Some thoughts on the way in which calculus is taught and learned

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Abstract

In this paper, the author reflects on the way in which calculus is taught and learned at Ocean University of China, and proposes some ideas for change. The changes suggested take account of students' differing abilities, and focus on a more student-centred approach. At the University of Sydney, the author has seen that dividing students into advanced and normal levels allows teachers to present the same material to two different groups, with a different emphasis and level of rigor for each group. So the first change suggested is to divide students into normal and advanced levels. Each group will have a distinctive focus and approach to suit the level of the student. Secondly, the author suggests that *MATLAB* be used as an aid for teaching and learning. Teachers can use *MATLAB* as a tool for vivid and precise demonstration. Students can use *MATLAB* as a tool for exploring with various concepts in calculus. The use of *MATLAB* also allows some simple mathematical modeling to be included. These changes are expected to make students more involved and active in learning. The third suggestion is to include some group work in order to develop students' communication skills and team spirit.

The present situation of teaching and learning of calculus in my university

Course description
Calculus is a two-semester compulsory service course for first year students whose major is strongly related to physics. It aims to provide students with a clear understanding of the concepts, theories, applications and principles of calculus, to develop computation skills, logical reasoning and problem-solving abilities, and to meet student needs for further courses. This course covers single variable differential and integral calculus, ordinary differential equations, analytic geometry, multivariable differential and integral calculus, infinite series and parametric generalized integration. The total number of lecture and tutorial hours is 180. Their proportion is about 3:1. Additionally, there is a fixed consultation time of two hours each week.

The current teaching and learning approach
The current teaching approach is teacher-centred. Teachers present lectures by introducing definitions, proving properties and theorems and giving some examples for computations and applications. On most occasions, we give the physics or geometric background of a concept before we write down its definition on the blackboard.

Students are asked to hand in their homework once a week. Teachers correct one third to one half of it and make some comments during the tutorial hours.

The primary ways students' learn calculus is by attending lectures, taking notes, reading textbooks, doing homework and asking questions. Good students have time to read some reference books and do more difficult problems. Many students are so busy struggling with their homework that they only look up related content in the textbook when they encounter difficulties with their homework.

The problems that we face
1. Each semester 10%-20% of the students fail.
2. Students are harder to teach than before.
3. Quite a number of students are learning at a surface level. They are ready to accept passively rather than enquire actively. They do not gain a deep understanding of key concepts and theories and some of them forget what they have learned soon after the final examination.

4. Some students feel calculus is too abstract and difficult to learn, and quickly lose interest.
5. The quality of teaching and learning calculus is decreasing.

Some thoughts for change

Divide students into two levels: Advanced and Normal

Teachers have found that current students are more difficult to teach. They are less motivated and not as hardworking as previous students. China is in transition with respect to social patterns. The values of people are becoming more diversified. Getting a good mark does not necessarily mean a bright future. Many of our students prefer more practical courses, for example, computer courses and English, which are more helpful to their future job-hunting. Like most of the universities in China, our university has increased enrolments by at least 30%-40% over the past few years. The gap between students' abilities has greatly increased. Even if the pace of the lectures has been slowed down, there are more students who cannot keep up. On the other hand, we really worry that good students would be bored and become impatient if we deliver unchallenging lectures.

'Teach students according to their aptitude' is an important educational thought of Confucius, who was a famous thinker, philosopher and educationalist in Chinese history. The situation could be expected to improve if we divide students into two levels: advanced and normal. Each level could then have its own focus and approach, relevant to the students' knowledge base and ability.

For normal level students, we will keep to the basic concepts, theories and applications, in preparation for meeting the needs of other courses. We will give more intuitive pictures of the key concepts and central ideas and omit some rigorous proofs. We will allow them more time to think mathematically by actively engaging them. That is, they will explore calculus by using mathematics software such as *MATLAB*, and build up and solve some simple mathematical models that are relevant to our real world. Although teaching content will be cut down, we expect our normal level students will learn more and be confident and interested in calculus rather than feeling anxious and frustrated.

For advanced level students, the content will keep most of its depth and breadth. Teachers will deliver lectures faster, explaining in less detail and leaving more space for students to think independently. The advanced level students, as well as the normal level students, will be asked to do some mathematical experiments.

