2 Unit Bridging Course – Day 8

Applications of the Exponential Function

Collin Zheng

THE UNIVERSITY OF SYDNEY

The General Exponential Function

Previously, we looked at the exponential function

$$y = e^x$$

which had the special property that

$$\frac{dy}{dx} = e^x.$$

That is, the derivative of the exponential function is equal to itself.

Our aim in this module is to employ variations of the exponential function to model and study real-world phenomena.

The General Exponential Function (ct.)

Consider the function

$$f(x) = Ae^{kx},$$

where A and k are constants.

It turns out that:

$$f'(x) = k \times Ae^{kx} = k\,f(x),$$

so $f(x)$ and its rate of change $f'(x)$ differ by a constant multiple.

Terminology-wise, we say that f' is **proportional** to f.

The General Exponential Function (ct.)

Consider the function

$$f(x) = Ae^{kx},$$

where A and k are constants.

It turns out that:

$$f'(x) = k \times Ae^{kx} = k\,f(x),$$

so $f(x)$ and its rate of change $f'(x)$ differ by a constant multiple.

Terminology-wise, we say that f' is **proportional** to f.

The General Exponential Function (ct.)

Consider the function

$$f(x) = Ae^{kx},$$

where A and k are constants.

It turns out that:

$$f'(x) = k \times Ae^{kx} = k\,f(x),$$

so $f(x)$ and its rate of change $f'(x)$ differ by a constant multiple.

Terminology-wise, we say that f' is **proportional** to f.

The General Exponential Function (ct.)

Consider the function

$$f(x) = Ae^{kx},$$

where A and k are constants.

It turns out that:

$$f'(x) = k \times Ae^{kx} = k\,f(x),$$

so $f(x)$ and its rate of change $f'(x)$ differ by a constant multiple.

Terminology-wise, we say that f' is **proportional** to f.

The General Exponential Function (ct.)

Consider the function

$$f(x) = Ae^{kx},$$

where A and k are constants.

It turns out that:

$$f'(x) = k \times Ae^{kx} = k\,f(x),$$

so $f(x)$ and its rate of change $f'(x)$ differ by a constant multiple.

Terminology-wise, we say that f' is **proportional** to f.

The General Exponential Function (ct.)

Since there exist many situations in the natural world where the rate of change of a quantity is proportional its size, f is well-positioned to act as a model for such phenomena.

Definition: The General Exponential Function

The function $f(x) = Ae^{kx}$ is called the **General Exponential Function**, with the property that f changes at a rate proportional to f itself.

That is, $f'(x) = k\,f(x)$ for all x.

A is called the **initial constant** and k is called the **proportionality constant**.

The General Exponential Function (ct.)

Since there exist many situations in the natural world where the rate of change of a quantity is proportional its size, f is well-positioned to act as a model for such phenomena.

Definition: The General Exponential Function

The function $f(x) = Ae^{kx}$ is called the **General Exponential Function**, with the property that f changes at a rate proportional to f itself.

That is, $f'(x) = k\,f(x)$ for all x.

A is called the **initial constant** and k is called the **proportionality constant**.

The General Exponential Function (ct.)

Since there exist many situations in the natural world where the rate of change of a quantity is proportional its size, f is well-positioned to act as a model for such phenomena.

Definition: The General Exponential Function

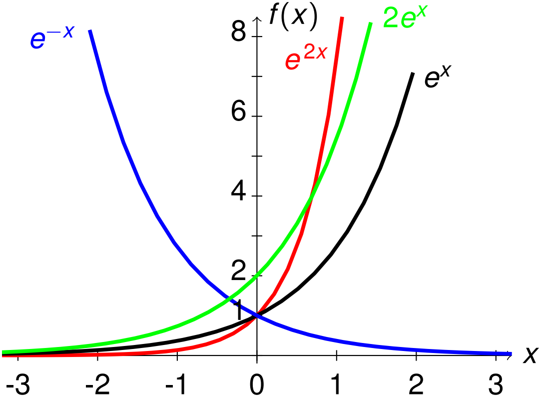
The function $f(x) = Ae^{kx}$ is called the **General Exponential Function**, with the property that f changes at a rate proportional to f itself.

That is, $f'(x) = k\,f(x)$ for all x.

