Mathematics Learning Centre

The University of Sydney

Derivatives: Gradient of a line, gradient of a curve

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1 Introduction

In day to day life we are often interested in the extent to which a change in one quantity affects a change in another related quantity. This is called a *rate of change*. For example, if you own a motor car you might be interested in how much a change in the amount of fuel used affects how far you have travelled. This rate of change is called *fuel consumption*. If your car has high fuel consumption then a large change in the amount of fuel in your tank is accompanied by a small change in the distance you have travelled. Sprinters are interested in how a change in time is related to a change in their position. This rate of change is called *velocity*. Other rates of change may not have special names like fuel consumption or velocity, but are nonetheless important. For example, an agronomist might be interested in the extent to which a change in the amount of fertiliser used on a particular crop affects the yield of the crop. Economists want to know how a change in the price of a product affects the demand for that product.

Differential calculus is about describing in a precise fashion the ways in which related quantities change.

To proceed with this booklet you will need to be familiar with the concept of the *slope* (also called the *gradient*) of a straight line. You may need to revise this concept before continuing.

1.1 An example of a rate of change: velocity

1.1.1 Constant velocity

Figure 1 shows the graph of part of a motorist's journey along a straight road. The vertical axis represents the distance of the motorist from some fixed reference point on the road, which could for example be the motorist's home. Time is represented along the horizontal axis and is measured from some convenient instant (for example the instant an observer starts a stopwatch).

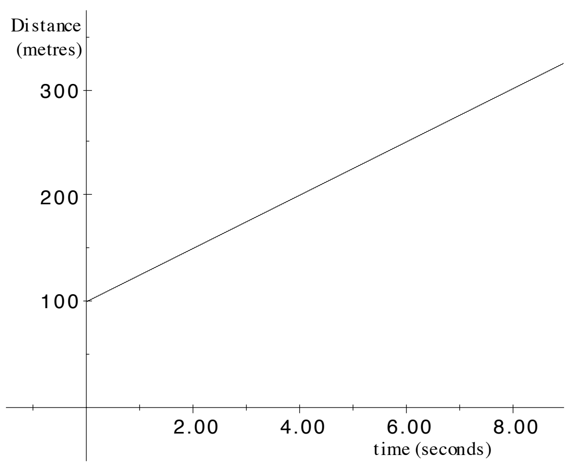

Figure 1: Distance versus time graph for a motorist's journey.

Exercise 1.1

How far is the motorist in Figure 1 away from home at time $t = 0$ and at time $t = 6$?

Exercise 1.2

How far does the motorist travel in the first two seconds (ie from time $t = 0$ to time $t = 2$)? How far does the motorist travel in the two second interval from time $t = 3$ to $t = 5$? How far do you think the motorist would travel in any two second interval of time?

The shape of the graph in Figure 1 tells us something special about the type of motion that the motorist is undergoing. *The fact that the graph is a straight line tells us that the motorist is travelling at a constant velocity.*

- At a constant velocity equal increments in time result in equal changes in distance.
- For a straight line graph equal increments in the horizontal direction result in the same change in the vertical direction.

In Exercise 1.2 for example, you should have found that in the first two seconds the motorist travels 50 metres and that the motorist also travels 50 metres in the two seconds between time $t = 3$ and $t = 5$.

Because the graph is a straight line we know that the motorist is travelling at a constant velocity. What is this velocity? How can we calculate it from the graph? Well, in this situation, velocity is calculated by dividing distance travelled by the time taken to travel that distance. At time $t = 6$ the motorist was 250 metres from home and at time $t = 2$ the motorist was 150 metres away from home. The distance travelled over the four second interval from time $t = 2$ to $t = 6$ was

$$\text{distance travelled} = 250 - 150 = 100$$

and the time taken was

$$\text{time taken} = 6 - 2 = 4$$

and so the velocity of the motorist is

$$\text{velocity} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{250 - 150}{6 - 2} = \frac{100}{4} = 25 \text{ metres per second.}$$

But this is exactly how we would calculate the slope of the line in Figure 1. Take a look at Figure 2 where the above calculation of velocity is shown diagramatically.