Using *MATLAB* as a teaching and learning aid

MATLAB is a very powerful technical computing program. It integrates computation, visualisation, and programming in an easy-to-use environment. So it is a convenient aid in the teaching and learning of calculus. Using *MATLAB*, teachers can demonstrate curves and surfaces quickly and precisely. The graphs can be zoomed in and out, rotated from various angles and viewed from different perspectives. Students can observe the features of graphs more easily.

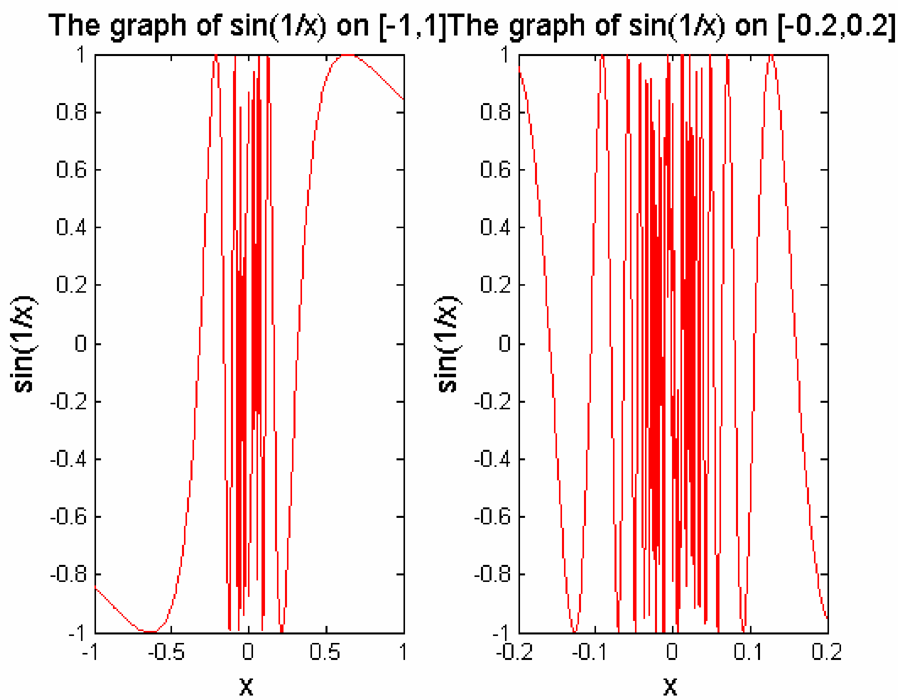
Examples:

