2 Unit Bridging Course

Day 9 - The Product Rule of Differentiation

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Products of functions

Another way of combining functions to make new ones is by multiplying then together to form a product.

Examples:

$$y = (x + 1)(x^2 + 3)$$

$$f(x) = \sqrt{x}(x^3 - 3x^2 + 7)$$

$$h(t) = t\,e^t.$$

We can differentiate the first two examples by multiplying out the brackets (note that $\sqrt{x} = x^{\frac{1}{2}}$), but that method does not work for the third example.

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Product rule

We need a rule to differentiate product functions in general.

The Product rule

If $y = u \times v$, where $u = f(x)$ and $v = g(x)$, then

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}.$$

Alternatively, If $h(x) = f(x) \times g(x)$ then

$$h'(x) = f(x)g'(x) + g(x)f'(x).$$

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Example

Let $y = (x+1)(x^2+3)$, then

$$\underset{\substack{\uparrow \\ u}}{(x+1)}\underset{\substack{\uparrow \\ v}}{(x^2+3)}$$

$$\begin{aligned}
\frac{dy}{dx} &= u \times \frac{dv}{dx} + v \times \frac{du}{dx} \\
&= (x+1) \times 2x + (x^2+3) \times 1 \\
&= 2x^2 + 2x + x^2 + 3 \\
&= 3x^2 + 2x + 3.
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Example

Consider $y = h(t) = te^t$. Let $u = t$ and $v = e^t$.

Notice that y, u and v are functions of t so the derivatives will be with respect to t.

Then

$$\begin{aligned} \frac{dy}{dt} &= u \times \frac{dv}{dt} + v \times \frac{du}{dt} \\ &= t \times e^t + e^t \times 1 \\ &= e^t(t+1). \end{aligned}$$

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Practice questions

Differentiate the following functions:

(i) $x^2 e^x$

(ii) $x^3\sqrt{x-1}$

(iii) $(x+1)e^{x^2}$ (Hint: you need the chain rule too).

Answers to practice questions

(i) $x^2e^x + 2xe^x$

(ii) $x^3 \times \frac{1}{2}(x-1)^{-\frac{1}{2}} + 3x^2(x-1)^{\frac{1}{2}}$

(iii) $2x(x+1)e^{x^2} + e^{x^2}(1)$.