Bridging the gap between elementary and advanced mathematics

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Abstract

'Mathematics is abstract and difficult, and is the subject that only gifted people can do.' Ideas such as these are held by the students and are the concern of most of the mathematics teachers in the universities of China. How can we make the students realise that mathematics is useful? How can we make the students aware of the relationship between elementary and advanced mathematics? This paper focuses on how to use the students' prior knowledge to teach the course *Basic Mathematics Research*, and so bridge the gap between elementary and advanced mathematics.

Most of the students at a Normal University in China will be middle school teachers after they graduate. What they learn in the university is very important for their future teaching careers. Firstly, this paper explores the gap between elementary and advanced mathematics. Secondly, it discusses how students learn and why it is important to use students' prior knowledge in constructing their new understanding. Thirdly, it explains how to use students' prior knowledge to teach *Basic Mathematics Research*, so as to bridge the gap between elementary and advanced mathematics. Finally, this paper reflects on mathematics teaching in a Normal University and reaches the conclusion: 'know the gap, then bridge it'.

Introduction

The main purpose of the Normal Universities in China is to train teachers for elementary and middle schools. Like other universities, students need to learn the courses *Mathematics Analysis*, *Advanced Algebra*, *Analytical Geometry*, *Real and Complex Analysis*, *Ordinary and Partial Differential Equations* and *Probability and Statistics.* They also need to learn *Mathematics Pedagogy*, *Mathematics Methodology* and *Basic Mathematics Research* to prepare for their future teaching. What is their opinion regarding the four years of advanced mathematics learning? Do they think advanced mathematics is important for their future teaching?

Some students choose mathematics as their major because of their high examination marks in mathematics in middle school. Among them there are some who truly like mathematics, yet there are others who only choose mathematics because their parents, their teachers or they themselves think mathematics is important even though they are not very interested in it. Students with differing attitudes to mathematics enter university to study mathematics.

Despite the differences, most students do not think middle school mathematics is difficult. But after they enter the mathematics department at a Normal University, they find it difficult to adapt to learning advanced mathematics. There are three reasons for this: content; teaching methods; and learning methods.

Content
There is a great deal of material to learn in advanced mathematics, such as algebraic, analytical, and geometrical mathematics. This content is much more abstract and difficult to learn when compared to middle school mathematics.

Teaching methods
The greatly increased amount of content requires university teachers to use different teaching methods. Teachers have to give a great deal of information in a period of two to four hours of continuous lecturing (occasionally with ten or twenty minute breaks between lectures). They do not have enough time to provide background material or to consider applications of abstract mathematics knowledge. They cannot give very much attention to individuals as there are so many students in the lecture (often more than 100 students, similar to the

University of Sydney, however there are fewer tutorials). This is a very different situation to middle schools where class sizes are small.

Learning methods
As there is much more content and different teaching methods, students are required to adjust their learning. Accordingly, students need more time to think and reflect, to understand the source and application of the knowledge, to know the nature of the knowledge visually and concretely, they need to communicate with the teachers and the students, they also need to have the ability and skills to find the useful information to support their learning. This is different from their middle school mathematics learning.

Faced with greater content and the different teaching methods, some students do not adjust very well. Even after four years of mathematics education in the Normal University, some students still think it is not useful for their career, as they do not understand why their future teaching should require so much abstract advanced mathematics. One of the reasons is that they cannot see the relationship between the abstract advanced mathematics and elementary mathematics.

In this paper, I shall address the problem of how to help students realise the relationship between elementary mathematics and advanced mathematics, and to recognise the instructive function of advanced mathematics to elementary mathematics teaching.

Before we talk about these, we should first understand how students learn.

How do students learn

A large amount of research data on students learning is available in the literature.

Behaviourist theory only relates learning to the change of behaviour, but does not consider of the inner change of mind. It acknowledges that learning is the process of changing outward behaviour. Learning should be in the small incremental steps, each step should build logically, one upon the next, (King, 2005).

Developmental theory relates learning to the inner change in mind and the outer change in behaviour. It believes that learners go through a series of conceptual stages associated with intellectual growth. Learning should also be small incremental steps but 'leaps in understanding' as a result of a build-up of knowledge and experience (Gestalt leap). Learners develop a 'schema' of their own. They learn by modifying this schema through the processes of 'assimilation' and 'accommodation'. 'Assimilation' is the process of relating new knowledge to the existing cognitive structure, while 'accommodation' is the process of alerting the existing cognitive structure, or creating a new one in order to fit new knowledge into it, (King, 2005).

Constructive theory regards new learning as always starting with what the learner brings with them into the new learning situation, whether or not what they know is right or wrong. The learning process is about the interaction between existing knowledge and beliefs and new experiences. The process of learning involves the constant construction and reconstruction of meaning, (King, 2005).