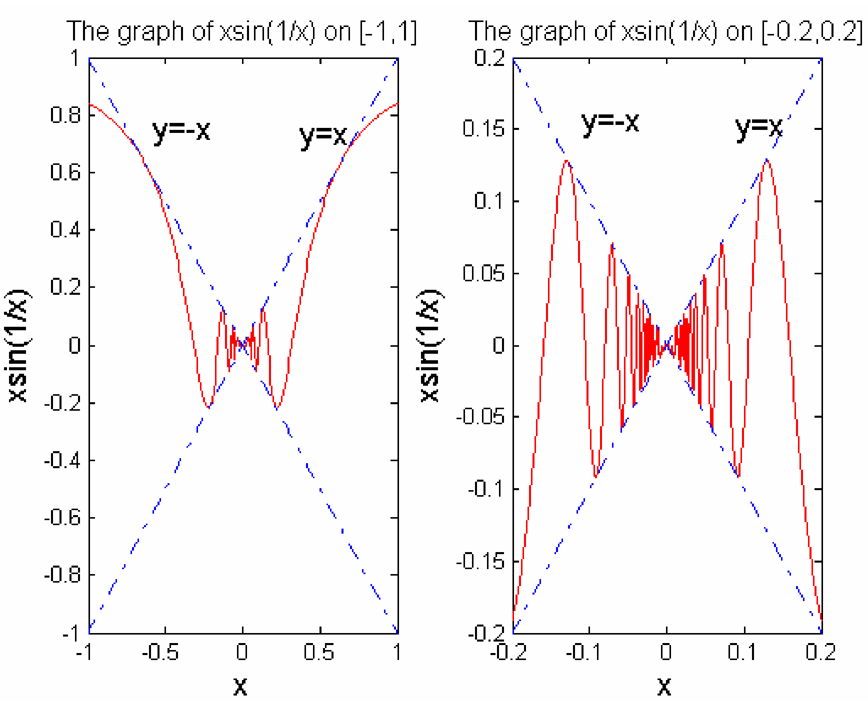
1. Some curves

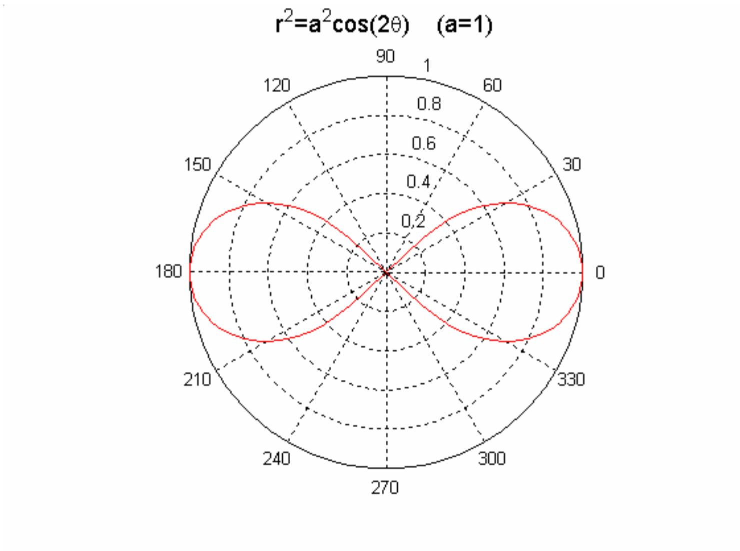
a. $y = \sin\frac{1}{x}$

b. $y = x\sin\frac{1}{x}$

c. $r^2 = a^2\cos 2\theta$



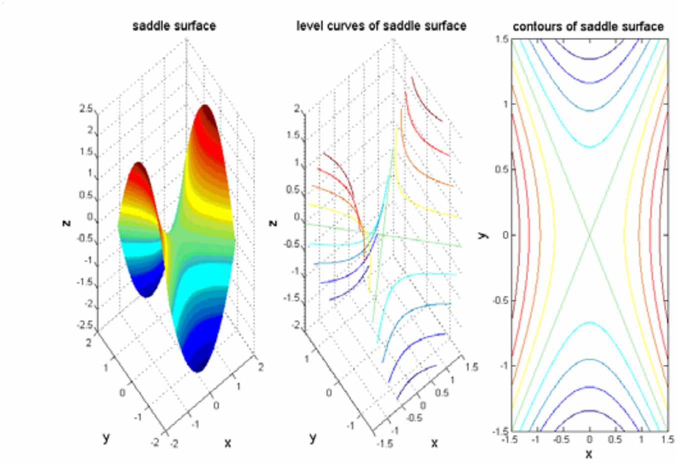


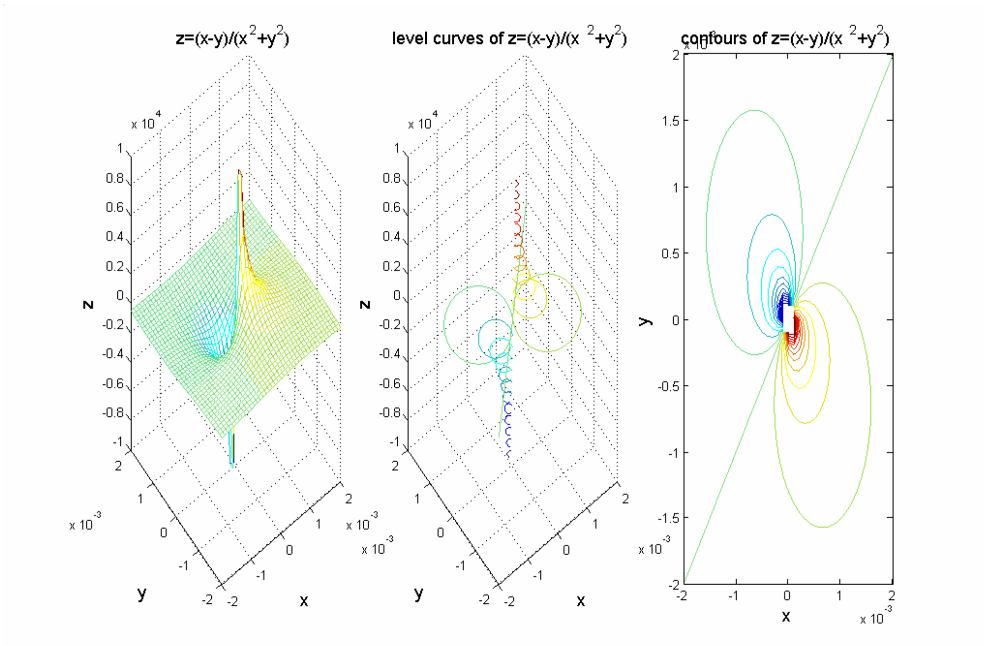


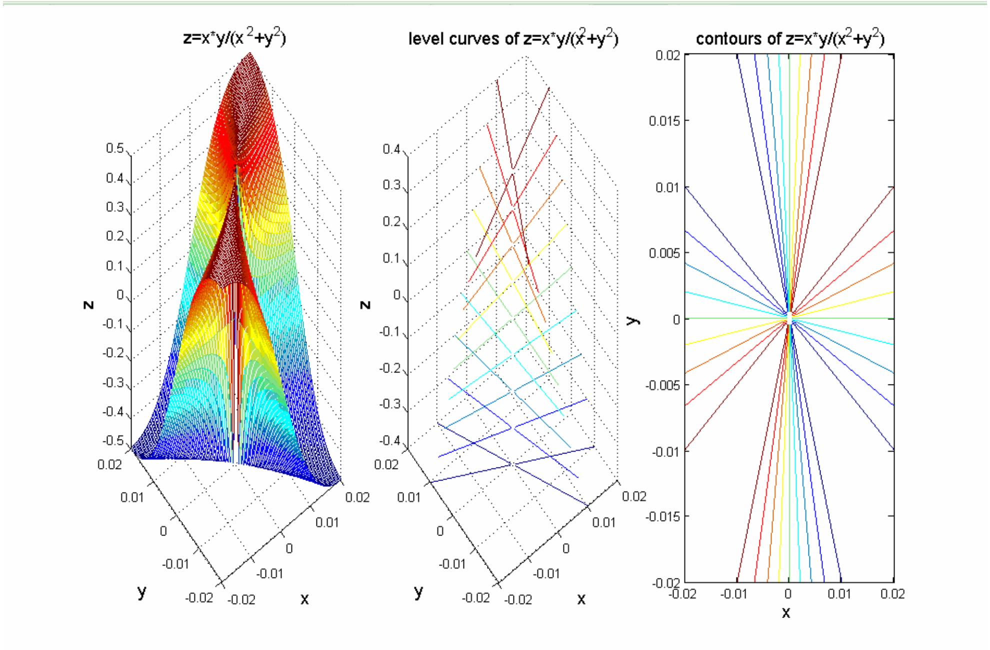
2. Some surfaces

a. $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ (*saddle surface*)

b. $z = \frac{x - y}{x^2 + y^2}$







