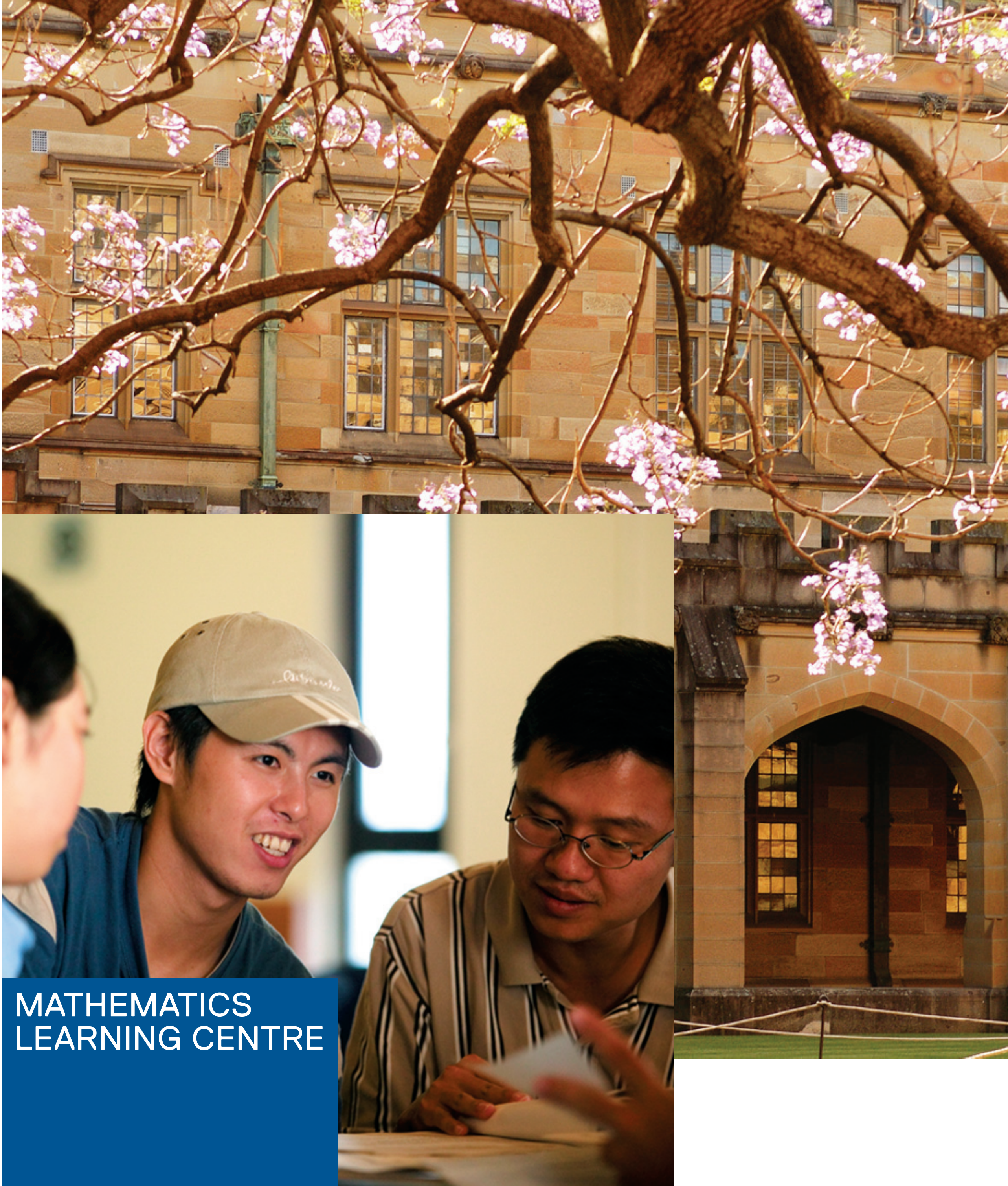

MATHEMATICS
LEARNING CENTRE

THE UNIVERSITY OF SYDNEY

INTRODUCTION TO PROBABILITY THEORY

Introduction to Probability Theory

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Sue Gordon
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Contents

1 Introduction

Probability Theory is a way in which we can study scientifically things that happen by chance. Consider the following questions:

1. What are your chances of winning a raffle in which 325 people have bought 1 ticket each?
2. If a coin is to be tossed 4 times and on the first 3 trials 'heads' comes up, what are the chances of getting 'tails' on the 4th trial?
3. If two dice are tossed is it more likely that you will get a '3' and a '4' thrown or a '1' and a '1'?
4. What are the chances that Labour will win the next election?

