1 Objectives

This experiment is designed to make you familiar with Fourier Transforms: a very powerful way of analysing signals, so that their frequency components are revealed.

2 Introduction

A Fourier transform converts a time varying signal into its frequency components. In this experiment we will be studying the Fourier Transform of electrical signals. The term electrical signal is often used to describe (for example) what goes into a TV receiver from its antenna. Signals have some information content. We can generalise the word signal to include non-electric signals such as those like sound, which we could convert to an electrical signal with a microphone. The inverse is of course possible; a loudspeaker can convert an electrical signal back into sound. Devices that can do these conversions are collectively called transducers.

It is possible (in principle) for any signal type to be converted into any other type. However, for the present purpose, we will need a signal that is already in electrical form and if it is non-electrical we will need to first put it through a transducer which converts the signal to an electrical form.
The present experiment studies the Fourier analysis process and will need a fast computer. If the signal is an analog signal then it must be put through an Analog to Digital Converter (ADC) next. There are various types of these ADC, the so called flash ADC wins hands-down for speed of operation (up to more than 20 Gigasamples per second for some models). The flash ADC uses a bank of \( n \) voltage comparators in a ‘ladder’ arrangement; each comparator indicates with its own digital output whether \( V_{in} \) is less than or greater than the particular voltage reference \( V_i \) of the comparator. So there is a need for a voltage divider chain that produces the \( V_i \). The comparator outputs go to a digital circuit which converts the pattern to a binary number. In the present PicoScope machine we have 256 comparator outputs to make a one byte (8 bits) number.

When fed with a series of \( k \) convert pulses, an ADC produces \( k \) digitised values of the input signal and that can be digitally stored and then displayed for as long as an observer wants this, making in effect a digital oscilloscope. This vector can be further processed in various ways, such as the sophisticated Fourier procedure explored in this experiment.

The PicoScope conveniently does the jobs described in the previous two paragraphs. The ADCs are there and so are the high speed memory chips needed to store the ADC data, which could be arriving at up to 1 GB/s. Moreover Fourier analysis is done at the PicoScope by a specialised processor. At this stage we need to examine the mathematics involved in the PicoScope acting in the frequency domain. We start by briefly examining the continuous Fourier Transform.

\[
H(f) = \mathcal{F}\{h\}(f) = \int_{-\infty}^{+\infty} h(t) \exp(-j2\pi ft)dt
\]

(1)

It involves first multiplying the input waveform \( h(t) \) by a complex exponential function \( \exp(-j2\pi ft) \).

\[
\exp(-j2\pi ft) = \cos(2\pi ft) - j \sin(2\pi ft)
\]

where \( j = \sqrt{-1} \) and then we integrate from \( t = -\infty \) to \( +\infty \). The exponential argument is imaginary so the exponential has a unit magnitude and a progressively decreasing phase. This is quite unworkable in a computer as we need the answer now, not an infinite time later. In addition, we have the infinitesimal differential \( dt \) involved too and this is equally unworkable.

Both problems are “solved” by sampling the signal at a finite number of times \( n \) in a sequence where the time between successive samples is a constant. The data recording time is then finite. These readings are then multiplied by the exponential which has a unit magnitude but has a phase angle starting from zero size and then increasing in steps. If we give the \( \phi \) in revolutions then we have 0, \( \phi \), 2\( \phi \), 3\( \phi \), 4\( \phi \), etc.

In mathematical notation we have

\[
X(m) = \mathcal{F}_D\{x\}(m) = \sum_{k=0}^{k=n-1} x(k) \exp(-j2\pi mk/n) \quad m = 0, 1, 2, 3, \ldots, n - 1
\]

(2)

In this we have \( n \) samples from \( k = 0 \) to \( k = n - 1 \). The input signal is \( x(k) \), and its Fourier Transform is \( X(m) \). In our case the time varying signal is represented by a sampled values \( x(k) \), \( X(m) \) is a frequency representation of the signal, known as the Discrete Fourier Transform (DFT). In the general case we would have a complex \( x(k) \) but in this experiment \( x(k) \) is real. \( X(m) \) is different in the sense that even for real \( x(k) \), it can be complex. However the PicoScope, when showing spectra, only displays the magnitudes of each component (the square root of the sums of squares of the real and imaginary part of each \( X(m) \)).
It is worth noting here that the DFT is implemented in the PicoScope’s software using a clever algorithm known as the Fast Fourier Transform (FFT). Equation (2) is a matrix equation and $\exp(-j2\pi mk/n)$ has $n^2$ elements but most are repeats because there are only $n$ different entries in this matrix. If $n$ is a composite (not prime) number then the matrix is factorisable and the FFT algorithm capitalises on this to dramatically reduce the computing time. For 1024 points the time drops by a factor of about 200 and it is even more dramatic for higher number of points. A slight disadvantage comes from having the condition that the algorithm is faster if the number of points is an integral power of 2. In most cases however, this is a trivial importance.

