, y, z) \rangle \left( \frac{4}{3} \pi R^3 \right)
\]

where \( \langle \ldots \rangle \) denotes an average value, and \( x^2 + y^2 + z^2 < R^2 \) represents the points within a sphere of radius \( R \) centred at the origin. For the purposes of this question, we shall assume such an integration and consider \( R = 1 \). The function to be integrated over the sphere is given by

\[
f(x, y, z) = \cos(x + y) + \sin^2(x + z)
\]

(c) Write a MATLAB function, called mean, which evaluates the function \( f(x, y, z) \).

(d) Write a MATLAB function, called randomsphere, which returns a random position \( (x, y, z) \) within the sphere of radius \( R = 1 \). To do this, calculate a sequence of random points in the range \(-1 \leq x \leq 1, -1 \leq y \leq 1 \) and \(-1 \leq z \leq 1 \), and accept the first point that is within the sphere.
(c) Write a MATLAB function, called `integration`, to which you pass an integer `n` and it returns an estimate of the Monte Carlo integration for `n` points. Ensure this function calls both `myfunc` and `randomsphere.

(15 marks)

THERE ARE NO MORE QUESTIONS.