The slope of a line is calculated by vertical rise divided by horizontal run and if we were to use the two points $(2, 150)$ and $(6, 250)$ to calculate the slope we would get

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{250 - 150}{6 - 2} = 25.$$

To summarise:

The fact that the car is travelling at a constant velocity is reflected in the fact that the distance-time graph is a straight line. The velocity of the car is given by the slope of this line.

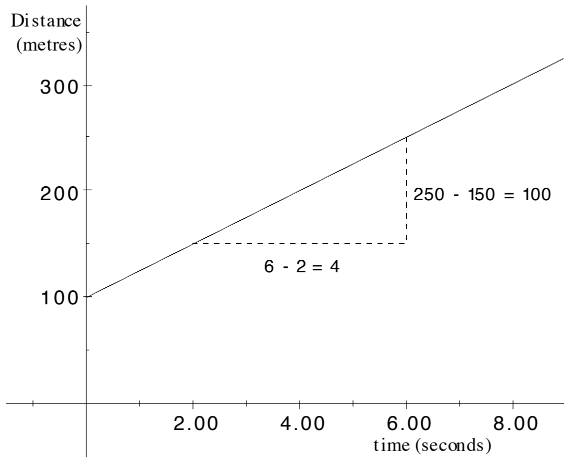

Figure 2: Calculation of the velocity of the motorist is the same as the calculation of the slope of the distance - time graph.

1.1.2 Non-constant velocity

Figure 3 shows the graph of a different motorist's journey along a *straight* road. This graph is not a straight line. The motorist is not travelling at a constant velocity.

Exercise 1.3

How far does the motorist travel in the two seconds from time $t = 60$ to time $t = 62$?

How far does the motorist travel in the two second interval from time $t = 62$ to $t = 64$?

Since the motorist travels at different velocities at different times, when we talk about the velocity of the motorist in Figure 3 we need to specify the particular time that we mean. Nevertheless we would still like somehow to interpret the velocity of the motorist as the slope of the graph, even though the graph is curved and not a straight line.

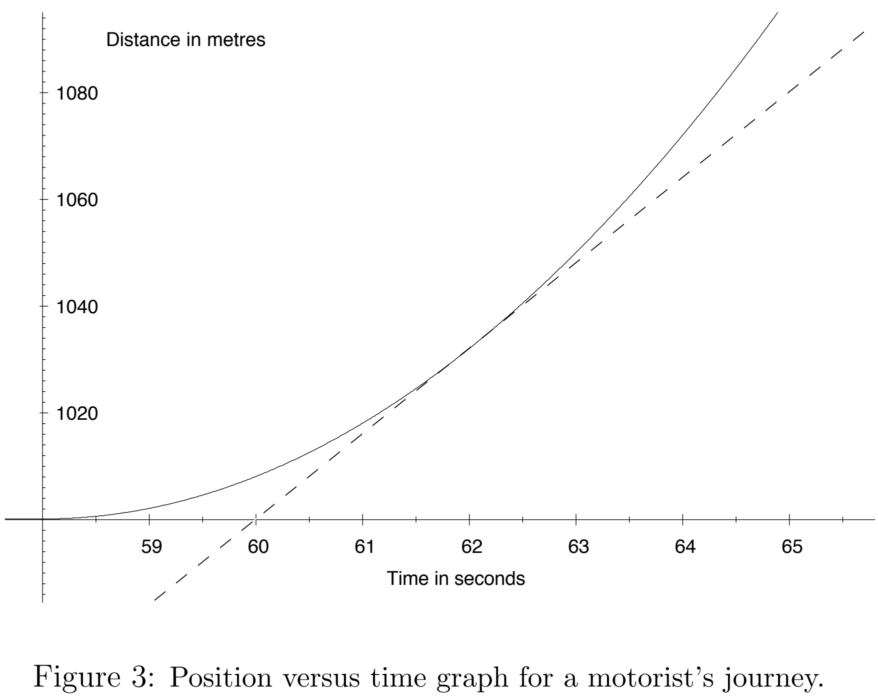

Figure 3: Position versus time graph for a motorist's journey.