The precise and vivid graphs, which *MATLAB* generates, are also helpful in analyzing the characteristics of graphs, such as symmetry, closed curves and special cross-sections.

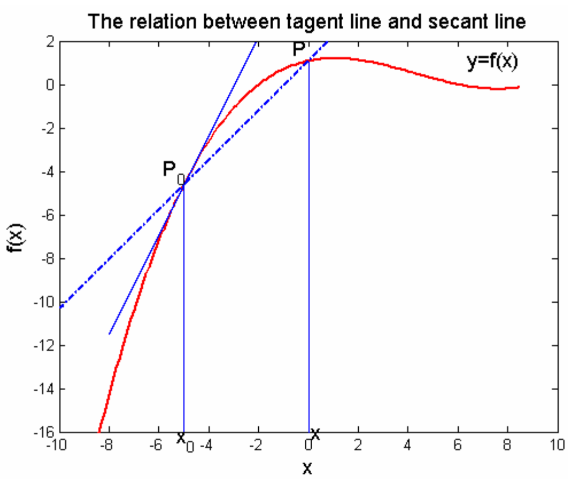
Using *MATLAB* as an intuitive tool for presenting and understanding some key concepts

Example 1

Tangent line of a curve at some point (animation)

The tangent line is the limit position of secant lines. We can use animation in *MATLAB* to demonstrate this fact.

