OBJECTIVES AND MARKING SCHEME
Question 1

(a) (i) Explain the concept of floating point overflow in C programming. Give an example of a calculation in which overflow could occur.

(ii) The area of a triangle can be expressed in terms of the lengths of its sides $a$, $b$, and $c$ as

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = (a + b + c)/2.$$  

Explain the concept of catastrophic cancellation in the context of implementing this equation. Give an example of when this would be a problem.

(b) Briefly explain the concept of local scope in C functions, using example code to demonstrate your explanation.

Objectives

- Understand basic C programming concepts and constructs, including the main program, the role and nature of variables, data types, the use of define directives, simple input/output to the screen, and iteration using a for loop
- Understand the properties of floating point numbers and some of the problems of floating point accuracy
- Understand floating point exceptions
- Recognize situations when rounding error becomes important in numerical calculations, and understand some ways to avoid this problem
- Understand the use of global variables

Marking Scheme

Total = 4 marks.

- (i) 1 mark for correct answer
- (ii) 1 mark for correct answer
- (iii) 2 marks for correct answer including example or demonstration

Partial marks for partially correct answers that mention main concept.
(b) The following program calculates the mean of a list of numbers read in from a file. The program compiles, but when it runs the answer it prints is incorrect. Identify and briefly explain four common C errors which appear in this code.

```c
#include <stdio.h>
define N 100

int main(){
    int i;
    int mean;
    float numbers[N];

    for(i = 0; i <= N; i++)
        scanf("%f", &numbers[i]);
    mean += numbers[i];

    mean /= N;
    printf("mean = %f\n", mean);
    return 0;
}
```

Objectives

- Understand basic C programming concepts and constructs, including the main program, the role and nature of variables, data types, the use of define directives, simple input/output to the screen, and iteration using a for loop
- Read data from a file using redirection

Marking Scheme

Total = 4 marks.

1 mark for each problem correctly identified.
(c) (i) The Pell numbers are an infinite series of integers that represent the denominators of the closest rational approximations to $\sqrt{2}$. The Pell numbers are defined by:

$$P_n = \begin{cases} 
0 & \text{if } n = 0; \\
1 & \text{if } n = 1; \\
2P_{n-1} + P_{n-2} & \text{otherwise.}
\end{cases}$$

The first few terms of the sequence are 0, 1, 2, 5, 12, 29, 70, . . .

Write a C program that asks the user for an integer $n$ and prints out the $n^{th}$ Pell number. Your program should work like this:

```
$ ./pell
Enter number: 4
Pell number 4 = 12
```

(ii) Will your C program give correct answers for all Pell numbers? Briefly explain a case where your program may give an incorrect answer.

(12 marks)

Objectives

- Understand basic C programming concepts and constructs, including the main program, the role and nature of variables, data types, the use of define directives, simple input/output to the screen, and iteration using a for loop
- Understand and be able to use control structures and their associated logic in C

Marking Scheme

Total = 4 marks.

- 3 marks for C code
- 1 mark for correct answer and explanation of (ii)
Question 2

(a) In about half a page, explain why the trapezoidal rule for integration is not exact for functions
higher than order 1. Demonstrate this by evaluating

\[ \int_{0}^{4} \frac{x^2}{2} \, dx \]

over 4 intervals and comparing it to the analytic solution. Explain how you could make this
estimate more accurate.

Objectives

- Understand and be able to implement the trapezoidal rule for numerical integration

Marking Scheme

Total = 4 marks.

- 2 marks for explanation of trapezoidal rule, including 1 mark for diagram or example
- 1 mark for evaluating integral correctly
- 1 mark for explanation of making the estimate more accurate
(b) Two numerical methods commonly used for finding the roots of an equation are the bisection method and the Newton-Raphson method.

(i) Briefly explain and derive the difference equation representing one iteration of the Newton-Raphson method:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \]

(ii) Briefly explain (including an example and diagram) one of the advantages of the bisection method compared to the Newton-Raphson method.

(iii) Briefly explain (including an example and diagram) one of the advantages of the Newton-Raphson method compared to the bisection method.

Objectives

- Understand the bisection method for numerical root finding for a function of one variable
- Understand the Newton-Raphson method for numerical root finding for a function of one variable
- Understand the relative advantages and disadvantages of the two root-finding methods

Marking Scheme

Total = 4 marks.

- 2 marks for explanation including 1 mark for derivation
- 1 mark for explanation of advantage in (ii)
- 1 mark for explanation of advantage in (iii)
(c) A disease epidemic in a population of susceptibles (S) and infected (I) and recovered (R) can be modelled by the following system of ODEs

\[
\frac{dS}{dt} = -rSI \\
\frac{dI}{dt} = rSI - aI \\
\frac{dR}{dt} = aI
\]

where the total population (S + I + R) remains constant.

(i) Explain what each term in this system of equations represents in a real world context.

(ii) Sketch a plot showing the possible behaviour of the three populations (R, S, and I) with time.

(iii) Explain how you could solve this system of equations numerically to predict the population of susceptibles, infected and recovered for some time \( t \).

(12 marks)

Objectives

- Understand the use of first order ODEs to model systems in science,
- Understand the method of treatment of coupled first order ODEs

Marking Scheme
Total = 4 marks.

- 1 mark for (i)
- 1 mark for plot in (ii)
- 2 marks for explanation of solution (iii)
Answer Question 3 (1002) OR Question 3 (1902). Do NOT answer both questions.

