THE UNIVERSITY OF SYDNEY
FACULTIES OF ARTS, EDUCATION, ENGINEERING AND
SCIENCE

COSC 1002
COMPUTATIONAL SCIENCE IN C

COSC 1902
COMPUTATIONAL SCIENCE IN C (ADV)

NOVEMBER 2010
Time allowed: NINETY minutes
MARKS FOR QUESTIONS ARE AS INDICATED
TOTAL: 35 MARKS

INSTRUCTIONS

• COSC 1002 students are to answer Questions 1, 2, and 3 (1002)
• COSC 1902 students are to answer Questions 1, 2, and 3 (1902)
• All answers should include explanations of reasoning
• Where a question says to provide a brief explanation, write less than half a page (excluding diagrams)
THE FOLLOWING FORMULAE MAY BE USEFUL

• One step of Euler’s method applied to \( \frac{dy}{dx} = f(x, y) \):
  \[
y_{i+1} = y_i + \Delta x f(x_i, y_i)
  \]

• One step of second order Runge-Kutta applied to \( \frac{dy}{dx} = f(x, y) \):
  \[
y_{i+1} = y_i + \Delta x f \left[ x_i + \Delta x, y_i + \frac{1}{2} \Delta x f(x_i, y_i) \right]
  \]

• The Taylor series expansion of \( f(x) \) about \( x = a \):
  \[
f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \cdots
  \]

• The extended Trapezoidal rule is given by:
  \[
  \int_a^b f(x) \, dx \approx \sum_{i=0}^{N-2} \frac{1}{2} \Delta x_i (f_i + f_{i+1})
  \]

• The following power series for \( |z| < 1 \), may be useful:
  \[
e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots
  \]
  \[
  \ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots
  \]
Question 1

(a) (i) Explain the concept of floating point overflow in C programming. Give an example of a calculation in which overflow could occur.

(ii) The area of a triangle can be expressed in terms of the lengths of its sides $a$, $b$, and $c$ as

$$A = \sqrt{s(s - a)(s - b)(s - c)}, \text{ where } s = (a + b + c)/2.$$ 

Explain the concept of catastrophic cancellation in the context of implementing this equation. Give an example of when this would be a problem.

(ii) Briefly explain the concept of local scope in C functions, using example code to demonstrate your explanation.

(b) The following program calculates the mean of a list of numbers read in from a file. The program compiles, but when it runs the answer it prints is incorrect. Identify and briefly explain four common C errors which appear in this code.

```c
#include <stdio.h>
#define N 100

int main(){
    int i;
    int mean;
    float numbers[N];
    
    for(i = 0; i <= N; i++)
        scanf("%f", &numbers[i]);
    mean += numbers[i];
    mean /= N;
    printf("mean = %f\n", mean);
    return 0;
}
```

(c) (i) The Pell numbers are an infinite series of integers that represent the denominators of the closest rational approximations to $\sqrt{2}$. The Pell numbers are defined by:

$$P_n = \begin{cases} 
0 & \text{if } n = 0; \\
1 & \text{if } n = 1; \\
2P_{n-1} + P_{n-2} & \text{otherwise.}
\end{cases}$$

The first few terms of the sequence are 0, 1, 2, 5, 12, 29, 70, . . . .

Write a C program that asks the user for an integer $n$ and prints out the $n^{th}$ Pell number. Your program should work like this:

```
$ ./pell
Enter number: 4
Pell number 4 = 12
```

(ii) Will your C program give correct answers for all Pell numbers? Briefly explain a case where your program may give an incorrect answer.

(12 marks)
Question 2

(a) In about half a page, explain why the trapezoidal rule for integration is not exact for functions higher than order 1. Demonstrate this by evaluating

\[ \int_0^4 \frac{x^2}{2} \, dx \]

over 4 intervals and comparing it to the analytic solution. Explain how you could make this estimate more accurate.

(b) Two numerical methods commonly used for finding the roots of an equation are the bisection method and the Newton-Raphson method.

(i) Briefly explain and derive the difference equation representing one iteration of the Newton-Raphson method:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \]

(ii) Briefly explain (including an example and diagram) one of the advantages of the bisection method compared to the Newton-Raphson method.