Questions 1, 2 and 3 can be answered exactly. We have enough theory at our disposal and the situations are sufficiently simple to be evaluated easily. We will answer them in the course of this unit. Question 4 is far more difficult, as the conditions affecting the outcome are complex, numerous and changing. Hence polls which try to give a simple answer to question 4 are often wrong.

1.1 How to use this book

You will not gain much just by reading this book. Have pen and paper ready and try to work through the examples before reading their solutions. Do **all** the exercises. Solutions to the exercises are at the back of the book. It is important that you try hard to complete the exercises on your own rather than refer to the solutions as soon as you are stuck.

Objectives

By the time you have worked through this book you should:

a. Understand what is meant by the concept of probability;

b. Be able to represent diagrammatically situations often encountered in probability problems;

c. Know how to calculate simple probabilities when there are a finite number of equally likely outcomes;

d. Understand what is meant by the terms 'Complementary Events', 'Incompatible Event', 'Conditional Probability' and 'Independence', and be able to use these concepts to compute probabilities.

Assumed knowledge

a. Operations with fractions.

b. Familiarilty with set notation. The set concepts needed are revised in Chapter 2.

2 Set Notation

You may omit this section if you are familiar with these concepts.

A set is a collection of objects.

We often specify a set by listing its members, or **elements**, in parentheses like this $\{\}$.

For example $A = \{2, 4, 6, 8\}$ means that A is the set consisting of numbers 2,4,6,8.

We could also write $A =$\{even numbers less than 9\}.

The **union** of A and B is the set of elements which belong to both A and B, and is denoted by $A \cup B$.

The **intersection** of A and B is the set of elements which belong to both A and B, and is denoted by $A \cap B$.

The **complement** of A, denoted by $\overline{A}$, is the set of all elements which do not belong to A. In making this definition we assume that all elements we are thinking about belong to some larger set U, which we call the **universal** set.

The **empty set**, written $\emptyset$ or $\{\}$, means the set with no elements in it.

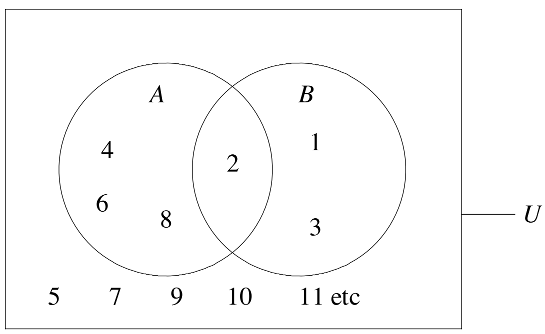
A set C is a **subset** of A if **all** the elements in C are also in A.

Diagrams called **Venn Diagrams** are often used to illustrate set concepts.

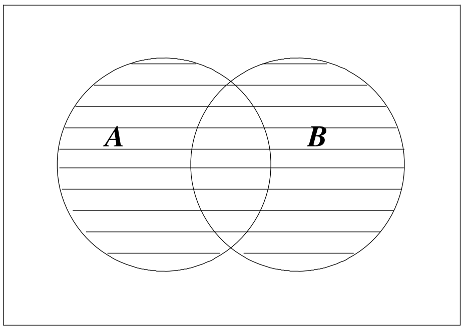
For Example, let

$$\begin{aligned} U &= \{\text{all positive numbers}\} \\ A &= \{2, 4, 6, 8\} \\ B &= \{1, 2, 3\} \end{aligned}$$

This may be represented in a Venn Diagram as follows:

