

# An Introduction to Matrix Algebra

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# Determinants

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# Determinants

Recall, we defined the determinant of the  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ as } \det A = ad - bc.$$

$$\text{We also write this as } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We can define the determinant of any square matrix.

Let's start with a  $1 \times 1$  matrix.

If  $B = [ b_{11} ]$ , then

$$\det B = | B | = b_{11}.$$

# Determinant of a $3 \times 3$ matrix

If  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$  then

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Notice the negative sign before  $b$  (to be discussed later).

The  $2 \times 2$  determinant after each coefficient is the determinant you get by deleting the row and column that coefficient is in.

Click to see how this works.

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## Determinant of a $3 \times 3$ matrix continued

Recall

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

We have expanded the determinant along the first row; our coefficients of the  $2 \times 2$  matrices are  $a$ ,  $b$  and  $c$ .

But the coefficient of  $b$  is negative and so to see why we need the following “matrix of signs”.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The “matrix of signs” tells us whether to multiply our coefficient by  $+1$  or  $-1$  according to its position.  $a$  and  $c$  are multiplied by  $+1$  while  $b$  is multiplied by  $-1$ .

## Expanding a determinant

We can use the “matrix of signs” to expand the determinant along any row or column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Suppose now we want to expand the determinant down the second column.

$$\det A = |A| = -b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + e \begin{vmatrix} a & c \\ g & k \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix}.$$

Click to see how we get the correct  $2 \times 2$  determinants.

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## Definition and an example

Expansion along any row or any column of a determinant always gives us the same answer, which is the value of the determinant.

Let  $A = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & -4 \\ 4 & 1 & -1 \end{bmatrix}$  then expanding along the first row gives

$$\begin{aligned} |A| &= -1 \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} - 5 \begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ &= -1(0 + 4) - 5(-1 + 16) + 2(1 - 0) = -77. \end{aligned}$$

Expanding down the second column gives

$$\begin{aligned} |A| &= -5 \begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 0 - 1 \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} \\ &= -5(-1 + 16) - 1(4 - 2) = -77. \end{aligned}$$

## Expanding larger determinants: a $4 \times 4$ example

Example: For  $4 \times 4$  matrices, the “matrix of signs” is

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ -4 & -1 & 3 & 1 \\ 0 & 2 & -1 & 1 \end{vmatrix} = 0 + 0 - 1 \begin{vmatrix} 3 & -1 & 2 \\ -4 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix} + 0$$
$$= (-1) \left[ -2 \begin{vmatrix} 3 & 2 \\ -4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ -4 & -1 \end{vmatrix} \right]$$
$$= (-1)[(-2)(3 + 8) + (-3 - 4)] = 29.$$

We expanded the  $4 \times 4$  determinant along the second row, and the  $3 \times 3$  determinant along the third row.