

An Introduction to Matrix Algebra

Jackie Nicholas

Mathematics Learning Centre



THE UNIVERSITY OF
SYDNEY

Matrix multiplication

Jackie Nicholas
Mathematics Learning Centre
University of Sydney

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Multiplying matrices

We can multiply matrices A and B together to form the product AB provided the number of columns in A **equals** the number of rows in B .

$$\text{If } A = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 0 & -1 \end{bmatrix}$$

then we can define AB as A has three columns and B has three rows.

Multiplying matrices

$$\text{If } A = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -5 \\ -1 & -4 \end{bmatrix}$$

then AB is not defined as A is a 2×3 matrix and B is a 2×2 matrix; the number of columns of A does not equal the number of rows of B .

On the other hand, the product BA **is** defined as the number of columns of B , 2, does equal the number of rows of A .

This tells us something very important; order matters!!

In most cases $AB \neq BA$. Here AB is not defined whereas BA is.

How to multiply matrices

In general, if A is a $m \times n$ matrix and B is a $n \times p$ matrix, the product AB will be a $m \times p$ matrix.

Let $C = AB$. It is a $m \times p$ matrix.

Recall that the entry in the i th row and j th column of C , ie the (i, j) th entry of C , is called c_{ij} .

The entry c_{ij} is the product of the i th row of A and the j th column of B as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \cdots + a_{in}b_{nj}.$$

Example of multiplying matrices 1

That probably looked a bit complicated so we will go through an example.

$$\text{Let } A = \begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$$

A is a 3×2 matrix and B is a 2×2 , so AB is defined.

$$\text{If } C = AB \text{ is then } C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} \text{ is a } 3 \times 2 \text{ matrix.}$$

Example of multiplying matrices 2

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

For the first entry c_{11} we multiply the **first** row of A with the **first** column of B as follows:

$$\begin{bmatrix} 0 & -5 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} 0 & \cdot \\ -3 & \cdot \end{bmatrix} = \begin{bmatrix} 0 \times 0 + -5 \times -3 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

ie $c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21} = 15$.

Example of multiplying matrices 3

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 15 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

For the entry c_{12} we multiple the **first** row of A with the **second** column of B as follows:

$$\begin{bmatrix} 0 & -5 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} \cdot & -1 \\ \cdot & 2 \end{bmatrix} = \begin{bmatrix} \cdot & 0 \times -1 + -5 \times 2 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

ie $c_{12} = a_{11} \times b_{12} + a_{12} \times b_{22} = -10$.

Example of multiplying matrices 4

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 15 & -10 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

For the entry c_{21} we multiple the **second** row of A with the **first** column of B as follows:

$$\begin{bmatrix} \cdot & \cdot \\ -1 & -4 \\ \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} 0 & \cdot \\ -3 & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ -1 \times 0 + -4 \times -3 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

ie $c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21} = 12$.

Example of multiplying matrices 5

So, to multiply two matrices we systematically work out each entry in this way, starting with the first entry.

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 15 & -10 \\ 12 & -7 \\ 6 & -10 \end{bmatrix}$$

Click to see how we get the other entries.

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Properties of matrix multiplication

Let A , B and C be matrices of dimensions for which the following expressions make sense, and let λ be a scalar. Then,

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$\lambda(AB) = (\lambda A)B = A(\lambda B)$$

Note also that $0A = 0 = A0$

The 0 in the last property could mean the scalar zero or the (appropriate) zero matrix. For example,

$$0 \begin{bmatrix} 2 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$