Some reflections on mathematics, mathematics education and mathematicians

Pan Shengliang
Department of Mathematics
East China Normal University
Shanghai 200062
People's Republic of China

slpan@math.ecnu.edu.cn

Introduction

In this paper some reflections on mathematics, mathematics education and mathematicians are presented. We have found that there is an imbalance between the three. Special emphasis is placed on the idea that mathematics should be taught enthusiastically, and that we as mathematics educators should strive to make the students more and more interested in mathematics.

This paper is organized as follows: firstly it will discuss briefly the nature of mathematics, mathematics education and mathematicians. Next it will discuss the current state of mathematics and mathematics education in universities. Finally it will discuss the role of mathematicians in the development of mathematics and mathematics education. The final section discusses how methods learnt at the University of Sydney may be useful for my own teaching of mathematics in China.

What is mathematics?

One of the oldest of all fields of study is that now known as mathematics. Often referred to, used, praised, and disparaged, it has long been one of the most central components of human thought, yet how many of us could describe what mathematics really is?

Roughly speaking, mathematics is the science which deals with numbers, diagrams, social systems and natural phenomena.

Kasner and Newman's point of view is that, 'Mathematics is the science which uses easy words for hard ideas'. According to Kant (1724-1804), 'the science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience'.

In fact, like other sciences, mathematics reflects the laws of the material world around us and serves as a powerful instructional tool for understanding nature. Mathematics reveals the hidden patterns that empower us to understand better the information-laden world in which we live. As a science of abstract objects, mathematics relies on logic rather than on observation for the purpose of stating truths, yet employs observation, simulation, and even experimentation as a means of discovering truth. Through its results, mathematics offers science both a foundation of truth and a standard of certainty. Mathematics offers distinctive modes of thought which are both versatile and powerful; including modeling, abstraction, optimization, logical analysis, inference from data, and use of symbols. Mathematics enables us to read critically, to identify fallacies, to detect bias, to assess risk, and to suggest alternatives. The resolution of mathematical problems supplies people with techniques, which can be used in different areas, even to everyday problems. Mathematical thinking is logical and strict, intuitive and creative, dynamic and changing.

As a mathematician, I would further say that mathematics is an awe-inspiring science, filled with mystery and wonder, and brimming with opportunities to make triumphant intellectual discoveries. It is truly one of the highest points of humankind's achievements. It offers everyone the chance to get a glimpse of the nature of the universe around us, and to learn and understand something more about the human condition. It is the most universal of our languages and the most useful of our tools; it is the most beautiful of our music and the most elegant of our poems; it is silent harmony and form; it is to some people an art.

As an undergraduate student I found the following results very beautiful:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...+\frac{1}{n^2}+...=\frac{\pi^2}{6},$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...+\frac{(-1)^n}{n^2}+...=\frac{\pi^2}{12}.$$

What do you feel about them? What about:

$$1+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}+...=?$$

To the best of my knowledge, this is an open problem.

What is mathematics education?

"Mathematics education … is devoted to the study of how people learn and do mathematics of any kind and of how this learning and doing can be influenced and fostered through teaching strategies." (Dörfler 2000) Mathematics education is fashioned to provide appropriate mathematical knowledge, understanding, and skills to diverse student populations. "Thus 'mathematics education' is clearly a distinct area of human activity whose content, objective and goal is mathematics itself at different levels and in different forms." This "should not be taken thereby to suggest an existence of mathematics outside and independent of the respective activities. Keeping this in mind, mathematics education can metaphorically be considered as the study of the relationships between mathematics and human beings, both taken in their whole variety." As a consequence, "mathematics education has to be interested in the early counting activities of children" and in the production of a proof of a theorem in Riemannian Geometry as well. "Acceptance of this description of mathematics education leads to acknowledging a very special role for mathematics education as a scientific discipline contributing to mathematics." (Dörfler 2000)

A central goal for all of mathematics education is the development of the general abilities arising from mathematical study in all students. Students should acquire an effective awareness of both the spirit and the uses of mathematics and should grow in their understanding of the breadth of the mathematical sciences and their deep interconnecting principles. In particular, the general abilities include: the ability to explore, conjecture, and reason logically; to solve non-routine problems; to connect concepts within mathematics and between mathematics and real world situations; and to read, write, listen, and speak mathematically. Acquiring these general abilities requires the development of personal self-confidence and a disposition to pursue, and to use quantitative and spatial information in solving problems and making decisions. Students' flexibility, perseverance, interest, curiosity, and creativity also affect the acquisition of general mathematics-related abilities.