2.1 Leakage and aliasing

The Discrete Fourier Transform has two potential deficiencies. These are called Leakage Fig. [3-1] and Aliasing Fig. [3-2]. Strictly speaking, aliasing is not unique to DTF as it occurs everywhere, where sampling of the measured value is used (that means any real measurement). We will call the sampling period $p$ and call $n$ the number of samples. Leakage results from the fact that the input data recording time $np$ is finite. We will illustrate it with a sine wave period $p'$ sampled many times per cycle. If $p'$ is a non-integral number of cycles is recorded ($p'$ does not divide $np$ exactly) then a “discontinuity” in the data is produced. The transform shows this in a characteristic manner (see EO Brigham’s book). The “cure” for leakage is to multiply the incoming data by a “window” function which is zero both at the start and end of the sampling period. The Hann window is a typical one; it is $x(t) = 0.5 - 0.5 \cos(2\pi t/T)$ where the data goes from $t = 0$ to $T$.

Aliasing is caused by the finite time interval $p$ between successive samples of the waveform being recorded. Let $p'$ be the period of a sine wave and if $p > p'/2$ then the signal reconstructed from the samples can be identified as it has a much larger period (or lower frequency) then the sampled original signal. This leads to peaks shown in a spectrum at false frequencies. To avoid aliasing we must assure that the sampled signal does not contain any components with a frequency greater then half of the sampling frequency (called the Nyquist frequency). Sampling with a frequency at least twice as high as highest frequency in the analysed signal avoids aliasing but it still will be not enough to properly represent the original signal. As a “rule of thumb” engineers say that the sampling frequency should be at least 5 times greater than the frequency of a signal that is sampled. For example, new standards for sampling high quality audio signals (20 Hz to 20 kHz) are 88.2, 96, 176.4 and 192 kSamples/s rather then older CD standard 44.1 kSamples/s or digital tape standard 48 kSamples/s.

**Question 1:** Prove that the Fourier Transform is a linear operator. **Hint:** Use equation (1).

**Question 2:** Is a DFT also linear? **Hint:** Use DFT definition given in equation (2).

**Question 3:** Write down the $4 \times 4$ DFT matrix $\exp(-j2\pi mk/n)$. How many different elements does this matrix contains?
Fig. 3-1 Sampling a periodic waveform: (a) the truncation interval \( n_1 p = 3p' \) does not produce distortion of the signal, (b) \( n_2 p = 3.5p' \) - obvious discontinuity of the waveform.

Fig. 3-2 Aliasing occurs when the sampling period \( p \) is larger than a half of the period \( p' \) of the input signal. Part (a) shows the original waveform and some samples taken at interval \( p = 0.75p' \). Part (b) shows the sampled values as they are recorded. Part (c) shows those samples interpreted as a waveform with period \( p'' = 3p' \).
3 The observations

3.1 Basics of the PicoScope

A PicoScope is the key instrument in this experiment.

1. Turn it and its controlling computer on and start the “PicoScope 6” application.

2. Connect the PicoScope’s CHA to the sine wave generator set to 10 kHz to see this signal in the time domain (voltage plotted against time). You will probably have to set the PicoScope’s other controls to see it clearly. So set the timebase speed to 500 µs per division (top of screen second row) and the horizontal zoom (to its right) to ×1 and the number of samples to 1 KS.

3. An important pair of buttons is at the left end of the screen’s bottom. They are: a green arrow which starts data acquisition and a red stop button which stops it. A list box near these controls triggering. Try out some of its effects.

4. The PicoScope can perform various important measurements on the input signal. For example to measure the frequency of the signal, from the menu choose Measurements → Add Measurement. The Add Measurement dialog box will appear on the screen. Select channel A to measure. Then choose type of measurement Frequency and Whole trace as the section of the graph that will be measured. Click OK to confirm the measurement settings. A measurements table should appear at the bottom of the window.