A is called the **initial constant** and k is called the **proportionality constant.**

The General Exponential Function (ct.)

Graphically, k modifies the steepness and orientation of the exponential function while A primarily serves to shift its y-intercept.



Exponential Growth

When the proportionality constant k is positive, $f(x)$ is said to describe **exponential growth.**

A good example of this is population growth.

Example

The population, $P(t)$, of an outback town is growing exponentially according to the formula

$$P(t) = 1000\, e^{0.2t},$$

where t is the number of years after the year 2000.

Find the population in 2000, 2010, and estimate the population in 2020.

Exponential Growth

When the proportionality constant k is positive, $f(x)$ is said to describe **exponential growth.**

A good example of this is population growth.

Example

The population, $P(t)$, of an outback town is growing exponentially according to the formula

$$P(t) = 1000\, e^{0.2t},$$

where t is the number of years after the year 2000.

Find the population in 2000, 2010, and estimate the population in 2020.

Exponential Growth

When the proportionality constant k is positive, $f(x)$ is said to describe **exponential growth.**

A good example of this is population growth.

Example

The population, $P(t)$, of an outback town is growing exponentially according to the formula

$$P(t) = 1000\, e^{0.2t},$$

where t is the number of years after the year 2000.

Find the population in 2000, 2010, and estimate the population in 2020.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\,e^{0.2\times 0} = 1000\,e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\, e^{0.2\times 0} = 1000\, e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

THE UNIVERSITY OF SYDNEY

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\, e^{0.2\times 0} = 1000\, e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\,e^{0.2\times 0} = 1000\,e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\, e^{0.2\times 0} = 1000\, e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\, e^{0.2\times 0} = 1000\, e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\,e^{0.2\times 0} = 1000\,e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\, e^{0.2\times 0} = 1000\, e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

Exponential Growth (cont.)

Year 2010: The year 2010 corresponds to $t = 10$. Hence the population in the year 2010 is:

$$P(10) = 1000\, e^{0.2\times 10} = 1000\, e^{2} = 7389.$$

Hence, the population has increased 7-fold in 10 years.

Year 2020: Finally, we can use the model to predict what the population will be in the year 2020:

$$P(20) = 1000\, e^{0.2\times 20} = 1000\, e^{4} = 54598.$$

That is, the population is expected to explode to roughly 55,000 in the subsequent 10 years.

Exponential Growth (cont.)

Year 2010: The year 2010 corresponds to $t = 10$. Hence the population in the year 2010 is:

$$P(10) = 1000\,e^{0.2\times 10} = 1000\,e^{2} = 7389.$$

Hence, the population has increased 7-fold in 10 years.

Year 2020: Finally, we can use the model to predict what the population will be in the year 2020:

$$P(20) = 1000\,e^{0.2\times 20} = 1000\,e^{4} = 54598.$$

That is, the population is expected to explode to roughly 55,000 in the subsequent 10 years.

Exponential Growth (cont.)

Year 2010: The year 2010 corresponds to $t = 10$. Hence the population in the year 2010 is:

$$P(10) = 1000\, e^{0.2\times 10} = 1000\, e^{2} = 7389.$$

Hence, the population has increased 7-fold in 10 years.

Year 2020: Finally, we can use the model to predict what the population will be in the year 2020:

$$P(20) = 1000\, e^{0.2\times 20} = 1000\, e^{4} = 54598.$$

That is, the population is expected to explode to roughly 55,000 in the subsequent 10 years.

Exponential Growth (cont.)

Year 2010: The year 2010 corresponds to $t = 10$. Hence the population in the year 2010 is:

$$P(10) = 1000\, e^{0.2\times 10} = 1000\, e^{2} = 7389.$$

Hence, the population has increased 7-fold in 10 years.

Year 2020: Finally, we can use the model to predict what the population will be in the year 2020:

$$P(20) = 1000\, e^{0.2\times 20} = 1000\, e^{4} = 54598.$$

That is, the population is expected to explode to roughly 55,000 in the subsequent 10 years.

THE UNIVERSITY OF SYDNEY

Exponential Growth (cont.)

Year 2010: The year 2010 corresponds to $t = 10$. Hence the population in the year 2010 is:

$$P(10) = 1000\, e^{0.2\times 10} = 1000\, e^{2} = 7389.$$

Hence, the population has increased 7-fold in 10 years.

