An Introduction to Matrix Algebra

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The inverse matrix

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The identity matrix

Recall, we defined an identity matrix as a square matrix with 1’s down the main diagonal and 0’s everywhere else.

So $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the $2 \times 2$ identity matrix,

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the $3 \times 3$ identity matrix, and

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the $4 \times 4$ identity matrix.
Identity matrices

Identity matrices are important as they have the property that when we multiply a matrix $A$ by the appropriate identity matrix, the product is $A$ itself.

If $A$ is a $2 \times 3$ matrix, then

$$I_2 \times A = A = A \times I_3.$$  

Example: If $A = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix}$ then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix}.$$  

Check that $A \times I_3 = A$ for yourself.
Inverse matrices

Let $A$ be an $n \times n$ matrix.

The inverse of a matrix $A$ is a matrix $B$ such that

$$AB = BA = I_n.$$ 

If $AB = BA$ then matrix $B$ must also be $n \times n$.

Thus only square matrices can have an inverse.
An inverse matrix is unique

If $A$ is a square $n \times n$ matrix which has an inverse, the inverse is unique.

Notation

The inverse of a matrix $A$, if it exists, is denoted by $A^{-1}$ so

$$AA^{-1} = A^{-1}A = I.$$ 

Note, not all square matrices have inverses.
The inverse of a $2 \times 2$ matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 \times 2$ matrix.

We define the *determinant* of $A$ as

$$\det A = |A| = ad - bc.$$ 

The matrix $A$ has an inverse $A^{-1}$ if

$$\det A = |A| = ad - bc \neq 0,$$

in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
Examples

Let \( A = \begin{bmatrix} 0 & -5 \\ 1 & -4 \end{bmatrix} \), then \( \det A = 0 - (-5)(1) = 5 \neq 0 \)

so the inverse \( A^{-1} \) exists.

\[
A^{-1} = \frac{1}{5} \begin{bmatrix} -4 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & 1 \\ -\frac{1}{5} & 0 \end{bmatrix}.
\]

Let \( B = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \), then \( \det B = 2(-3) - (-6)(1) = 0 \).

So \( B \) is not invertible, ie \( B^{-1} \) does not exist.