5. You can also view the input signal in the frequency domain: from the menu choose Views → AddView → Spectrum. You are now seeing the 10 kHz signal in the

Fig. 3-3 10 kHz signal observed in the time domain (top) and in the frequency domain (bottom). A frequency measurement is also displayed.
frequency domain (Voltage plotted against frequency). There should be a series of peaks on the trace and they should move if the sine wave generator’s frequency is varied. Note that you have a decibel scale for the vertical axis.

6. Now click the Spectrum Options button in the second row of buttons (it is a set of four vertical lines). Try out the various options there (do not forget to press Apply to accept each option).

7. At the beginning of the X axis there is a small square box that you can drag along the axis. This is a ruler handle. When a ruler is dragged out from its ’parking position’ a ruler legend is displayed showing time or frequency (depending on the view type). You can have up to two horizontal or vertical rulers in each view. To access vertical rulers drag a blue ruler handle placed at the top of the Y axis on left hand side.

### 3.2 Hands on leakage and aliasing

Let us start observation of leakage.

1. To see the effect of a window by further examining the sine wave of say 10 kHz in the frequency domain. Keep the decibel Y axis scale for this. The wide range of signal sizes for this scale is important here. Use the rectangular window function first. This is effectively a ’no window’ setting.

2. Then try all the other window settings. The ’no window’ setting narrows the top of peaks but makes a ’skirt’ under each peak. A window tends to ’blunt’ and widen the top of peaks but reduces skirt widths. Some windows are better in this than others. Take a look at Brigham’s book [1, chapter 9] for more explanation.

3. Print out the PicoScope’s display for the rectangular window and at least one other case. Comment on your results.

Now take a closer look at aliasing.

4. Examine a sine wave in the frequency domain again. Start with its frequency set lower than the upper limit of the range you are using namely \( F_{\text{max}} \). Increasing the sine generator’s frequency will shift the peak to the right but this motion will reverse when the peak reaches the right side of the screen and the peak will come back towards the left edge of the screen and its frequency as indicated on the scale below will be wrong (we’ll call this frequency \( f_w \)). The correct value of the signal’s frequency is obtained by doubling \( F_{\text{max}} \) and subtracting \( f_w \) from it. (It will be convenient to use the setting/measurements facility to display \( f_w \) at the screen bottom). Continuously increasing the frequency causes the peak go back and forth (and you would have to upgrade the formula for the signal’s real frequency to accommodate this ’see saw’ effect!)

5. It is worth pointing out that the PicoScope digital oscilloscope function is like any digital oscilloscope, in that it can suffer from aliasing problems in the time domain mode of operation as well as it might in the spectrum mode. Input a sinusoidal signal into the oscilloscope with a frequency of 10 kHz and look at it in the time domain.
6. Set the timebase speed to 500 µs per division and then slowly turn the sinusoidal generator’s frequency control knob. Set Measurements to get the oscilloscope displaying what it ‘thinks’ the frequency is (at the bottom of its screen). You will find settings where the screen shows a coarse sine wave with the PicoScope thinking its frequency is wildly different from that produced by the sine wave generator.

7. Print out the screens as graphs. Does the spectrum display add anything of interest to what you see in the time domain picture?

This ‘time domain’ aliasing is virtually the same as experienced in the frequency domain. A dramatic way to emphasize what can happen is to imagine a digital oscilloscope in the time domain; which had been somehow set to sample a sine wave but only did it for every time that the voltage of this wave was zero. The trace would be a completely misleading straight line!

It is relevant here to quote part of E.O. Brigham’s summary at the end of his chapter 6:- “We have shown that if care is exercised, then there exist many applications where the discrete Fourier transform can be employed to derive results essentially equivalent to the continuous Fourier transform.” He goes on to write that:- “If one will always remember that the N samples of the time domain function represent one of a periodic function, then application of the discrete Fourier transform should result in few surprises.” The two key words here are “periodic function”. Brigham spells this out in the preceding paragraph of his text. In a sense we can view the data set \( x(k) \) as effectively being repeated over and over as far as equation (2) is concerned. The DFT of a train of identical \( x(k) \) sets is effectively the same as a DFT of a single \( x(k) \) data set!