There is no absolutely right or wrong in learning theory. Sometimes we apply behaviourist theory to practical teaching. For example, we can get information on a correlation between the students' actual achievements and their examination marks, their response to the teachers' questions, their homework etc. However this is not enough. Learning is not just a change of outward behaviour; it also involves a change of the mind or a change of the cognitive structure. Learning new knowledge means that one should adjust the previous cognitive structure to the new situation. Students are not the containers waiting to be filled with water, nor are they the paper margin waiting for the teachers to write on. They should learn based on their previous cognitive structures, (King, 2005).

We know that students' existing knowledge or cognitive structure is important for their learning. University students already have some knowledge and life experience, thus are more likely to 'construct interpretations of newly encountered problems and phenomena in ways that agree with their own prior knowledge even when those interpretations conflict with what a teacher has attempted to teach' (National Research Council of the National Academies, 2003). With the interpretations constructed by the students themselves using their prior knowledge, it is a great challenge or motivation for the students' to learn the new knowledge. Therefore, we university mathematics teachers ought to pay more attention to the students' previous experience in life and knowledge in elementary mathematics, in order to find ways to build new knowledge. In the following I will discuss how to use students' prior knowledge to construct new understanding in my course, so as to bridge the gap between elementary and advanced mathematics.

Teaching *Basic Mathematics Research*: Bridging the gap

We have talked about the gap between elementary and advanced mathematics, and have discussed some learning theories. Based on these, especially on the research result that students always use their prior knowledge to construct their new understandings, I will talk about how to bridge this gap in my course *Basic Mathematics Research*.

About this course
This course is specifically designed for the Normal University mathematics department students, usually for grade three or grade four. The course mainly uses the content, thinking or methods of advanced mathematics to deal with the problems of elementary mathematics. The special topics of the course cover many areas in mathematics, including number theory, Euclid geometry and spherical geometry, fractal geometry, length and measure, calculus etc.

This is a one semester course, 54 hours lectures, including a mid-term examination and a final examination. There are no tutorials during semester, but arrangements are made for answering questions from students at the end of the semester.

Overview of previous work
I have taught this course several times, and I have found that students were not very clear about the relationship between advanced mathematics and elementary mathematics. They thought advanced mathematics was too abstract and profound to understand. They believed that they should learn something from the advanced mathematics thinking and method, however they did not think abstract advanced mathematics knowledge had any direct relationship with the elementary mathematics that they would be teaching in the future.

Now reviewing past teaching, I can clearly see that there is a need to construct a bridge between elementary and advanced mathematics.

Using prior knowledge to construct new understandings: bridging the gap
Students have some advanced mathematics knowledge from their first two years in university, they have been engaged in proving theorems, trying to understand or just remember. It is virtually impossible for them to see the relationship between elementary and advanced mathematics clearly.

How can I help the students understand advanced mathematics knowledge better and grasp the link of elementary and advanced mathematics? One of the most effective ways is to use the students' prior knowledge to construct their understanding.

Firstly, I would pose questions or problem situations. When students apply their prior knowledge to solve or explain it, they may have some difficulties. It would be necessary to give the students some advanced mathematics background knowledge to assist them, the students could even apply this advanced mathematics knowledge to other elementary problems. In this way, students can easily grasp the relationship between elementary and advanced mathematics.

Here are some examples.

Example 1
Is the 'number' of rational numbers the same as that of real numbers?

Some students in the middle school may be wondered about this problem even though it exceeds the syllabus of the middle school mathematics. Therefore, the Normal University students should know the advanced mathematics background for this, so as to improve their future teaching.

Using their prior knowledge, students may answer 'yes', but there are also students who think the answer is 'no', based on intuition and not exact mathematical explanations. Actually this problem relates to the knowledge of 'sets'. I will start my first lesson on 'set theory' by exploring this problem. This relates to the advanced mathematics knowledge such as 'set', 'cardinality', 'countable set' and 'uncountable set', 'the continuum hypothesis', 'the comparison of the cardinality' etc. By proving the rational numbers set is countable and the real numbers set is uncountable, we know the cardinality of rational numbers set is $\aleph_0$ while real numbers set is $\aleph$, by the relationship $\aleph_0 < \aleph < 2^{\aleph} < \cdots$, we know the 'number' of rational numbers is smaller than that of real numbers. After this advanced mathematical knowledge, I will lead students back to other elementary problems:

Is the 'number' of rational numbers the same as that of irrational numbers?
Is the 'number' of points in a big circle greater than that in a small circle?