Question 3 (1002)

(a) Consider the initial value problem

\[
\frac{dy}{dx} = e^{-y}, \quad y(0) = 0.
\]

(i) Show that \( y = \ln(1 + x) \) is the analytic solution to this initial value problem.

(ii) Apply Euler’s method twice to this problem, with a step \( \Delta x \), to determine approximate expressions for \( y(\Delta x) \) and \( y(2\Delta x) \).

(iii) For the approximate expression for \( y(2\Delta x) \) in part (b), use the formula sheet to express the result as a power series correct to second order. Using the formula sheet express \( \ln(1 + x) \) as a power series correct to second order.

(iv) What is the error in the second order term after two steps?

Objectives

- Understand first order ordinary differential equations (ODEs) and the concept of an initial value problem
- Understand how first order ODEs appear in simple science problems, e.g. the description of population growth
- Understand and implement Euler’s method to solve first order ODEs

Marking Scheme

Total = 4 marks.

- 1 mark per part for (i) to (iv)
(b) Consider the function defined by

\[
C(x) = \int_0^x \cos \left( \frac{\pi}{2} t^2 \right) dt.
\]

Briefly explain how \( C(1) \) can be evaluated by numerically solving an ordinary differential equation. Include a statement of the initial value problem required to be solved, and suggest a suitable numerical method.

Objectives

- Understand the relationship between numerical integration and the solution of a first order ODE, and be able to numerically integrate a function by solving an ODE
- Understand first order ordinary differential equations (ODEs) and the concept of an initial value problem

Marking Scheme

Total = 3 marks.

- 1 mark for statement of initial value problem
- 2 marks for explanation including mention of numerical method
(c) A ‘modified’ logistic model for population growth is

\[
\frac{dN}{dt} = aN \left[1 - \left(\frac{N}{N^*}\right)^2\right],
\]

where \( a > 0 \) and \( N^* > 0 \) are constants.

(i) Carefully sketch the direction field for the above ODE, for all \( N > 0 \) and \( t > 0 \).

(ii) Briefly explain the concept of stability, and why it is important in integrating ODEs.

(iii) For what values of \( N \) is the above ODE unstable?

(11 marks)

Objectives

- Understand the importance of numerical accuracy and stability in the solution of first order ODEs, and be able to perform a simple test for accuracy
- Understand a criterion for local stability of a first order ODE
- Understand the use of a direction field to provide qualitative information about the solution to a first order ODE

Marking Scheme

Total = 4 marks.

- 1 mark for direction field
- 2 marks for explanation of stability and implications
- 1 mark for solutions
Answer Question 3 (1002) OR Question 3 (1902). Do NOT answer both questions.

Question 3 (1902)

(a) Consider the initial value problem

\[ \frac{dy}{dx} = e^{-y}, \quad y(0) = 0. \]

(i) Show that \( y = \ln(1 + x) \) is the analytic solution to this initial value problem.

(ii) Apply the second order Runge-Kutta method once to this problem, with a step \( \Delta x \), to determine approximate expression for \( y(\Delta x) \).

(iii) For the approximate expression for \( y(\Delta x) \) in part (b) use the formula sheet to express the result as a power series correct to third order. Using the formula sheet express \( \ln(1 + x) \) as a power series correct to third order. By comparing the two power series show that in this case the method is indeed a second order method.

(iv) What is the error in the third order term after one step?

Objectives

- Understand first order ordinary differential equations (ODEs) and the concept of an initial value problem
- Understand how first order ODEs appear in simple science problems, e.g. the description of population growth
- Understand the origin of second order Runge-Kutta, a more accurate scheme for integrating ODEs than Eulers method

Marking Scheme

Total = 4 marks.

- 1 mark for each part (i) to (iv)
(b) The Planck spectrum is
\[ B_\nu = \frac{2\hbar \nu^3}{c^2} \left[ \exp\left( \frac{\hbar \nu}{k_B T} \right) - 1 \right]^{-1} \]

(i) Show that the peak of this spectrum in \( \nu \) (for a fixed \( T \)) is given by the solution to
\[ x = 3(1 - e^{-x}), \]
where \( x = h\nu/(KT) \).

(ii) Apply the Newton-Raphson method to this solution, with the initial guess \( x_0 = 2 \). Show that \( x_2 \approx 2.82 \).

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**Objectives**

- Numerically solve various transcendental equations that arise in scientific problems, e.g. to locate the peak of the Planck spectrum

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**Marking Scheme**

**Total = 4 marks.**

- 2 marks for (i)
- 2 marks for (ii)
(c) A possible model for population growth is

\[ \frac{dN}{dt} = aN^2 - bN, \]

where \( a > 0 \) and \( b > 0 \) are constants.

(i) Carefully sketch the direction field for the above ODE, for all \( N \geq 0 \) and \( t \geq 0 \).

(ii) Briefly explain the concept of stability, and why it is important in integrating ODEs.

(iii) For what values of \( N \) is the above ODE unstable?

(11 marks)

Objectives

• Understand the importance of numerical accuracy and stability in the solution of first order ODEs, and be able to perform a simple test for accuracy

• Understand a criterion for local stability of a first order ODE

• Understand the use of a direction field to provide qualitative information about the solution to a first order ODE

Marking Scheme

Total = 4 marks.

• 1 mark for direction field

• 2 marks for explanation of stability and implications

• 1 mark for solutions