(iii) Briefly explain (including an example and diagram) one of the advantages of the Newton-Raphson method compared to the bisection method.

(c) A disease epidemic in a population of susceptibles (S) and infected (I) and recovered (R) can be modelled by the following system of ODEs

\[
\begin{align*}
\frac{dS}{dt} & = -rSI \\
\frac{dI}{dt} & = rSI - aI \\
\frac{dR}{dt} & = aI
\end{align*}
\]

where the total population (S + I + R) remains constant.

(i) Explain what each term in this system of equations represents in a real world context.

(ii) Sketch a plot showing the possible behaviour of the three populations (R, S, and I) with time.

(iii) Explain how you could solve this system of equations numerically to predict the population of susceptibles, infected and recovered for some time \(t\).

(12 marks)
Answer Question 3 (1002) OR Question 3 (1902). Do NOT answer both questions.

Question 3 (1002)

(a) Consider the initial value problem
\[ \frac{dy}{dx} = e^{-y}, \quad y(0) = 0. \]

(i) Show that \( y = \ln(1 + x) \) is the analytic solution to this initial value problem.

(ii) Apply Euler’s method twice to this problem, with a step \( \Delta x \), to determine approximate expressions for \( y(\Delta x) \) and \( y(2\Delta x) \).

(iii) For the approximate expression for \( y(2\Delta x) \) in part (b), use the formula sheet to express the result as a power series correct to second order. Using the formula sheet express \( \ln(1 + x) \) as a power series correct to second order.

(iv) What is the error in the second order term after two steps?

(b) Consider the function defined by
\[ C(x) = \int_0^x \cos \left( \frac{\pi}{2} t^2 \right) dt. \]

Briefly explain how \( C(1) \) can be evaluated by numerically solving an ordinary differential equation. Include a statement of the initial value problem required to be solved, and suggest a suitable numerical method.

(c) A ‘modified’ logistic model for population growth is
\[ \frac{dN}{dt} = aN[1 - (N/N_*)^2], \]
where \( a > 0 \) and \( N_* > 0 \) are constants.

(i) Carefully sketch the direction field for the above ODE, for all \( N > 0 \) and \( t > 0 \).

(ii) Briefly explain the concept of stability, and why it is important in integrating ODEs.

(iii) For what values of \( N \) is the above ODE unstable?

(11 marks)
Answer Question 3 (1002) OR Question 3 (1902). Do NOT answer both questions.

**Question 3 (1902)**

(a) Consider the initial value problem

\[ \frac{dy}{dx} = e^{-y}, \quad y(0) = 0. \]

(i) Show that \( y = \ln(1 + x) \) is the analytic solution to this initial value problem.

(ii) Apply the second order Runge-Kutta method once to this problem, with a step \( \Delta x \), to determine approximate expression for \( y(\Delta x) \).

(iii) For the approximate expression for \( y(\Delta x) \) in part (b) use the formula sheet to express the result as a power series correct to third order. Using the formula sheet express \( \ln(1 + x) \) as a power series correct to third order. By comparing the two power series show that in this case the method is indeed a second order method.

(iv) What is the error in the third order term after one step?

(b) The Planck spectrum is

\[ B_\nu = \frac{2h\nu^3}{c^2} \left[ \exp \left( \frac{h\nu}{k_BT} \right) - 1 \right]^{-1} \]

(i) Show that the peak of this spectrum in \( \nu \) (for a fixed \( T \)) is given by the solution to

\[ x = 3(1 - e^{-x}), \]

where \( x = h\nu/(KT) \).

(ii) Apply the Newton-Raphson method to this solution, with the initial guess \( x_0 = 2. \) Show that \( x_2 \approx 2.82 \).

(c) A possible model for population growth is

\[ \frac{dN}{dt} = aN^2 - bN, \]

where \( a > 0 \) and \( b > 0 \) are constants.

(i) Carefully sketch the direction field for the above ODE, for all \( N \geq 0 \) and \( t \geq 0 \).

(ii) Briefly explain the concept of stability, and why it is important in integrating ODEs.

(iii) For what values of \( N \) is the above ODE unstable?

(11 marks)