An Introduction to Matrix Algebra

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Determinants

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Determinants

Recall, we defined the determinant of the $2 \times 2$ matrix

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ as } \det A = ad - bc. \]

We also write this as

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \]

We can define the determinant of any square matrix.

Let’s start with a $1 \times 1$ matrix.

If $B = \begin{bmatrix} b_{11} \end{bmatrix}$, then

\[ \det B = \left| B \right| = b_{11}. \]
Determinant of a $3 \times 3$ matrix

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ then

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$  

Notice the negative sign before $b$ (to be discussed later).

The $2 \times 2$ determinant after each coefficient is the determinant you get by deleting the row and column that coefficient is in.

Click to see how this works.
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Determinant of a $3 \times 3$ matrix continued

Recall

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\]

We have expanded the determinant along the first row; our coefficients of the $2 \times 2$ matrices are $a$, $b$ and $c$.

But the coefficient of $b$ is negative and so to see why we need the following “matrix of signs”.

\[
\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}
\]

The “matrix of signs” tells us whether to multiply our coefficient by $+1$ or $-1$ according to its position. $a$ and $c$ are multiplied by $+1$ while $b$ is multiplied by $-1$. 
Expanding a determinant

We can use the “matrix of signs” to expand the determinant along any row or column.

\[
\begin{bmatrix}
  + & - & + \\
  - & + & - \\
  + & - & + \\
\end{bmatrix}
\]

Suppose now we want to expand the determinant down the second column.

\[
\det A = |A| = -b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + e \begin{vmatrix} a & c \\ g & k \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix}.
\]

Click to see how we get the correct 2 × 2 determinants.

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\]
Definition and an example

Expansion along any row or any column of a determinant always gives us the same answer, which is the value of the determinant.

Let \( A = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & -4 \\ 4 & 1 & -1 \end{bmatrix} \) then expanding along the first row gives

\[
|A| = -1 \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} - 5 \begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\
= -1(0 + 4) - 5(-1 + 16) + 2(1 - 0) = -77.
\]

Expanding down the second column gives

\[
|A| = -5 \begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 0 - 1 \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} \\
= -5(-1 + 16) - 1(4 - 2) = -77.
\]
Expanding larger determinants: a $4 \times 4$ example

Example: For $4 \times 4$ matrices, the “matrix of signs” is

$$\begin{bmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & + \\
\end{bmatrix}$$

$$\begin{vmatrix}
3 & -1 & 0 & 2 \\
0 & 0 & 1 & 0 \\
-4 & -1 & 3 & 1 \\
0 & 2 & -1 & 1 \\
\end{vmatrix} = 0 + 0 - 1 \begin{vmatrix}
-4 & -1 & 1 \\
0 & 2 & 1 \\
\end{vmatrix} + 0$$

$$= (-1) \begin{vmatrix}
-2 & 3 & 2 & 1 \\
-4 & 1 & 4 & -4 \\
\end{vmatrix} + 3 \begin{vmatrix}
-4 & -1 \\
1 & 1 \\
\end{vmatrix}$$

$$= (-1)[(-2)(3 + 8) + (-3 - 4)] = 29.$$

We expanded the $4 \times 4$ determinant along the second row, and the $3 \times 3$ determinant along the third row.