Suppose P0 is a fixed point on the curve of *f(x)* and *P* is a moving point which is in the neighborhood of P_0 . Let *P* move along the curve of *f(x)*, then the corresponding secant line changes its position accordingly. When *P* is sufficiently close to P_0, the secant line approaches some fixed straight line, i.e. tangent line.



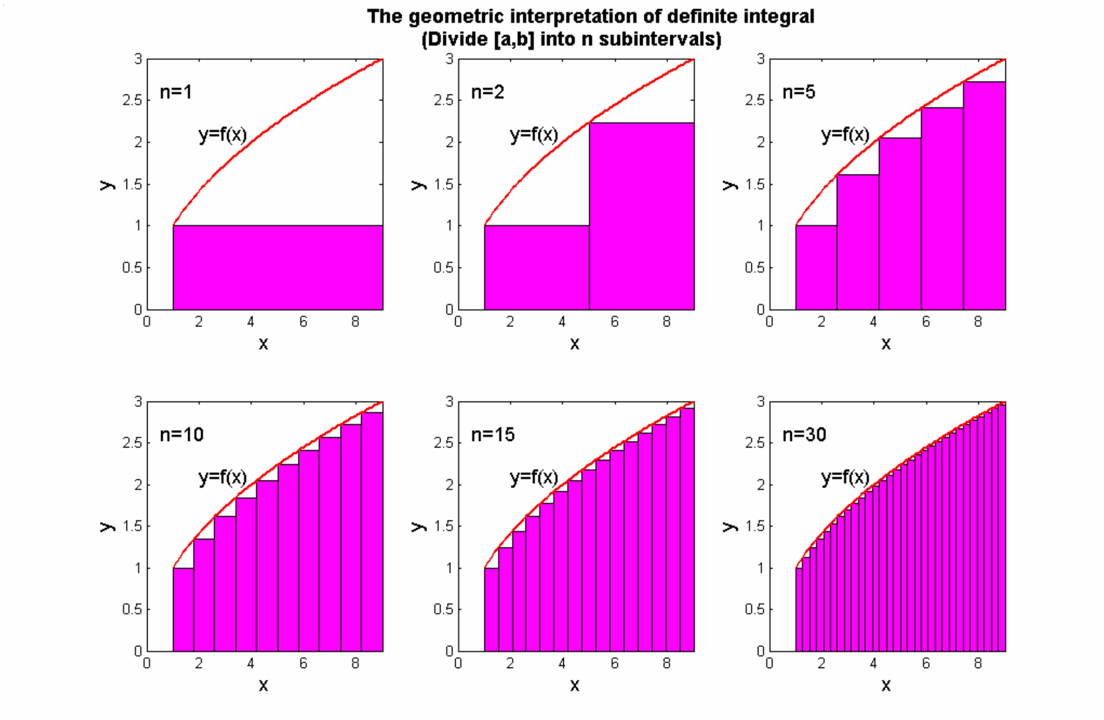
Example 2

Geometric interpretation of definite integral

Suppose *f(x)* is a positive and continuous function on *[a,b]*, then $\int_a^b f(x)dx$ is the area of the region which is bounded by the curve of *f(x)*, the *x* axis and the straight lines *x=a* and *x=b*.

To show this, we divide *[a,b]* into n subintervals. For each subinterval, we approximate the area of the small region by that of the corresponding shaded small rectangle. So we can use the sum of n rectangular areas to approximate the area under the curve *f(x)* over *[a,b]*.

By increasing n, we get better approximations to the area. When n becomes sufficiently large, the difference between them almost vanishes.



Using *MATLAB* as a tool for students to explore calculus

By plotting graphs or making computations with *MATLAB*, students can preview lessons more effectively than if they only use pen and paper. In our experience of teaching Fourier series, we have found many students could not perceive it well. Although teachers had strongly emphasized the importance of distinguishing between the function and its corresponding Fourier series, there were still a great number of students who took them as the same thing and did not analyze the value to which the Fourier series converges, nor when a function can be represented by its corresponding Fourier function.

Learning by doing is one of the important strategies for fostering a deep learning approach. Modern educational research tells us that students retain:

- 10% of what they read,
- 26% of what they hear,
- 30% of what they see,
- 50% of what they see and hear,
- 70% of what they say, and
- 90% of what they say as they do something.

The more involved, the more active and motivated students will be. On the basis of this idea, students will be asked to do a mathematical experiment before the lecture on Fourier series, by plotting and observing graphs of some functions and several of the partial sums of their corresponding Fourier series. Students will work in groups and report back their group conclusions.