### 3.3 The comb

1. Connect the pulse generator on the multiplier box into the PicoScope and examine the spectra you get while the pulse generator’s duty cycle is varied. This is the ratio of the pulse width to the period of the pulses. The duty cycle is easily measured in the time domain display. Note that the voltage output from the pulse generator alternates between a positive value and a value which is effectively zero. So the waveform has a non-zero average value and there then is a non-zero DC component of the signal. As described above, there is consequently a finite value for the zero frequency component of the signal in the frequency domain. Check that this is happening (easier to do if the duty cycle is not too small).

2. Now examine the spectrum as the duty cycle is reduced. Most of the harmonics of the pulse’s frequency are visible but when the duty cycle is reduced, the rate of fall off of their amplitudes as you go up the spectrum is progressively reduced until you approach asymptotically the stage where all the harmonics have equal amplitude, making a comb (it really does resemble that implement for making your hair more tidy!)

**Question 4:** If the period of the signal from the pulse generator is \( T \) (in the time domain), write the formula for the frequency of the \( n \)-th harmonic (in the frequency domain).
### 3.4 Balanced modulation and amplitude modulation

This section is about the Frequency Shifting theorem (sometimes called the modulation theorem) and is very important in the Fourier analysis of signals. Brigham states this theorem as:- If the transform of \( h(t) \) is \( H(f) \) then the transform of \( h(t) \exp(j2\pi f_0 t) \) is \( H(f - f_0) \).

There is a far more revealing way to describe this theorem. For an example of this theorem’s operation, we multiply two cosine waves with different frequencies \( f_1 \) and \( f_2 \) (and let’s have \( f_1 > f_2 \)). The result is \( \cos(2\pi f_1 t) \cos(2\pi f_2 t) = \cos[2\pi(f_1 + f_2)t] + \cos[2\pi(f_1 - f_2)t] \). In this case we have no peak at \( f_1 \). The missing peak can be reinstated by replacing the product by \( \cos(2\pi f_1 t)[1 + d \cos(2\pi f_2 t)] \) and this is called an AM (Amplitude Modulated) radio signal.

A constant value \( d : 0 < d < 1 \) is called depth of modulation. The \( \cos(2\pi f_1 t) \cos(2\pi f_2 t) \) signal is called balanced modulation and \( \cos[2\pi(f_1 + f_2)t] \) or \( \cos[2\pi(f_1 - f_2)t] \) represent SSB (Single SideBand) modulation.

1. Use a multiplier circuit to make these signals using the outputs of two sine wave generators. Inspect them in both the time and frequency domains. AM is obtained by turning on the offset control of one of the sine wave generators and adjusting this offset. As usual, you should make hard copies of the measurements.

### 3.5 Radio station spectra

![Fig. 3-4 Spectrum of Sydney’s AM radio stations.](image)

The next exercise in this experiment is to examine the spectra of the AM and FM radio stations. Let us start with MF (Middle Frequency) stations. All these stations broadcast signals that are AM.

2. Connect cable marked as “AM RADIO” to the PicoScope.

3. Switch the PicoScope to Spectrum Mode and set Spectrum Range to 1.953 MHz.
Fig. 3-5  Spectrum of the 2BL radio station. A carrier (702 kHz) with side bands (produced by amplitude modulation) are clearly visible.

Fig. 3-6  Spectrum of Sydney’s FM radio stations.
Fig. 3-7  Spectrum of the 2ABCFM radio station. Relatively quiet music. Clearly visible stereo pilot at 19 kHz.

Fig. 3-8  Spectrum of the 2ABCFM radio station. The signal is strongly modulated by loud music.
4. Look at the region between 500 and 1600 kHz and try to identify Sydney’s AM radio stations. You should see a spectrum very similar to that in Fig. 3-4.

5. Print this spectrum and write on it the names and frequencies of all station that you recognised. See appendix B for a list of Sydney’s radio stations.

6. Choose one radio station and increase the horizontal scale so that just a few kHz around the carrier frequency is displayed.

7. Set the AM radio to this station and observe the correlation between the sound you are listening to and the shape of both sidebands. Your spectrum should be similar to Fig. 3-5.

8. Use the Waveform Buffer to choose one spectrum that you like. Print it and comment on what you observe.

Next we will observe radio stations that broadcast in the VHF (Very High Frequency) range. Those stations modulate the frequency of the carrier by an acoustic signal.

9. Plug the pick up device (i.e. an antenna!) marked as “FM RADIO” into your recording system.

10. Set a Spectrum Range of 125 MHz and look at the region between 90 and 109 MHz. Your spectrum should look like Fig. 3-6.