Students can solve these elementary problems by the newly established advanced mathematical knowledge. The detail teaching process is illustrated in Figure 1.

By exploring an elementary mathematics problem, students can learn some advanced mathematics, and using this advanced knowledge, they can consider other elementary problems. Students not only pay attention to the abstract mathematical definitions and theorems and proofs, they also get an overall picture of the chapter 'set theory', including the relationship between elementary and advanced mathematics.

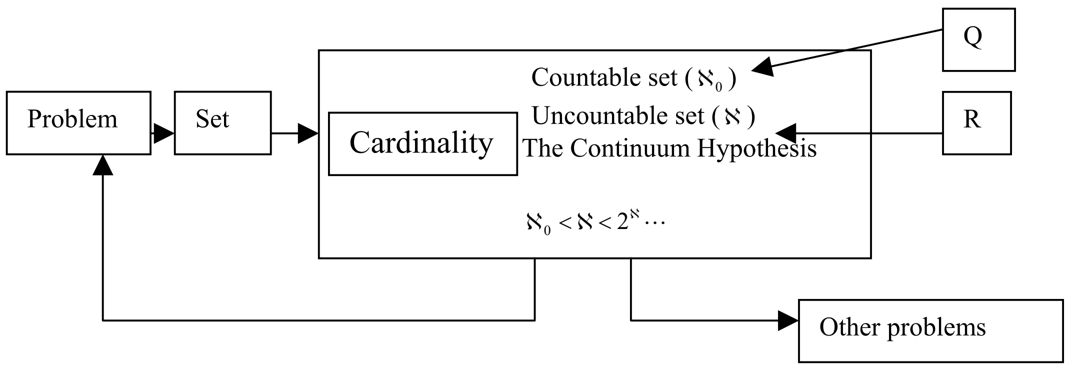


Figure 1. Teaching process for solving elementary problems

When using prior knowledge to solve this elementary problem, students will discover some conflicts with the proper solution. This provides a good opportunity for teaching. I can give the advanced mathematics knowledge when the students are eager to know the truth, in this way, students can also easily grasp the connection of elementary and advanced mathematics, they can further understand the meaning of 'cardinality' linked with some elementary mathematics problems.

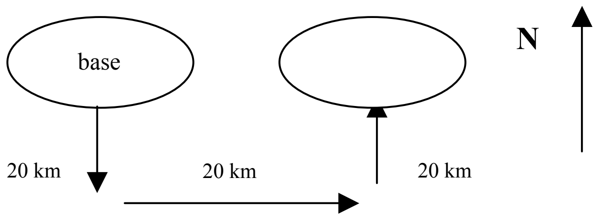
Example 2

I will give students the following puzzle before starting a new topic. The puzzle (Easdown, 2005) states:

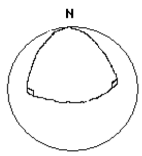
1. I am a bear hunter;
2. I move 20 km due south from the base camp;
3. I swivel 90 degree and move 20 km due east and find a bear; and
4. The bear and I swivel 90 degree and move 20 km due north arriving at the base camp.

Question: What is the colour of the bear?

We can draw a diagram to help us understand this puzzle:



The anticipated answer is that 'The bear is white because it is at the North Pole'.



Then I would give my next question: *Is the sum of the three angles in a triangle always equals to 180 degrees? How about a quadrilateral?*

In China, the recently established middle school mathematics syllabus has added new content about spherical geometry. It is necessary for the Normal University students to learn something about this. We live on Earth, a planet whose surface is a sphere rather than a plane; university students should know something about non-Euclid geometry, some properties that are different from Euclid geometry. With the puzzle and the question, I can start the topic 'spherical geometry'.

There are also other examples supporting the link between elementary and advanced mathematics, *'Can we use an unlimited scale to measure a limit scale?'*, this leads to the topic of fractal geometry. *'What's the natural number? Can we define it by axiom*', this in turn leads to Peano's axioms. Students can recall or actively study some advanced mathematics knowledge to solve these problems.

Other ways to bridge the gap

There are also other ways in which I could bridge the gap between elementary and advanced mathematics in this course. For example, I can use problem based learning, more tutorial time and concept mapping to help student understanding however they are not discussed in this paper.

Conclusion

I have discussed how to use prior knowledge to construct new understanding, and so bridge the gap between elementary and advanced mathematics. University teachers should know how students learn and what students already know in order to teach effectively. All colleges in Normal Universities believe that advanced mathematics help to improve students' career prospects.

However, just one special course is not sufficient to convince students. I would like to see more special mathematics courses introduced in Normal Universities in China, which will help to bridge the gap between elementary mathematics and advanced mathematics. This is important for education in China as a whole.

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