What is a mathematician?

In **my** opinion, a **mathematician** is a person who not only studies mathematics but also does research in mathematics. Some mathematicians do research as well as teach mathematics.

Mathematicians are typically interested in finding and describing patterns which may have originally arisen from problems of calculation, but have now been abstracted to become their own problems. From much published research work of mathematicians, it may look as if the primary approach of a mathematician is to start with some given assumptions, often called axioms, and then proceed to prove other facts which follow from the assumptions according to exact rules of logic. That, however, is the finished product that gets published; it is not work in progress.

Contrary to popular belief, mathematicians are not typically any better at adding or subtracting numbers, or figuring the tip on a restaurant bill, than members of any other profession, in fact, some of the best mathematicians are notoriously bad at these tasks!

A mathematician uses numbers and symbols in many ways, from creating new theories to translating scientific and technical problems into mathematical terms. There are two types of researching mathematicians: the theoretical mathematicians, who work with pure mathematics to develop and discover new mathematical principles and theories without regard to their possible applications; and applied mathematicians, who use mathematical methods to solve practical problems in diverse areas.

To some extent, people give differing definitions of the mathematician, probably owing to the nature of their own work. We cite some examples.

- A mathematician is a machine for turning coffee into theorems, P. Erdos (1913-1996) (Rose 1988).
- A person who can, within a year, solve

  $$x^2-92y^2=1,$$

  is a mathematician, Brahmagupta (598-668) (Ernest 1991).
- I have hardly ever known a mathematician who was capable of reasoning, Plato (429-347 BC) (Rose 1988).
- To be a scholar of mathematics you must be born with talent, insight, concentration, taste, luck, drive and the ability to visualize and guess (Halmos 1985).
- Mathematics is a dangerous profession; an appreciable proportion of us go mad, and then this particular event would be quite likely (Littlewood 1953).

- Mathematicians are like lovers. Grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another, B. B. Fontenelle (1657-1757) (Larney 1975).
- Young men should prove theorems, old men should write books (Hardy 1941).

The present situation of mathematics and mathematics education in the universities

Mathematics has developed into an immense system comprising, according to *Mathematical Reviews* (1992), more than 60 categories of mathematical activity. There are currently about 200,000 new theorems published every year. In fact, the publication of new discoveries in mathematics continues at an immense rate in hundreds of scientific journals, many of them devoted to mathematics and many devoted to subjects to which mathematics is applied. Mathematics by nature is both a pure, theoretical adventure of the mind and a practically applied science. This dichotomy allows theoretical mathematicians to 'do mathematics for mathematics sake' and applied mathematicians to 'use mathematics as a tool' to solve real problems. Mathematics has infiltrated almost every discipline of both the natural and the social sciences. The usefulness of mathematics in physics, mechanics, astronomy, computer science, military science, engineering, industries, etc. is evident. The combination of mathematics with biology forms biological mathematics; with ecology creates ecological mathematics; with economics creates mathematical economics; with finance forms mathematical finance (also called financial mathematics), etc. This infiltration in turn helps mathematics evolve. Mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, analysis, proof and with mathematical models generally.

Mathematical ideas have an unusually long life. The Babylonian solution for quadratic equations is as useful today as it was 4,000 years ago. Mathematical ideas are both enduring and expanding. New mathematical ideas are built on older mathematical ideas or propositions. An analogy can be made to the notion of 'continuous improvement', where current practices (in this case, ideas) can be improved upon, given new effort and time. Usually improvement does not occur without effort, and it typically does not occur quickly. Often problems are solved, and new areas of mathematics created, by looking at old problems in new ways.

The present situation of mathematics is just as Robert Oppenheimer (1904-1967) said, 'Today, it is not only that our kings do not know mathematics, but our philosophers do not know mathematics and – to go a step further – our mathematicians do not know mathematics'.

There are an enormous number of research areas of mathematics. Therefore, it is very difficult for students to understand and embark upon one of the new areas of mathematics; they may feel that studying mathematics is a tiresome matter.