What do we mean by the slope of a curve? Suppose for example that we are interested in the velocity of the motorist in Figure 3 at time $t = 62$. In Figure 3 we have drawn in a dashed line. Notice that this line just grazes the curve at the point on the curve where $t = 62$. The dashed line is in fact the *tangent* to the curve at that point. We will talk more about tangents to curves in Section 2. For now you can think of the dashed line like this: if you were going to draw a straight line through this point on the curve, and if you wanted that straight line to look as much like the curve near that point as it possibly could, this is the line that you would draw. This solves our problem about interpreting the slope of the curve at this point on the curve.

The slope of the curve at the point on the curve where $t = 62$ *is the slope of the tangent to the curve at that point: that is the slope of the dashed line in Figure 3.*

The velocity of the motorist at time $t = 62$ is the slope of the dashed line in that figure. Of course if we were interested in the velocity of the motorist at time $t = 64$ then we would draw the tangent to the curve at the point on the curve where $t = 64$ and we would get a different slope. At different points on the curve we get different tangents having different slopes. At different times the motorist is travelling at different velocities.

1.2 Other rates of change

The situation above described a car moving in one direction along a straight road away from a fixed point. Here, the word *velocity* describes how the distance changes with time. Velocity is a *rate of change.* For these type of problems, the velocity corresponds to the rate of change of distance with respect to time. Motion in general may not always be in one direction or in a straight line. In this case we need to use more complex techniques.

Velocity is by no means the only rate of change that we might be interested in. Figure 4 shows a graph representing the yield a farmer gets from a crop depending on the amount of fertiliser that the farmer uses.

The shape of this graph makes good sense. If no fertiliser is used then there is still some crop yield (50 tonnes to be precise). As more fertiliser is used the crop yield increases,

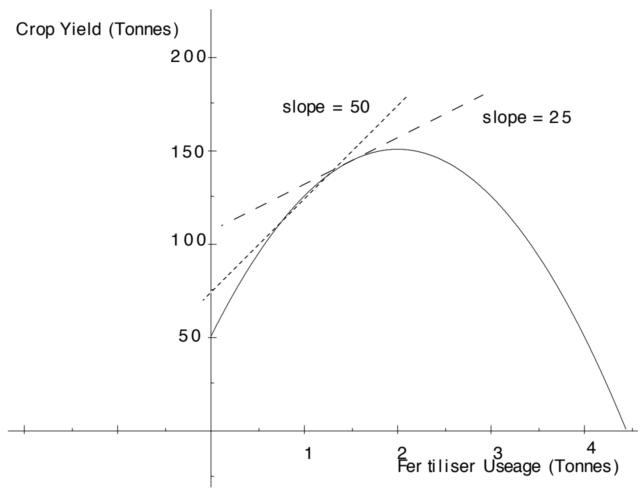

Figure 4: Crop yield versus fertiliser useage for a hypothetical crop.

as you would expect. Note though that at a certain point putting on more fertiliser does not improve the yield of the crop, but in fact decreases it. The soil is becoming poisoned by too much fertiliser. Eventually the use of too much fertiliser causes the crop to die altogether and no yield is obtained.

On the graph the tangents to the curve corresponding to fertiliser usage of 1 tonne (the dotted line) and of 1.5 tonnes (the dashed line) are drawn. The slope of these tangents give the rate of change of crop yield with respect to fertiliser usage.

The slope of the dotted tangent is 50. This means that if fertiliser usage is increased from 1 tonne by a very small amount then the crop yield will increase by 50 times that small change. For example an increase in fertiliser usage from 1 tonne (1000 kg) to 1005 kg will increase the crop yield by approximately $50 \times 5 = 250$ kg. If we are using 1 tonne of fertiliser then the rate of change of crop yield with respect to fertiliser useage is quite high. On the other hand the slope of the dashed tangent is 25. The same increase (by 5 kg) in fertiliser useage from 1500 kg (1.5 tonnes) to 1505 kg will increase the crop yield by about $25 \times 5 = 125$ kg.

