# Solutions to Selected Exercises 9

**1.** **ii** $y = \sqrt{5 - x^2}$

First rewrite the equation as $y = (5 - x^2)^{\frac{1}{2}}$.

This is a composite function with $u = f(x) = 5 - x^2$ and $g(u) = u^{\frac{1}{2}}$,

so $g(f(x)) = (5 - x^2)^{\frac{1}{2}}$.

So,

$$\frac{dy}{dx} = \underbrace{\frac{1}{2}(5 - x^2)^{-\frac{1}{2}}}_{g'(f(x))} \times \underbrace{(-2x)}_{f'(x)}.$$

**2.** **ii** $y = \dfrac{4x}{x^2 + 1}$

This is quotient so we will use the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2+1)\frac{d}{dx}(4x) - (4x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} \\ &= \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} \\ &= \frac{4 - 4x^2}{(x^2+1)^2}. \end{aligned}$$

**3.** **b** $y = 2x^2(x^2 - 5)$

Using the product rule we get

$$\frac{dy}{dx} = 2x^2 \times \frac{d}{dx}(x^2-5) + (x^2-5) \times \frac{d}{dx}(2x^2) = 2x^2 \times 2x + (x^2-5) \times 4x = 4x^3 + 4x^3 - 20x = 8x^3 - 20x.$$

**h** $y = \dfrac{1}{(2x+3)^5}$

First we will rewrite the equation as $y = (2x+3)^{-5}$ and use the chain rule. So,

$$\frac{dy}{dx} = -5(2x+3)^{-6}\frac{d}{dx}(2x+3) = -5(2x+3)^{-6}(2) = -10(2x+3)^{-6} = \frac{-10}{(2x+3)^6}.$$

**j** $y = (x^2+1)(x^2-6)$
Here we will use the product rule.

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)\frac{d}{dx}(x^2-6) + (x^2-6)\frac{d}{dx}(x^2+1)\\ &= (x^2+1)(2x) + (x^2-6)(2x)\\ &= 2x^3+2x+2x^3-12x = 4x^3-10x.\end{aligned}$$

**s** $y = \dfrac{x}{\sqrt{x^2+1}}$
First we rewrite the equation as $y = \dfrac{x}{(x^2+1)^{\frac{1}{2}}}$ and use the quotient rule to differentiate. We'll need the chain rule too.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+1)^{\frac{1}{2}} \times \frac{d}{dx}(x) - x \times \frac{d}{dx}\left((x^2+1)^{\frac{1}{2}}\right)}{\left((x^2+1)^{\frac{1}{2}}\right)^2}\\ &= \frac{(x^2+1)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x}{x^2+1}\\ &= \frac{(x^2+1)^{\frac{1}{2}} - \frac{x^2}{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\\ &= \frac{\frac{(x^2+1)-x^2}{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\\ &= \frac{1}{(x^2+1)^{\frac{3}{2}}}.\end{aligned}$$

**4. ii** $f(x) = e^{-2x}$

$$f'(x) = e^{-2x} \times \frac{d}{dx}(-2x) = -2e^{-2x}.$$

**vi** $f(x) = e^{x^2-2x+7}$

$$f'(x) = e^{x^2-2x+7} \times \frac{d}{dx}(x^2-2x+7) = (2x-2)e^{x^2-2x+7} = 2(x-1)e^{x^2-2x+7}.$$

**viii** $f(x) = x^2e^x$

$$f'(x) = x^2 \times \frac{d}{dx}(e^x) + e^x \times \frac{d}{dx}(x^2) = x^2e^x + 2xe^x = xe^x(x+2) = x(x+2)e^x.$$

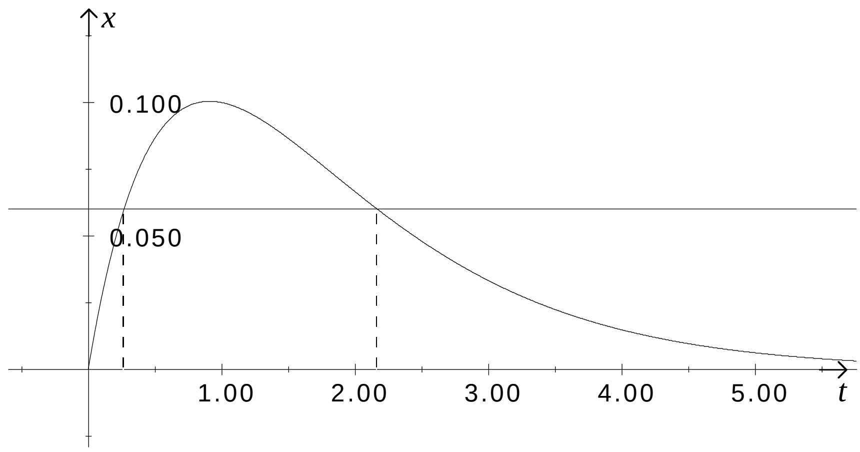
**5.** The maximum concentration of the drug in the blood occurs when the derivative of $x = 0.3te^{-1.1t}$ equals zero.