Some tasks for students:

1. Observe the difference between the graph of a function *f(x)* and some of the partial sums $S_n(x)$ of its corresponding Fourier series *S(x)*.

$$f(x) \approx \frac{a_0}{2} = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}) = S(x)$$

The partial sums are:

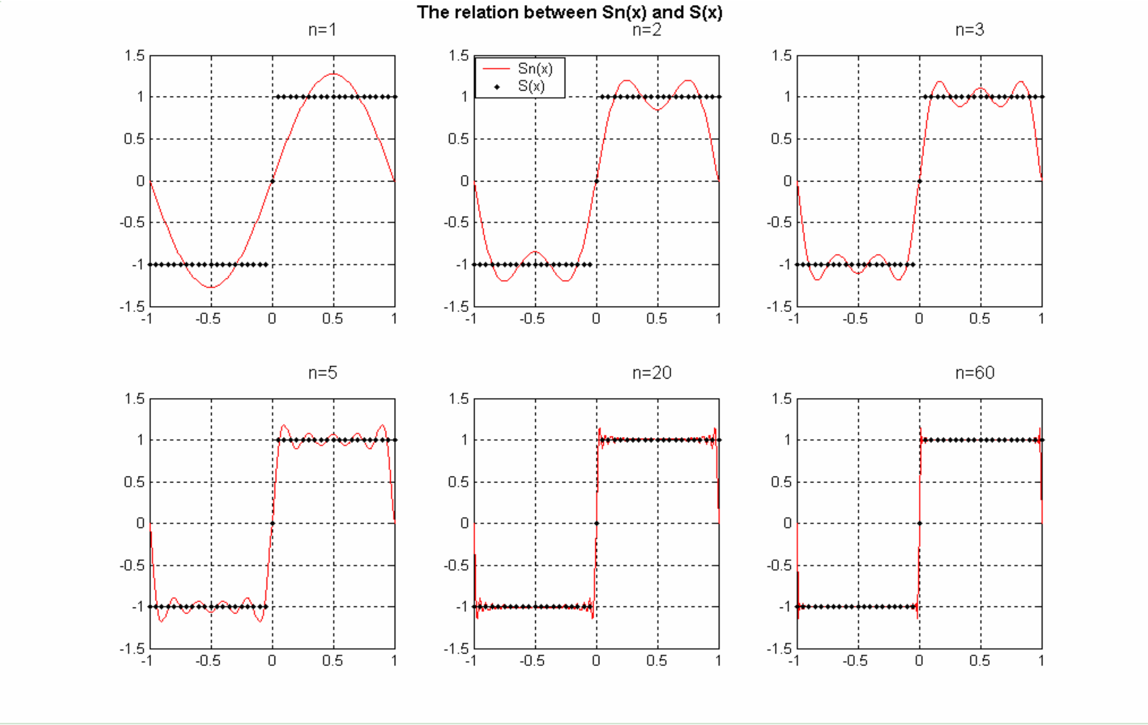
$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l}), \qquad n = 1,2,3,\ldots$$

2. Observe the trend of the partial sums $S_n(x)$ when n increases.
3. Observe the graphs of *f(x)* and *S(x)*.
4. Does the sum of Fourier series necessarily preserve the corresponding analytical properties which the partial sums $S_n(x)$ possess, such as continuity?

We can have students observe the graphs without asking them questions, and let them draw their own conclusions. That will be more challenging and student-centred.

Example:

$$Suppose \quad f(x)=\begin{cases}1 & 0<x\le 1\\ 0 & x=0\\ -1 & -1\le x<0\end{cases},$$



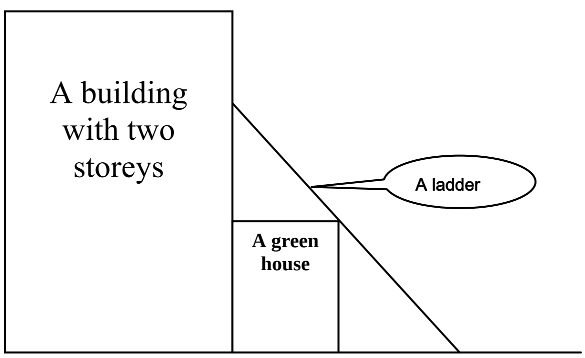
Including some simple mathematical modelling which is relevant to real life