Unfortunately, many people have been frightened away from mathematics by teachers who were perhaps well-meaning, but whose understanding of mathematics was at best nasty, brutish, and short. Many have been led to believe that mathematics is a cold and lifeless subject, filled with formulae and theorems that have only a passing connection to one another, and no connection whatsoever to the outside world. Many have been taught that the best way to learn mathematics is to memorize one formula after another. Mathematics comes over as a dead subject, studied only by people of god-like intellectual stature who have either died or have ensconced themselves in impenetrable ivory towers into which there is no possible trespass, in effect sealing themselves off from humanity.

In China, the difficulties our students have when learning Mathematics have become more and more evident. Several worrying aspects of the problem have been identified. The first year students of mathematics departments are usually unable to write down a correct proof of a proposition. They only know that the conclusion of a proposition is correct, but they do not know how to give a detailed proof of the proposition correctly. If they take a multiple choice examination, they can get a very high score, but if they take an examination requiring mathematical analysis, they score poorly. The difficulties are: grave deficiencies in the mathematical knowledge new students bring from high school, lack of motivation, and lack of training in problem solving skills amongst the students. A standard approach to the teaching of mathematics has generally been in use from the time of Diesterweg, but it has almost never been used in teaching higher mathematics: it has been assumed that for undergraduate students, mathematics should be taught in a strict, logical and deductive way. For example, the teaching of mathematical disciplines in universities begins with a detailed account of logical and set-theoretic foundations; the linear algebra courses began with the general theory of vector spaces, etc. However, modern experience has shown that the strict logical and deductive teaching of mathematics is inappropriate for undergraduate mathematics teaching.

The role of mathematicians in the development of mathematics and mathematics education

Now, mathematics must be made alive! Mathematics must be made to live again!

We should let the students of mathematics know that they are not the only ones who have had difficulties with mathematics. It took years for Archimedes, Newton, Leibnitz, Gauss, Euler, and Lagrange to understand some of the concepts that nowadays will take a first semester calculus student a week to learn. It took centuries for mathematics to reach the point at which it rests today, and only a handful of the towering 'giants' in mathematics' history have left as a legacy any more than a single small theorem tucked away in some musty old book on a forgotten library shelf.

We are all human.

To the student we should say, 'You can do mathematics!' It takes practice, but you can do mathematics. It takes work, but you can do mathematics. It is accessible. It is understandable. You can do it. If you spend enough time you can become a mathematician, even a great mathematician. You can read Alfréd Rényi, 'If I feel unhappy, I do mathematics. If I am happy, I do mathematics to keep happy'; or Siméon Poisson (1781-1840), 'Life is good for only two things, discovering mathematics and teaching mathematics', and Henri Poincaré, 'Mathematical discoveries, small or great are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labour, both conscious and subconscious'.

Very likely, mathematicians do not regard this as their role, despite the fact that many of them are involved in the teaching of mathematics at the university level. They themselves consider their main task to be to develop mathematical theories or to apply them in various contexts. This itself does give a possible research area for mathematics education – to investigate the mathematical activity of mathematicians, its conditions, forms, means, goals, intentions, etc. However, to study mathematical research in this way would constitute a fairly passive approach.

How to use the methods learnt here to aid my own teaching in the future?

Thanks to the lectures given by the teachers at the Centre of English Teaching and the School of Mathematics and Statistics of the University of Sydney, I have learnt some teaching methods through watching the teaching here. I will try to combine these methods with those I have learnt in China, and to use them in my practice in the future.

I will concentrate on the possible role for mathematics in mathematics education, its opportunities and obstacles. I take as mathematics that which in the course of history has evolved as the product of the activity of mathematicians and has to a great extent been standardised, conventionalised and corroborated by extended experience and manifold practical usages. It is the concepts, methods, notations, basic assumptions, etc. which rather unanimously are considered to be mathematical that make up 'mathematics'. I admit that possibly those concepts and methods might differ depending on basic views about the nature of mathematics. Yet, the general tenet thereby is not weakened: in mathematics education one has to study and investigate mathematics as it is currently practiced, produced and used in all its forms. "We should never lose sight of the fact that mathematics education is also about human beings. Possibly, this aspect of mathematics education could be termed 'mathematicology'. This kind of 'mathematicology' thus has to investigate mathematics as one of the two sides of mathematical activity and should not be detached from the people who carry it out. Topics for this kind of research to pursue could be:

