How to teach *Linear Algebra* effectively

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Abstract

Linear Algebra is a very important course for university students, but it's also assumed to be quite difficult from the students' point of view. In this context, how to teach linear algebra in a more effective way is obviously a serious problem. This paper is written to express some thoughts on this issue by the author. In this paper, optimal teaching sequence, usage of computers and applications of linear algebra are discussed.

Description of the course

In Chang'an University, *Linear Algebra* is a compulsory course for most 2nd year students with about 40 lecture hours. Every year around 4,000 students take this course, and about 20 teachers give the lectures. It covers content areas such as determinants, matrices, vector spaces, systems of linear equations, similar matrices and quadratic forms. All engineering students have a uniform set of examination problems, while students who major in accounting have another set. There are no planned tutorials for the students but before the final examination many teachers often manage office hours for students with difficulties. Though calculation is a big part of this course, most teachers don't teach their students any algorithms or usage of computers. Few applications are mentioned.

Not surprisingly, *Linear Algebra* is abstract and difficult for most students. Maybe this is because of its mathematical essence. But our approach has made it even worse. Our students complain that *Linear Algebra* is difficult, dull, and they don't know how to use it, or even whether it is useful. Since algebra is really very useful in many different areas, we must have given our students some wrong impressions about linear algebra. And we need of a better way to teach this important course more effectively. This is not an easy problem by any means, but I hope my thinking here will be helpful.

On the sequence of teaching material

In general, *Linear Algebra*, as well as other mathematical courses, can be taught step-by-step only. For example, we can't teach students eigenvalues and eigenvectors of matrices before we teach matrices and vectors. This is because of the nature of mathematics. Mathematics is such a science that the whole mansion is built on a minimum foundation. From the point of view of constructivists, new learning always starts with what the learner brings with them into the new learning situation. This doesn't mean that there's a unique sequence in which we should teach linear algebra. In fact, for some directly related topics, the sequence is fixed; but for other topics, the sequence is changeable. So we do have space to design an optimal sequence of topics to teach, so that the students might link the newer materials with their former knowledge to get an intuitive impression and a better understanding.

Second year students do know something about systems of linear equations and low-dimension vectors before they take this course. That means we might choose systems of linear equations or vectors as the starting point of the course. From this point of view, a textbook written by Howard Anton (*Contemporary Linear Algebra*) is very good: vectors and systems of linear equations are the first two chapters! In fact, the sequence of the book is well designed, with concepts related to the students' former knowledge in the beginning, leading to the most important entry, matrices, and then determinants, in the first part of the book. Before the next difficult concept, there is a chapter called matrix models, which focuses on some applications of linear algebra, in order to motivate the interests of the students.

Besides the materials the course syllabus currently covers, I think we need to add some history of mathematics. Like other branches of science, mathematics is still developing. It is an open system, always absorbing new material from the real world. It is quite normal that non-mathematicians make great contributions to mathematics. It's also natural that mathematicians might sometimes make mistakes. Without mentioning its history, students can't find any clues connecting mathematics to reality, or how mathematical theories are developed.

On the usage of computers

Using computers, which are so powerful, it is possible to change the teaching and learning of *Linear Algebra* totally.

In fact, computers might be used in this course for many different reasons. The teacher might use a computer to give students a demonstration, or to give them online support, and students might use computers to finish those dull calculations and focus on the ideas.

At first, using animation techniques, linear algebra might be shown on the computer screen geometrically (of course in cases of 2-D or 3-D), so it is much more intuitive rather than abstract, and easier for students to understand.

For example, the power method is used to calculate the dominant eigenvalue and eigenvector associated with a symmetric matrix. To prove this method is out of the range of this course. However, by using a computer, the teacher can make it plausible geometrically in the 2*2 case. He may use another method to calculate all the eigenvalues of a 2*2 symmetric matrix that is chosen randomly (it is quite easy to do this by hand also). Usually, this matrix will have two distinct eigenvalues. (If it fails, try another matrix.) Then he may find one unit eigenvector associated with each eigenvalue. The two eigenvectors he gets are always perpendicular. (This is a main property of symmetric matrix.) After that he may choose a random vector in R^2 and split it into the sum of those two perpendicular eigenvectors. In this case the effect of multiplying the vector by the matrix might be shown clearly on the screen: the vector is 'dragged' toward the direction of the eigenvector of the dominant eigenvalue! Consequently, by repeatedly normalizing and multiplying the resulting vector by the matrix, the trend of the vectors is obvious. They will converge to an eigenvector associated with the dominant eigenvalue.

Secondly, some useful software has been developed, so that it is quite possible for the students to get support from their computers. For example, there's an online assessment system called *eGrade* that contains a large bank of skill-building problems and solutions. Teachers can now automate the process of assigning, delivering, and grading all kinds of homework, quizzes, and tests while providing students with immediate scoring and feedback on their work. This system is attainable on the Internet. At The University of Sydney, there is a similar system called *Maplix* being used, and the effect is quite good. In Chang'an University, even though there's a forum for teachers and students to contact each other, the system is not suitable for discussing mathematical problems. It would be desirable to introduce a better system such as *eGrade* to my university. What's more, it's quite easy to find some useful materials such as project problems on the Internet; so we can share the existing materials, and not repeat the work that has already been done.