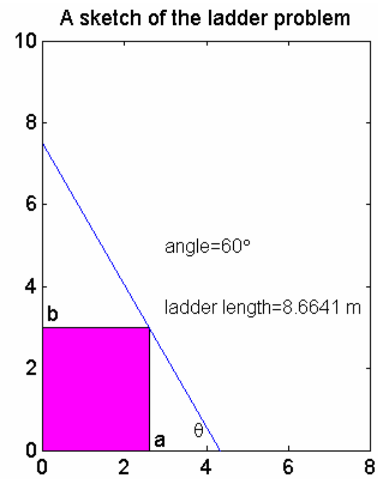
The reason that a lot of students do not like mathematics is not only because it is hard to learn, but also because it seems distant from real life. They cannot see its practical use.

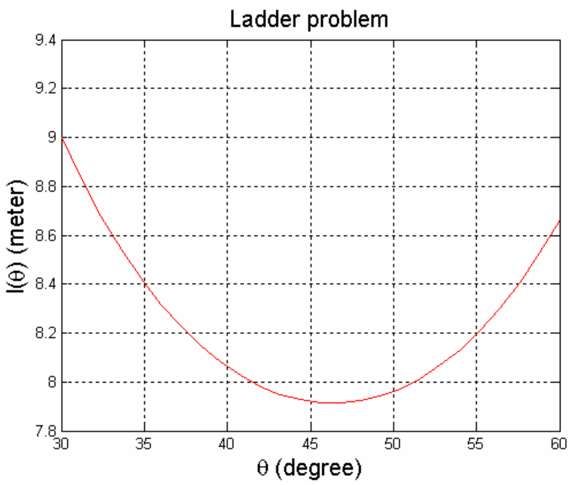
In fact, mathematics is very useful because it has infiltrated into every field of natural science, also engineering, social science and daily life. We can convince our students of this by including some simple mathematical modeling tasks involving calculus which are quite relevant to real life.

Example 1

Suppose there is a big house with two storeys. A greenhouse made of glass is next to it. There is a window on the bigger house above the glass house. Assume we lean a ladder against the wall over the glass house, so that we can clean the window by standing on top of the ladder. What is the shortest length of the ladder?







We will ask students to solve this problem using several methods.

Firstly, we need to establish a mathematical model.

Build up a Cartesian coordinate system as shown in the figure titled 'A sketch of the ladder problem'.

Suppose θ is the angle between the ladder and the ground and a, b are respectively the width and height of the greenhouse. The length of the ladder is l. Therefore,

$$l(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$$

Taking the real situation into account, we can reasonably assume that: $\theta \in [\frac{\pi}{6}, \frac{\pi}{3}]$

Hence the mathematical model is reduced to finding the minimum of $l(\theta)$ on $[\frac{\pi}{6}, \frac{\pi}{3}]$.

Students will be asked to solve the model using as many methods as possible.

Analytical solution

Find θ such that $l'(\theta)$ =0, i.e. $\theta = \tan^{-1}\sqrt[3]{\frac{b}{a}}$.

Numerical solution

Use the *MATLAB* command *fmin*.

Graphical solution

Plot the graph of $l(\theta)$ and estimate θ such that $l(\theta)$ is the minimum on $[\frac{\pi}{6}, \frac{\pi}{3}]$.

For a practical problem, it is usually very hard to find an analytical solution. So we will strongly recommend the graphical method, which is one of the greatest advantages of *MATLAB*.

In addition, we will give the following optional task to students:

Use animation in *MATLAB* to display the position, the length of the ladder and the corresponding angle simultaneously.

Although this task is not necessary for solving the problem, students will be excited about the animation and hence take more delight in using *MATLAB*.