11. Identify all stations that you observe.

12. Print the spectrum and write the names and frequencies of all recognised stations.

13. Choose one radio station and increase the horizontal scale to get a spectrum as shown in Fig. 3-7 or Fig. 3-8.

14. Set the FM radio to this station and observe the correlation between what you hear and what you see.

15. Print one of the Waveform Buffers and comment on it.

**Question 5:** Are both sidebands around the carrier in AM signal symmetrical? Comment on your result.

**Question 6:** What kind of modulation is used in FM radio? Give a mathematical formula for it.

### 3.6 My own Fourier investigation (Optional)

If time permits you might like to do some other experiments using the existing equipment. Discuss your proposal with a tutor. If accepted, perform your experiment and describe in detail what you are doing. Write down your results and interpret their meaning.
A Cool mathematics of DFT

This is the place for some more observations on the mathematics of Discrete Fourier Transforms.

As described in equation (2), the Discrete Fourier Transform takes a series of numbers $x(k)$, (actually a vector) representing perhaps some measurement process. It transforms these to another vector $X(m)$ by multiplying it by a square matrix $W = \exp(-j2\pi mk/n)$ which has complex elements, all of them having magnitudes of unity. For the present we will consider an eight point transform. The square matrix $W$ can be represented by an $8 \times 8$ square array of numbers, because $n = 8$.

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\
0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\
0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{bmatrix}
$$

where the numbers give the phase angles of the complex numbers,

0 is 'east’ i.e. $0^\circ$
1 is 'north east’ i.e. $45^\circ$
2 is 'north’ i.e. $90^\circ$
3 is 'north west’ i.e. $135^\circ$
4 is 'west’ i.e. $180^\circ$
5 is 'south west’ i.e. $225^\circ$
6 is 'south’ i.e. $270^\circ$
7 is 'south east’ i.e. $315^\circ$

The 'backwards’ progression in the second row comes from the minus sign in the argument of the complex exponential and we’ve used the fact that $\exp(j\theta + 2\pi) = \exp(j\theta)$ to get all the phase angles to go into the range: $0^\circ$ and $360^\circ$.

The $W$ matrix top row and left column are all 'east’ pointing vectors, i.e. $1 + j0$. The second row ($m = 1$) shows the left entry progressively rotating clockwise each step by $45^\circ$ ($\pi/4$) because we have $n = 8$. The rotation is clockwise (the convention for such diagrams sets the negative direction in this way and this direction is needed because of the negative sign in the argument of the exponent in equation (2)). The third row’s vectors ($m = 2$) all rotate clockwise but with steps twice the size of those in the second row, for the fourth row ($m = 3$) it’s three times, etc.

How does the transform work? It is helpful to consider $x(k)$ being a 'pure’ $\cos + j \sin$ wave of a certain wavelength $p$. Then there is a simple operation done. The matrix multiplying operation being done in the left hand side of equation will then involve a series of simple operations where two complex numbers (we’ll temporarily call them $y$ and $z$) are multiplied together and they both have magnitudes of unity. We’ll put them in polar form:
\[ y = \cos(\theta) + j \sin(\theta) \] and \[ z = \cos(\phi) + j \sin(\phi) \]

the product \( yz \) is \( \cos(\theta + \phi) + j \sin(\theta + \phi) \). In other words we have to add \( \theta \) and \( \phi \).

For division we have \( y/z = \cos(\theta - \phi) + j \sin(\theta - \phi) \) and if \( \theta = \phi \) then \( y/z = \cos(0) + j \sin(0) = 1 \). This is, in effect, a 'cancellation' situation.

Recall that there is a negative sign in the argument of the complex exponent shown in equation (2). This indicates that if \( x(k) \) is 'sinusoidal'; equation (2) is an operation where each row of \( W \) is 'inspected' to find which rows generate a 'cancellation' (\( y/z \) above) and which rows do not.

When \( p = 8 \) we can say that the data \( x(k) \) can be represented (as in same way as the matrix above) as the sequence \((0, 1, 2, 3, 4, 5, 6, 7)\). Equation (2) operation multiplies them in turn by what’s in each row of \( W \). Using the above representation for \( W \) and replacing multiplication with adding, we get another pattern:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\
0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\
0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix} +
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\
0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\
0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6
\end{pmatrix}
\]

"(total)" :
\[
\begin{pmatrix}
0 \\
8 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Note that the 'additions' are modulo 8. The bottom column gives the 'vectorial' result and thus if we add 8 vectors of equal magnitude, with directions placed every 45°, 90° or 180° around the circle; then we get a zero result, whilst adding 8 unit vectors of equal magnitude all pointing 'east' give a result of 8.