- quality and structure of mathematical concepts and theories;
- methods of concept formation;
- notational systems;
- symbols and symbolization in mathematics;
- the role of diagrams;
- abstraction and generalization;
- mathematization and applications of mathematics;
- specificities of the mathematical discourse; and
- the role of cultural tools, means and media." (Dörfler 2000)

"In short, mathematicology should try to develop theoretical descriptions and models of mathematical activities and processes. This will involve reflecting upon exactly what it is that mathematicians and learners of mathematics do. Once this has been made explicit, it will be open to being monitored, changed and influenced." "Of course, there is already available a comprehensive body of research of this kind which perhaps is not sufficiently taken into account, and which deserves to be extended in a systematic way." (Dörfler 2000)

"This deepened understanding is in my view not just knowledge of mathematics per se but the kind of knowledge about mathematics that I want to indicate by the above description of 'mathematicology'." Mathematicology would need to "integrate the expertise of mathematics education researchers with that of teachers and also of students." In the kind of mathematicology I envisage, "the focus would be on human subjects as the agents of the mathematical activities. Foundational research to the contrary excludes the human actor completely. The methods employed in 'mathematicology-style' studies might vary from theoretical analysis of mathematical texts to empirical studies of the mathematical activities of individuals and groups. The overarching goal is to gain a deepened understanding of mathematics as a human activity." (Dörfler 2000)

Acknowledgements

The author would like to thank The University of Sydney and China Scholarship Council who organized and carried out the program of 'Teaching Science in English'. And also to thank Associate Professors Mike King, Mary Peat and Lloyd Dawe and Dr Lindsay Grimison for their lectures, to thank Ms Sandra Britton and several mathematicians for their help and all the activities in the School of Mathematics and Statistics, to thank June Hammond and David Yardley for reading the drafts of this paper, and to thank our groups for the wonderful days in Sydney.

References

Anglin, W. S. (1980) Mathematics and history. *Mathematical Intelligencer*, **4**(4).

Davis, P. and Hersh, R. (1981) *The Mathematical Experience*. Boston: Birkhäuser.

Dörfler, W. (2000) Mathematics, Mathematics Education and Mathematicians: An Unbalanced Triangle. International Commission on Mathematical Instruction http://www.emis.de/mirror/IMU/ICMI/bulletin/49/Mathematics.html

Ernest, P. (1991) *The Philosophy of Mathematics.* London: The Falmer Press.

Eves, H. (1988) *Return to Mathematical Circles*. Boston: Prindle, Weber & Schmidt.

Halmos, P. R. (1985) *I Want To Be A Mathematician*. Washington: MAA Spectrum.

Hardy, G. R. (1941) *A Mathematician's Apology*. London, Cambridge University Press.

Kasner, E. and Newman, J. (1940) *Mathematics and the Imagination*. New York: Simon and Schuster.

Kulm, G. (Ed.) (1990) *Assessing Higher Order Thinking in Mathematics,* Washington: AAAS.

Larney, V. H. (1975) *Abstract Algebra: A First Course*. Boston: Prindle, Weber and Schmidt.

Littlewood, J. E. (1953) *A Mathematician's Miscellany*. Methuen and Co. Ltd.

Mathematical Intelligencer, **13**(1) (1991).

Mathematical Sciences Education Board (1989) *Everybody Counts: A Report to the Nation on the Future of Mathematics Education.* Washington, DC: National Academies Press.

Newman, J. R. (Ed.) (1956) *The World of Mathematics*. New York: Simon & Schuster.

Rose, N. (1988) *Mathematical Maxims and Minims*. Raleigh NC: Rome Press Inc.

Rowlands, S. and Graham, T. (1997) *The Nature of Mathematics and the State of Mathematics Education*. [Online] http://s13a.math.aca.mmu.ac.uk/Chreods/Issue_11/SRowlandsTGraham.html.

Schifter, D. (1996) A constructivist perspective on teaching and learning mathematics. In C. T. Fosnot (Ed.) *Constructivism: Theory, Perspectives and Practice*. New York, Teachers College Press.

Solow, D. (1990) *How to Read and Do Proofs: An Introduction to Mathematical Thought Processes*. New York: Wiley.

Wood, L. N. and Perrett, G. (1997) *Advanced Mathematical Discourse*. Sydney, Australia: University of Technology, Sydney.