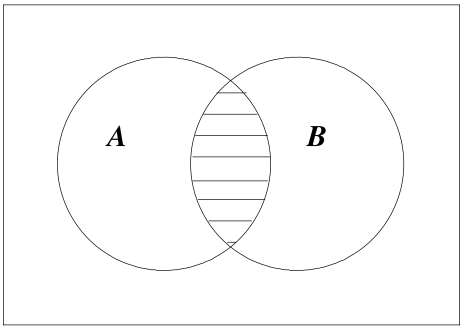


$A \cup B = \{1, 2, 3, 4, 6, 8\}$



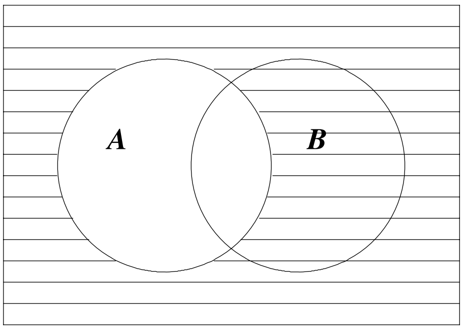
$A \cup B$ is shaded

$A \cap B = \{2\}$



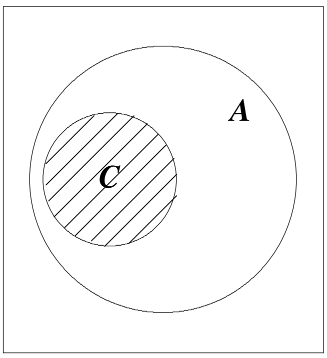
$A \cap B$ is shaded

$\overline{A} = \{1, 3, 5, 7, 9, 10, 11, 12, ...\}$



$\overline{A}$ is shaded

Let $C = \{6, 8\}$ Then C is a subset of A



C is shaded

3 Finite Equiprobable Spaces

In loose terms, we say that the probability of something happening is $\frac{1}{4}$ if, when the experiment is repeated often under the same conditions, the stated result occurs 25% of the time.

For the moment, we will confine our discussion to cases where there are a finite number of equally likely outcomes.

For example, if a coin is tossed, there are two equally likely outcomes: heads (H) or tails (T). If a die is tossed, there are six equally likely outcomes: 1,2,3,4,5,6.

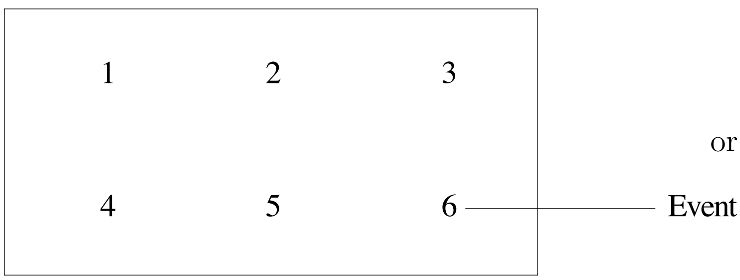
3.1 Some notation

The set of all possible outcomes of the given experiment is called the **sample space**. An **event** is a subset of a sample space.

3.2 Calculating probabilities

Look again at the example of rolling a six faced die. The possible outcomes in this experiment are 1,2,3,4,5,6, so the sample space is the set {1,2,3,4,5,6}. The 'event' of 'getting a 6' is the subset {6}. We can represent this in many ways.

For example:



or $\{1, 2, 3, 4, 5, \mathbf{6}\}$

There are six possibilities in the sample space and only one of these corresponds to getting a 6, so the probability of getting a 6 when you roll a die is $\frac{1}{6}$.

We say that the probability of an event A occuring is

$$P(A) = \frac{\text{Number of elements in } A}{\text{Total number of elements in the sample space}}$$

Example

If a fair coin is tossed, it is clear from our definition of probability above that P (obtaining a head) $= \frac{1}{2}$.

Example

A card is selected at random from a pack of 52 cards. Let $A =$ 'the card is a heart' and $B =$ 'the card is an ace'.

Find $P(A)$, $P(B)$.

Solution

$P(A) = \frac{13}{52}$ since there are 13 hearts in the pack. $P(B) = \frac{4}{52}$ since there are 4 aces in the pack.

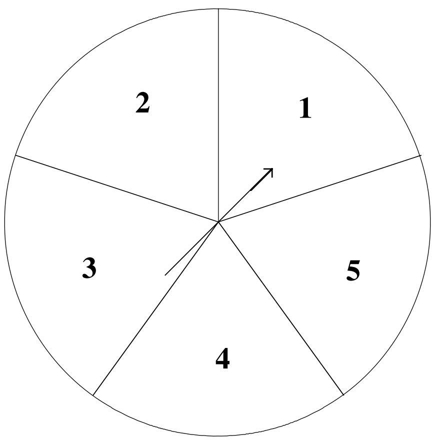
To calculate the probability of an event, we simply need to find out the total number of possible outcomes of an experiment and the number of outcomes which correspond to the **given** event.

Exercise 1

What are your chances of winning a raffle in which 325 tickets have been sold, if you have one ticket?

Exercise 2

A cursor is spun on a disc divided into five equal sectors as shown below. The position of the pointer is noted. (If it is on a line the cursor is spun again.)



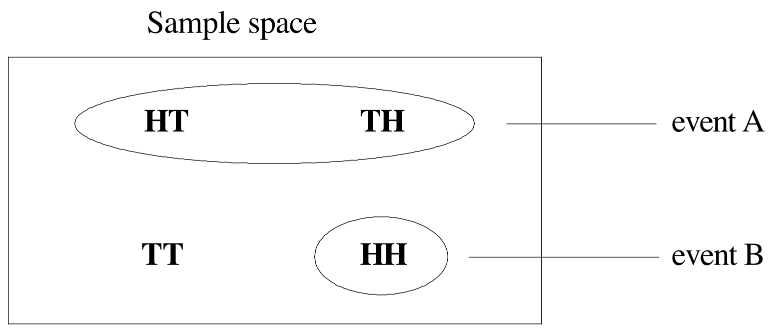
Let A be the event 'pointer is in the first sector' and B the event 'pointer is in the 2nd or 4th sector'.
Find $P(A)$, $P(B)$.