2 What is the derivative?

If you are not completely comfortable with the concept of a function and its graph then you need to familiarise yourself with it before continuing. The booklet *Functions* published by the Mathematics Learning Centre may help you.

In Section 1 we learnt that differential calculus is about finding the rates of change of related quantities. We also found that a rate of change can be thought of as the slope of a tangent to a graph of a function. Therefore we can also say that:

Differential calculus is about finding the slope of a tangent to the graph of a function, or equivalently, differential calculus is about finding the rate of change of one quantity with respect to another quantity.

If we are going to go to all this trouble to find out about the slope of a tangent to a graph, we had better have a good idea of just what a tangent is.

2.1 Tangents

Look at the curve and straight line in Figure 5.

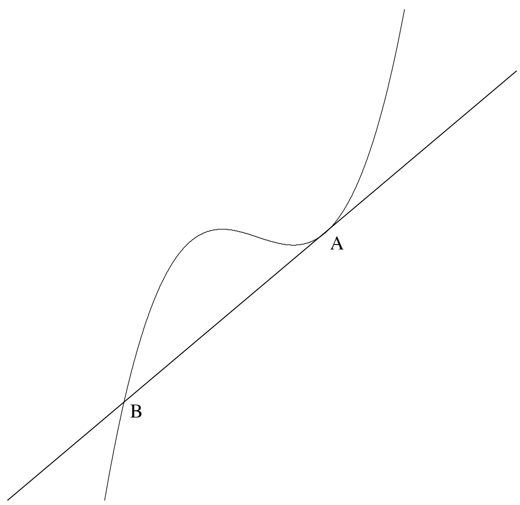

Figure 5: The line is tangent to the curve at point A but not at point B.

Imagine taking a very powerful magnifying glass and looking very closely at this figure near the point A. Figure 6 shows two views of this curve at successively greater magnifications. The closer we look at the curve near the point A the straighter the curve appears to be. The more we zoom in the more the curve begins to look like the straight line. This straight line is called the *tangent to the curve at the point* A. If we want to draw a straight line that most resembles the curve near the point A, the tangent line is the one that we would draw. It is pretty clear from Figure 5 that no matter how closely we look at the curve near the point B the curve is never going to look like the straight line we have drawn in here. That line is tangent to the curve at A but not at B. The curve does have a tangent at B, but it is not shown on Figure 5.

Note that it is not necessarily true that the tangent line only cuts the curve at one point or that curve lies entirely on one side of the line. These properties hold for some special curves like circles, but not for all curves, and certainly not for the one in Figure 5.

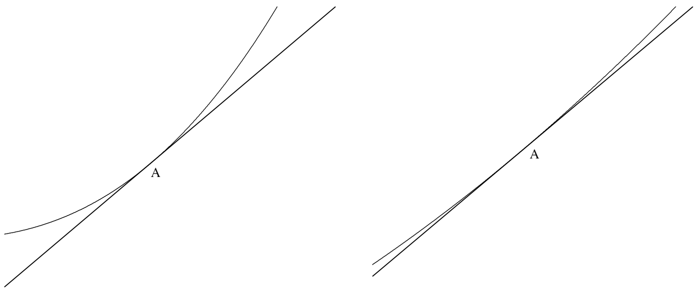

Figure 6: Two close up views of the curve in Figure 5 near the point A. The closer we look near the point A the more the curve looks like the tangent.

2.2 The derivative: the slope of a tangent to a graph

Terminology The slope of the tangent at the point $(x, f(x))$ on the graph of f is called the *derivative of f at x* and is written $f'(x)$.

Look at the graph of the function $y = f(x)$ in Figure 7. Three different tangent lines have been drawn on the graph, at A, B and C, corresponding to three different values of the independent variable, $x = -0.5$, $x = 0.5$ and $x = 1.5$.