The entries in the single column marked “(totals)” has the spectrum \( X(m) \) running from \( m = 0 \) (the DC component of the signal \( x(k) \)) to \( m = n - 1 \) as you go down the column. The conversion of \( m \) to frequency is done by \( f = m / (\text{the time interval between successive samples}) \).

Now we need to know how (3) changes when \( x(k) \) has been altered from \( x(k) = \exp(+j2\pi kp/n) \)
to \( x(k) = \exp(-j2\pi kp/n) \). The result is that the single 'peak' shifts from \( X(1) \) to \( X(7) \):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\
0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\
0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
8
\end{pmatrix}
\]

\( \text{"(total)"} : \)

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\
0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\
0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix}
\]

We now invoke (arguably) the most powerful theorem in the Fourier fold. It is the Linearity theorem which says:

\[
\mathcal{F}\{aG + bH\} = a\mathcal{F}\{G\} + b\mathcal{F}\{H\}
\]

where \( a, b \) are constants and \( G, H \) are complex functions.

The transform of a linear combination of functions is equal to the linear combination of transforms (the proviso is that the additions and multiplications must be in complex algebra where necessary). The truth of this theorem is evident after one inspects the nature on the summing operation in equation (2).

Here we use the linearity theorem to find the transform of \( \cos \theta \). We use \( \cos \theta = 0.5 \exp(+j\theta) + 0.5 \exp(-j\theta) \) and find that \( X(m) \) for this \( x() \) has all zeroes except \( X(1) = 8/2 = 4 \) and also \( X(7) = 8/2 = 4 \).

There is an important symmetry here which applies to real variables (such as this one: \( \cos \theta \) or any one you could 'feed' to the PicoScope). Its transform may be complex but a plot of its magnitudes has a symmetry about the channel which is a half of the number of points \( n \). For \( n = 8 \) and a cosine wave we have \( X(4 - 3) = X(4 + 3) \).

The PicoScope has a choice of 'the number of spectrum bands'. This is in effect \( 2n \) and the significance of this is that there is no use showing the magnitude of the transform beyond the right side of the screen; it’s just the mirror image of the part already on the screen.

We have used the Linearity theorem above but there is another which should be mentioned here. The Time Shifting theorem gives the effect of sampling the inputed waveform at different times \( t_0 \). If the time origin shifts by \( t_0 \), then the Fourier transform is multiplied by \( \exp(-j2\pi ft_0) \). This means that the magnitude does not change, only the phase changes and what is more telling is that nothing changes on the PicoScope screen showing the spectrum because this screen does not indicate the phases of spectrum components. Sine and cosine waves look the same on the spectrum display!
For further insight it is worthwhile to look back at equation (3). It says the same thing as equation (2). We examine the effect of time shifting; by replacing the input vector \( x(k) = (0, 1, 2, 3, 4, 5, 6, 7) \), by the cyclic shifted sequence \( (7, 0, 1, 2, 3, 4, 5, 6) \).

The second row of equation (3) now reads:

\[
(0 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1) + (7 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6) = (7 \ 7 \ 7 \ 7 \ 7 \ 7 \ 7 \ 7) \quad \text{"(total)" : (8)}
\]

There is apparently no change to the result. It is still a vector of magnitude 8 but instead of pointing 'east', it now points 'south east'. It represents the complex number \( 8(1/\sqrt{2} - j/\sqrt{2}) \), in accordance with the time shifting theorem. All other \( X(m) \) remain equal to zero.

Look in the reference book of E.O.Brigham [1] for more information on other theorems.