Example 2

Suppose we ignore the year of a person's birthday. What is the probability *p(n)* of at least two people having the same birthday in a group of n people? (n<=365)

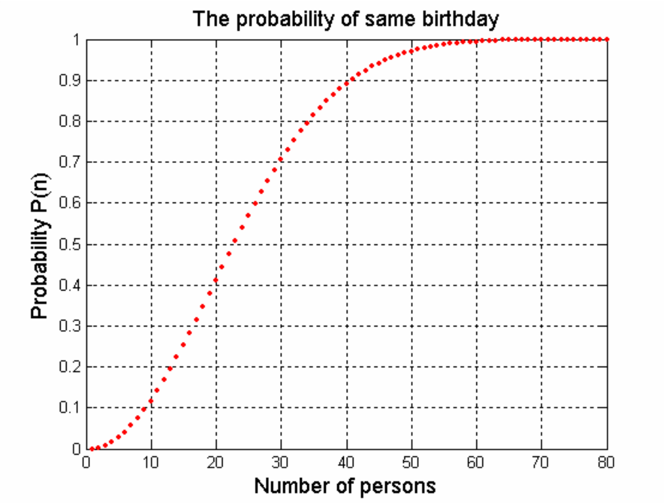
Consider the simplified case:

Assume there are 365 days in a year.

It is not difficult to find its analytical solution.

$$P(n) = 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}$$

By using *MATLAB*, we can easily calculate the probabilities of at least two persons having the same birthday in a group of different n people (2≤n≤365) and display the graph.



Another fascinating approach is to use stochastic simulation to solve this problem. We can generate n uniformly distributed random integers between 2 and 365, which stand for the birthdays of n different persons. After that, we can check whether there exist at least two persons who have the same birthday. Using a loop statement, we can repeat the procedure m times and figure out the frequency. When m is sufficiently large, the frequency will be a good approximation for the corresponding probability.

Including group work

In today's world, communication and cooperation have become increasingly important. Unfortunately, Chinese students have limited opportunities to be involved in this kind of training. Most of them have never given a presentation. Generally speaking, their oral communication skills are poor, as are their writing skills. Additionally, with the implementation of the one-child policy in China, more students become self-centred and lack team spirit. Some educationalists worry about this situation very much. But we should not just blame our students. Instead, we should take responsibility for creating more opportunities for them to cooperate. Thus, group work has great significance for Chinese students. It is not only a good approach for student-centred learning, but a good way to cultivate their generic skills, especially presentation and writing skills. Students are expected to learn cooperation by cooperating.

Each group will consist of 4-5 students and have a leader. Every group member will receive the same marks. Some possible group work could include:

1. correcting assignments with each other;
2. learning important concepts by doing multiple-choice questions and giving examples or counter-examples;
3. developing and solving simple mathematical modeling tasks;
4. writing group reports; and
5. giving group presentations.

Adjusting assessment

Assessment is a significant component of teaching and learning. We will adjust current assessment to take account of the new group work component.

The new assessment will consist of:

1. Homework 10%
2. Semi-term examination 20%
3. Final examination 60%
4. Group report and group presentation 10%

In the past, the final examination contributed 70% to the assessment. In future, we will take 10% from the final examination and allocate it to the group report and group presentation. Although the group work is very important and meaningful, we had better be cautious with it, because some problems will probably arise.

Firstly, there is a heavy workload involved in the assessment for group presentation. Usually, our class size is 90 students. If each group consists of five students, there will 18 groups. If each group is permitted 20 minutes for its presentation, six hours will be needed. Furthermore, it is difficult for students to control the timing of their presentations, especially at the beginning. In addition, it is hard for teachers to give an entirely objective mark to a group presentation. So in order to implement the group presentation and its assessment smoothly, teachers should read the group reports carefully, identify their problems and talk to each group before the presentation. As to the fairness of the assessment for group presentation, we can take the weighted average mark, which will consist of the teacher's mark and other groups' mark.

Summary

The new approach will give students more autonomy for their learning and we hope more of our students will develop independent learning skills which are important for life-long learning. It also provides students with a new way to study mathematics. But we may come across some problems.

1. In order to guarantee the learning outcomes, we will leave 2 hours laboratory work each week for students' to get familiar with *MATLAB* or do "mathematical experiments".

2. When we develop students' experience of doing mathematics, we should be careful not to weaken mathematical thinking, such as abstract and critical thinking. Students will be encouraged to use their hands as well as their heads in their learning. Special attention should be paid to advanced level students.

3. Some students will probably be lazy and dependent on others when doing group work. Hence they will achieve trivial and superficial learning outcomes. Teachers should often remind them to be independent and maintain self-respect. Each group member can randomly be asked to give his or her group presentation.

4. A most important prerequisite for the successful implementation of the new approach is that students should take more responsibility for their own learning. They should show initiative and motivation. Otherwise, the desired outcomes of teaching and learning will not be satisfied.

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