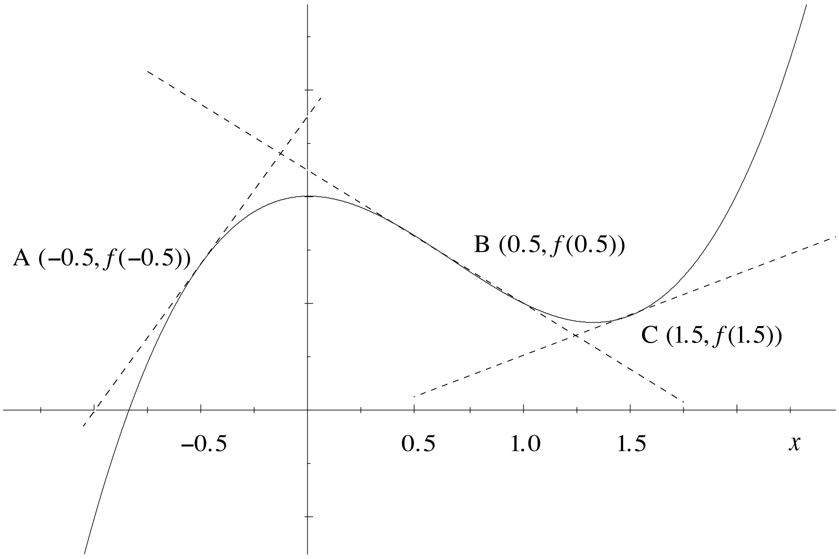

Figure 7: Tangent lines to the graph of $f(x)$ drawn at three different points on the graph.

If we were to make careful measurements of the slopes of the three tangents shown we would find that $f'(-0.5) \approx 2.75$, $f'(0.5) \approx -1.25$ and $f'(1.5) \approx 0.75$. Here the symbol $\approx$ means 'is approximately'. We can only say approximately here because there is no way that we can make completely accurate measurements from a graph, and no way even to draw a completely accurate graph. However this graphical approach to finding the approximate derivative is often very useful, and in some situations may be the only technique that we have.

At different points on the graph we get different tangents having different slopes. The slope of the tangent to the graph depends on where on the graph we draw the tangent. Because we can specify a point on the graph by just giving its x coordinate (the other coordinate is then $f(x)$), we can say that *the slope of the tangent to the graph of a function depends on the value of the independent variable* x, or the value of $f'(x)$ depends on x. In other words, f' is a *function* of x.

Terminology The function f' is called the *derivative* of f.

Terminology The process of finding the derivative is called *differentiation.*

The derivative of a a function f is another function, called f', which tells us about the slopes of tangents to the graph of f. Because there are several different ways of writing functions, there are several different ways of writing the derivative of a function. Most of the ways that are commonly used are expressed in the following table.

Function	**Derivative**
$f(x)$	$f'(x)$ or $\frac{df(x)}{dx}$
f	f' or $\frac{df}{dx}$
y	y' or $\frac{dy}{dx}$
$y(x)$	$y'(x)$ or $\frac{dy(x)}{dx}$

Exercise 2.1 (You will find this exercise easier to do if you use graph paper.)

Draw a careful graph of the function $f(x) = x^2$. Draw the tangents at the points $x = 1$, $x = 0$ and $x = -0.5$. Find the slopes of these lines by picking two points on them and using the formula

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

These slopes are the (approximate) values of $f'(1)$, $f'(0)$ and $f'(-0.5)$ respectively.

Exercise 2.2

Repeat Exercise 2.1 with the function $f(x) = x^3$.

3 Solutions to exercises

Exercise 1.1

From the graph, at time and at time $t = 6$ the motorist is 250 metres from home.

Exercise 1.2

At time $t = 0$ the motorist is 100 metres from home and at time $t = 2$ the motorist is 150 metres from home, so in the first 2 seconds the motorist has travelled $150 - 100 = 50$ metres. At time $t = 3$ the motorist is 175 metres from home and at time $t = 5$ the motorist is 225 metres from home so in the time from $t = 3$ to $t = 5$ the motorist has travelled $225 - 175 = 50$ metres.

Exercise 1.3 A time $t = 60$ the motorist is 1008 metres from home and at time $t = 62$ the motorist is 1032 metres from home so in the 2 second interval from time $t = 60$ to time $t = 62$ the motorist travelled $1032 - 1008 = 24$ metres. A time $t = 64$ the motorist is 1072 metres from home so in the 2 second interval from time $t = 62$ to time $t = 64$ the motorist has travelled $1072 - 1032 = 40$ metres.