### B Sydney’s radio stations

Radio and Television Broadcasting Stations (January 2008)

<table>
<thead>
<tr>
<th>Call sign</th>
<th>Frequency (kHz-MF)</th>
<th>Purpose</th>
<th>Polarisation</th>
<th>Latitude</th>
<th>Longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>2RN</td>
<td>576 (kHz-MF) National</td>
<td>V OD</td>
<td>50k</td>
<td>-33 56 37</td>
<td>150 53 03</td>
</tr>
<tr>
<td>2PB</td>
<td>630 (kHz-MF) National</td>
<td>V OD</td>
<td>10k</td>
<td>-33 56 36</td>
<td>150 53 11</td>
</tr>
<tr>
<td>2BL</td>
<td>702 (kHz-MF) National</td>
<td>V OD</td>
<td>50k</td>
<td>-33 56 37</td>
<td>150 53 03</td>
</tr>
<tr>
<td>2GB</td>
<td>873 (kHz-MF) Commercial</td>
<td>V OD</td>
<td>5k</td>
<td>-33 49 29</td>
<td>151 04 55</td>
</tr>
<tr>
<td>2UE</td>
<td>954 (kHz-MF) Commercial</td>
<td>V OD</td>
<td>5k</td>
<td>-33 51 15</td>
<td>151 03 54</td>
</tr>
<tr>
<td>2KY</td>
<td>1017 (kHz-MF) Commercial</td>
<td>V OD</td>
<td>5k</td>
<td>-33 50 23</td>
<td>151 03 40</td>
</tr>
<tr>
<td>2EA</td>
<td>1107 (kHz-MF) National</td>
<td>V OD</td>
<td>5k</td>
<td>-33 50 29</td>
<td>151 04 36</td>
</tr>
<tr>
<td>2CH</td>
<td>1170 (kHz-MF) Commercial</td>
<td>V OD</td>
<td>5k</td>
<td>-33 50 29</td>
<td>151 04 36</td>
</tr>
<tr>
<td>2RPH</td>
<td>1224 (MHz-VHF) Community</td>
<td>V DA</td>
<td>5k</td>
<td>-33 48 34</td>
<td>150 54 49</td>
</tr>
<tr>
<td>2SM</td>
<td>1269 (MHz-VHF) Commercial</td>
<td>V OD</td>
<td>5k</td>
<td>-33 50 01</td>
<td>151 04 12</td>
</tr>
<tr>
<td>2MHM</td>
<td>1539 (MHz-VHF) HPON</td>
<td>V OD</td>
<td>1k</td>
<td>-33 50 57</td>
<td>151 04 48</td>
</tr>
<tr>
<td>2MFM</td>
<td>92.1 (MHz-VHF) Community</td>
<td>M OD</td>
<td>6k</td>
<td>-33 51 53</td>
<td>151 12 37</td>
</tr>
<tr>
<td>2ABCFM</td>
<td>92.9 (MHz-VHF) National</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
<tr>
<td>2LND</td>
<td>93.7 (MHz-VHF) Community</td>
<td>M DA</td>
<td>50k</td>
<td>-33 52 29</td>
<td>151 12 30</td>
</tr>
<tr>
<td>2FBI</td>
<td>94.5 (MHz-VHF) Community</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
<tr>
<td>2PTV</td>
<td>95.3 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
<tr>
<td>2SYD</td>
<td>96.9 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
<tr>
<td>2SBFSFM</td>
<td>97.7 (MHz-VHF) National</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
<tr>
<td>2OOO</td>
<td>98.5 (MHz-VHF) Community</td>
<td>M DA</td>
<td>25k</td>
<td>-33 50 28</td>
<td>151 12 21</td>
</tr>
<tr>
<td>2RPH</td>
<td>100.5 (MHz-VHF) Community</td>
<td>M DA</td>
<td>1k</td>
<td>-33 52 39</td>
<td>151 13 17</td>
</tr>
<tr>
<td>2UUS</td>
<td>101.7 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 48 25</td>
<td>151 10 46</td>
</tr>
<tr>
<td>2MBS</td>
<td>102.5 (MHz-VHF) Community</td>
<td>M DA</td>
<td>50k</td>
<td>-33 51 56</td>
<td>151 12 36</td>
</tr>
<tr>
<td>2CBA</td>
<td>103.2 (MHz-VHF) Community</td>
<td>M DA</td>
<td>50k</td>
<td>-33 48 25</td>
<td>151 10 49</td>
</tr>
<tr>
<td>2DAY</td>
<td>104.1 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 48 25</td>
<td>151 10 46</td>
</tr>
<tr>
<td>2DAY</td>
<td>104.1 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
<tr>
<td>2MMEM</td>
<td>104.9 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 48 25</td>
<td>151 10 46</td>
</tr>
<tr>
<td>2MMEM</td>
<td>104.9 (MHz-VHF) Commercial</td>
<td>M DA</td>
<td>150k</td>
<td>-33 49 17</td>
<td>151 11 01</td>
</tr>
</tbody>
</table>
References