Year 2020: Finally, we can use the model to predict what the population will be in the year 2020:

$$P(20) = 1000\, e^{0.2\times 20} = 1000\, e^{4} = 54598.$$

That is, the population is expected to explode to roughly 55,000 in the subsequent 10 years.

Exponential Decay

When the proportionality constant k is negative, $f(x)$ is said to describe **exponential decay.**

A good example of this is the decomposition of radioactive material, which decays at a rate proportional to its size.

Example

An amount of radioactive carbon A, measured in kilograms, is decaying exponentially according to the formula

$$A(t) = 25\, e^{-0.00012t},$$

where t is the number of years after the year 2000.

Find the initial amount, and the amount of radioactive carbon present in the year 3000.

Exponential Decay

When the proportionality constant k is negative, $f(x)$ is said to describe **exponential decay.**

A good example of this is the decomposition of radioactive material, which decays at a rate proportional to its size.

Example

An amount of radioactive carbon A, measured in kilograms, is decaying exponentially according to the formula

$$A(t) = 25\,e^{-0.00012t},$$

where t is the number of years after the year 2000.

Find the initial amount, and the amount of radioactive carbon present in the year 3000.

Exponential Decay

When the proportionality constant k is negative, $f(x)$ is said to describe **exponential decay.**

A good example of this is the decomposition of radioactive material, which decays at a rate proportional to its size.

Example

An amount of radioactive carbon A, measured in kilograms, is decaying exponentially according to the formula

$$A(t) = 25\,e^{-0.00012t},$$

where t is the number of years after the year 2000.

Find the initial amount, and the amount of radioactive carbon present in the year 3000.

Exponential Decay (cont.)

Year 2000: The initial amount corresponds to $A(0)$, which we know is the initial constant $A = 25$. Hence initially, we have 25kg of radioactive carbon.

Year 3000: In the year 3000, the amount of radioactive carbon remaining is given by

$$A(1000) = 25\,e^{-0.00012\times1000} = 25\,e^{-0.12} = 22.17.$$

That is, less than 3kg of the radioactive material has decomposed after a whole millenium, with almost 90% still remaining!

Exponential Decay (cont.)

Year 2000: The initial amount corresponds to $A(0)$, which we know is the initial constant $A = 25$. Hence initially, we have 25kg of radioactive carbon.

Year 3000: In the year 3000, the amount of radioactive carbon remaining is given by

$$A(1000) = 25\, e^{-0.00012 \times 1000} = 25\, e^{-0.12} = 22.17.$$

That is, less than 3kg of the radioactive material has decomposed after a whole millenium, with almost 90% still remaining!

Exponential Decay (cont.)

Year 2000: The initial amount corresponds to $A(0)$, which we know is the initial constant $A = 25$. Hence initially, we have 25kg of radioactive carbon.

Year 3000: In the year 3000, the amount of radioactive carbon remaining is given by

$$A(1000) = 25\,e^{-0.00012\times 1000} = 25\,e^{-0.12} = 22.17.$$

That is, less than 3kg of the radioactive material has decomposed after a whole millenium, with almost 90% still remaining!

Exponential Decay (cont.)

Year 2000: The initial amount corresponds to $A(0)$, which we know is the initial constant $A = 25$. Hence initially, we have 25kg of radioactive carbon.

Year 3000: In the year 3000, the amount of radioactive carbon remaining is given by

$$A(1000) = 25\,e^{-0.00012\times 1000} = 25\,e^{-0.12} = 22.17.$$

That is, less than 3kg of the radioactive material has decomposed after a whole millenium, with almost 90% still remaining!

Summary

- The *General Exponential Function* $f(x) = Ae^{kx}$ holds the property that f changes at a rate proportional to f itself. A is called the *initial constant* and k the *proportionality constant*.
- The formula $f(t) = Ae^{kt}$, for some quantity $f(t)$ at time t, serves as an excellent model for real-world phenomena following a pattern of *exponential growth* ($k > 0$) or *decay* ($k < 0$). Here, A represents the value of f at the time $t = 0$.
- Examples of exponential growth includes population growth, while examples of exponential decay includes the decomposition of radioactive material.