Exercise 2.1 Refer to Figure 8. We have used the indicated points on the lines to calculate the slopes. You may have chosed different points, but your answers should be close to those here. Remember this is only an approximate way of finding the slopes, so you shouldn't consider yourself wrong if you don't get exactly the same answers as here.

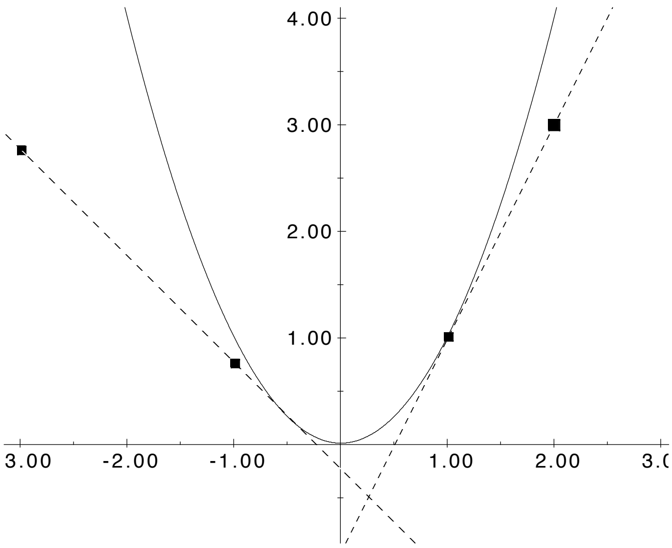

Figure 8: Tangents to graph of $f(x) = x^2$.

Slope of tangent to $f(x) = x^2$ at $x = 1$ is

$$f'(1) \approx \frac{3-1}{2-1} = 2.$$

Slope of tangent to $f(x) = x^2$ at $x = -0.5$ is

$$f'(-0.5) \approx \frac{2.75 - 0.75}{-3-(-1)} = -1.$$

The tangent at $x = 0$ is the x-axis, which has slope 0, so $f'(0) = 0$.

Exercise 2.2

Refer to Figure 9.

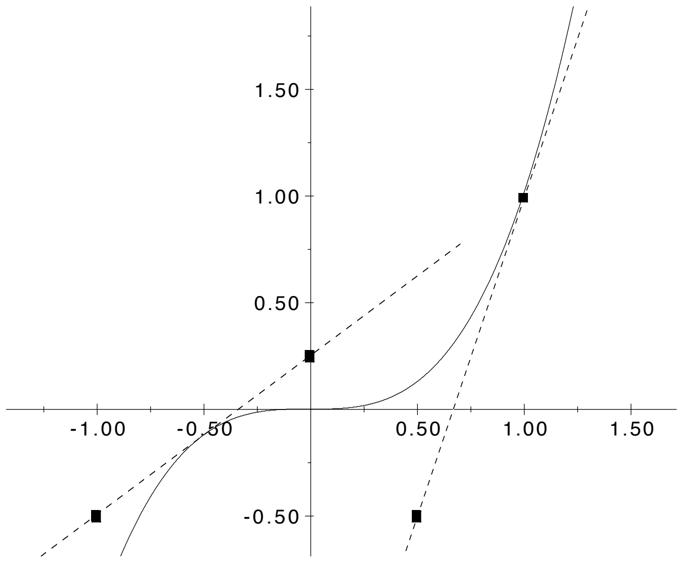

Figure 9: Tangents to graph of $f(x) = x^3$.

Slope of tangent to $f(x)$ at $x = 1$ is

$$f'(1) \approx \frac{1 - (-0.5)}{1 - 0.5} = 3.$$

Slope of tangent to $f(x)$ at $x = -0.5$ is

$$f'(-0.5) \approx \frac{0.25 - (-0.5)}{0 - (-1)} = 0.75.$$

As in Exercise 2.1, the tangent at $x = 0$ is the x-axis which has slope 0, so $f'(0) = 0$.