$$x' = 0.3e^{-1.1t} + 0.3t(-1.1)e^{-1.1t} = 0.3e^{-1.1t}(1 - 1.1t) = 0 \quad \text{ie when} \quad t = \frac{1}{1.1} = \dot{9}\dot{0}.$$

Note that $e^{-1.1t} > 0$ for all values of $t$. The following table confirms we have a maximum when $t = \dot{9}\dot{0}$.

| $t$ | $< \dot{9}\dot{0}$ | $\dot{9}\dot{0}$ | $> \dot{9}\dot{0}$ |
|---|---|---|---|
| $x'$ | $+ve$ | $0$ | $-ve$ |
| $y$ | ↗ | $\frac{0.3e^{-1}}{1.1}$ | ↘ |

The maximum value of $\frac{0.3e^{-1}}{1.1}$ units is achieved when $t = \dot{9}\dot{0}$ ie when $t = 54.5$ minutes.




From my sketch, the drug will kill germs between about 0.25 hours and 2.15 hours after it is taken. (This is when the graph of $x = 0.3te^{-1.1t}$ is above the line $x = 0.06$.) So, the length of time the drug is able to kill germs is about 1.9 hours.

**6.** $y = e^{-x^2}$

The first thing to note about this function is that it is always positive. ($e^{-x^2} > 0$ for all values of $x$.) So, the graph of $y = e^{-x^2}$ is above the $x$-axis and never crosses it.

$$\frac{dy}{dx} = e^{-x^2} \times \frac{d}{dx}(-x^2) = -2xe^{-x^2}.$$

This is equal to zero when $x = 0$. So, $(0, 1)$ is the only stationary point.

| $x$ | $< 0$ | $0$ | $> 0$ |
|---|---|---|---|
| $y'$ | $+ve$ | $0$ | $-ve$ |
| $y$ | ↗ | $1$ | ↘ |

The table tells us that $(0, 1)$ is a maximum.

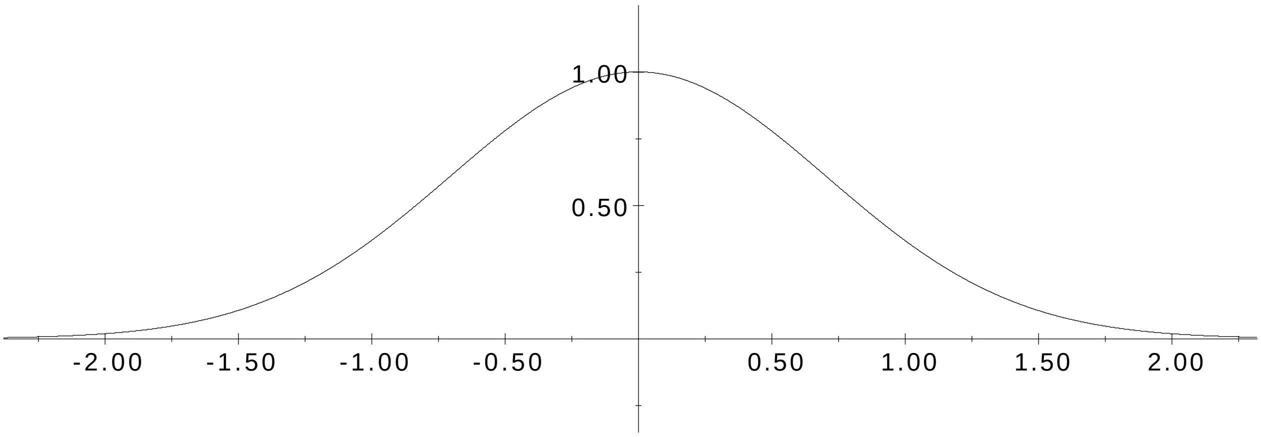
We differentiate again to see if there are any points of inflection.

$$\frac{d^2y}{dx^2} = (-2x)e^{-x^2}(-2x) + e^{-x^2}(-2) = 2e^{-x^2}(2x^2 - 1).$$

This equals zero when $x^2 = \frac{1}{2}$, ie when $x = \pm\frac{1}{\sqrt{2}}$.

| $x$ | $< -\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | $> -\frac{1}{\sqrt{2}}, < \frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $> \frac{1}{\sqrt{2}}$ |
|---|---|---|---|---|---|
| $y''$ | $+ve$ | 0 | $-ve$ | 0 | $+ve$ |
| $y$ | concave up | $e^{-\frac{1}{2}}$ | concave down | $e^{-\frac{1}{2}}$ | concave up |

The table tells us that there are two points of inflection at $(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$ and $(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$.




Note that the graph does not touch the $x$-axis but gets closer and closer to it as $x$ gets large in magnitude.