The next reason is maybe more important. In *Linear Algebra*, it is quite often the case that when the method is quite easy, large calculations are involved. For example, to find the inverse of a 10*10 matrix (this is quite small) requires some 300 multiplications. A little carelessness may result in big mistakes. In other cases the only practical method available is a numerical method. For example, to find all the eigenvalues of a 10*10 matrix, you need to solve a polynomial equation of degree 10. Since there's no formula for solving polynomial equations of degree bigger than 3, the only way to do it is to use a computer. Sometimes it is just impossible to solve the problem by hand in an acceptable time period. Fortunately, a modern computer can do most of the routine work involved in linear algebra. And many commercial software programs will do this. Students have to be familiar with at least one of these software programs if they want to apply linear algebra in the future. There is another benefit here. By using this kind of software, students needn't do all the boring routine calculations; the only things they need to do is to type in their data and choose the right set of commands. So the students might concentrate on the concepts and the ideas, not on those boring calculations. I think this is quite important for students.

Though introducing numerical methods or some algorithms in the class sounds good, it's not very practical because of time restrictions. A better solution is to introduce suitable software for students to use. After all, linear algebra itself is quite enough for most students to struggle with.

We need to be careful of a tendency to rely too much on computers also. Though computers are very useful, they can't replace the thinking of human beings. For example, intuitive thinking is obviously a human characteristic that is very useful sometimes. I have a good example from Dr Lindsay Grimison. At first sight it might seem like a game. The problem is:
Is there a route for a knight to travel through all the 64 small squares on a chessboard consecutively from its initial position? If the answer is 'yes', try to find this route.

This is a typical Non-deterministic Polynomial-time Complete (NPC) problem in graph theory, and I can compose a program to solve it with a computer. Since there's no effective algorithm for any NPC problems, the time cost of this program is extremely huge. Surprisingly, by recognising some simple principles and using some intuition, I found the solution without real difficulties.

On applications of linear algebra

Interesting applications are very helpful for motivating the students, and motivation is one of the most important factors of effective learning. Linear algebra is widely used in many areas of engineering, computer science, communication science, and some other branches of mathematics as well, so there are abundant examples from which to choose. The only problem here is how to choose. Remember, the focus of the course is linear algebra itself, not its applications. The students will have enough chances to apply this powerful tool later.

In my opinion, there's a chance to use some contemporary approaches to teaching here.

In my university, teachers try to teach everything in the class. (We know this is due to their strong tendency to rely on the theory of behaviourists.) Students need only follow their teachers. There is little space for them to think. In this case, the teaching is teacher-centred, and the students are passive accepters.

By dividing the students into small groups, and assigning them some well-prepared projects, students may engage in the learning process deeply. So they are discovering knowledge rather than learning, and hence deep level processing of information is a natural result.

Adapting to new change

Since 1999, the government has encouraged universities to enroll more and more students. Chinese higher education is at the brink of a big transformation, from teaching only the talented students to teaching almost everyone. This means that the average initial level of knowledge of the students has decreased. A successful education cannot ignore this change.

If we apply a unique standard for all our students, an obvious result is that some students complain that the material is too difficult and at the same time some top students think that it is not challenging. I think the design of courses at The University of Sydney is very good. There are two levels of courses, normal and advanced. Students can choose the courses that suit them. This is much more flexible. The strong students might learn more, and the weak students do not lose their self-confidence.

In Chang'an University and other universities, along with the increasing number of enrolled students, another problem has arisen also: shortage of qualified staff. Every year, young graduates without experience are recruited. Even though their knowledge of mathematics is adequate, to teach mathematics is another question. There is a well known saying 'boil dumplings in a tea pot', which means lacking the ability to express ideas. But we could say that this is also a chance for us to improve. The fact that these young academics have no background in pedagogy means that they can accept these newer theories without disturbing old ones. And usually it is much easier for them to apply new technology because their grasp of computer application and foreign language is much better.

Teaching in English might be a chance to reform. Since this is an experiment itself, different criteria might be applied. So better textbooks and contemporary education theory might be tested as well.

Conclusion

It is too early to draw conclusions at this time.

There are so many different approaches to teaching, one is not superior to the others in every circumstance. Each of the approaches has its benefits and shortcomings at the same time. We need to compare those different methods and test them in practice. Maybe a combination of different methods is a better solution for us. Any way, practicing is the only way to find out.

Some kind of innovation is needed, and we need to experiment on a relatively small scale first in order to accumulate experience. The whole process should be promoted step-by-step. Simply introducing other's successful experience may not be the best solution.

There's another big problem. Although I have learnt some new ideas on teaching, it will take a lot of time to master the methods. 'Rome was not built in one day.'

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