Example

Consider the following problem. Two coins are tossed. Let A be the event 'one head and one tail is obtained', and, B be the event 'two heads are obtained'.

Find $P(A)$, $P(B)$.

Solution



Since there are 4 outcomes in the sample space.
$P(A) = \frac{2}{4} = \frac{1}{2}$
$P(B) = \frac{1}{4}$

Notice that HT and TH must be regarded as **different** events.

Example - Rolling two dice.

If two dice are rolled, each of them can show the numbers 1,2,3,4,5, or 6. Suppose one die is yellow and one is white. Then we can represent the sample space as follows:

White Die						
6	1, 6	2, 6	3, 6	4, 6	5, 6	6, 6
5	1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
4	1, 4	2, 4	**A** 3, 4	4, 4	5, 4	6, 4
3	1, 3	2, 3	3, 3	**B** 4, 3	5, 3	6, 3
2	1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1	**C** 1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
	1	**2**	**3**	**4**	**5**	**6**

Yellow Die

Counting the number of points in the sample space, we obtain 36.

A '3' on the yellow die and a '4' on the white die is represented by box **A**.

A '4' on the yellow die and a '3' on the white die is represented by box **B**.

Thus if T is the event 'rolling a three and a four', the probability of T is $P(T) = \frac{2}{36} = \frac{1}{18}$.

If N is the event 'rolling two ones' then this is represented by box **C** on the diagram and $P(C) = \frac{1}{36}$.

Exercise 3

Two fair dice are rolled. What is the probability that at least one 3 is showing?

Often we can conveniently represent the possible outcomes on a diagram and count directly. We will also develop some techniques and rules to assist in our calculations.

Now let us see what happens in reality. Try the following experiment:

Roll a die 50 times and record the number of each of the quantities 1,2,3,4,5,6.

Continue rolling and record the number of each quantity after 100 rolls. Now record the number after 200 rolls. Find the **relative frequency** of each quantity after 50, 100 and 200 rolls.

For example calculate $\frac{\text{the number of times `1' occurs}}{\text{total number of rolls}}$.

Does it get closer to $\frac{1}{6}$, i.e. 0.17?

4 Complementary Events

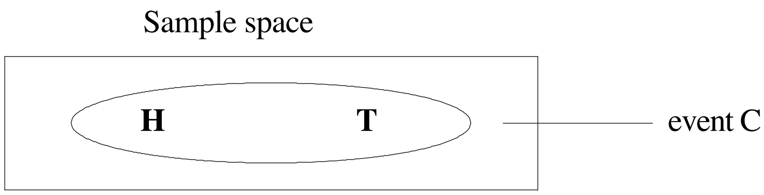
4.1 Certain and impossible events

If an event is a **certainty**, then its probability is one. In common language we often say it is 100% certain (which is the same thing).

For example, in the coin tossing experiment, let C be the event 'obtaining a head or a tail'.

The sample space is {H, T}. The event is {H, T}.

So $P(C) = \frac{2}{2} = 1$.



Example

If a normal die is rolled, what is the probability that the number showing is less than 7?

Solution

Sample space = {1,2,3,4,5,6}

Event = {1,2,3,4,5,6}

Hence the probability (number is less than 7) $= \frac{6}{6} = 1$.

If an event is **impossible**, then its probability is zero.

Example

Find the probability of throwing an 8 on a normal die.

Here there are **no** possible outcomes in the event.

i.e. Sample space = {1,2,3,4,5,6}

Event = {}, i.e. the empty set.

Hence the probability of throwing an 8 is $\frac{0}{6} = 0$.

4.2 Complementary events

If the event is neither impossible nor certain, then clearly its **probability is between 0 and 1**.

Two events are **complementary** if they cannot occur at the same time and they make up the whole sample space.

Example

When a coin is tossed, the sample space is $\{H, T\}$ and the events H = 'obtain a head' and T = 'obtain a tail' are complementary.

If we calculate the probabilities we find that
$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ and $P(H) + P(T) = 1$.

Example

A die is rolled. Let A be the event 'a number less than 3 is obtained' and let B be the event 'a number of 3 or more is obtained'.

Then $P(A) = \frac{2}{6}$, and $P(B) = \frac{4}{6}$.

So that $P(A) + P(B) = 1$.

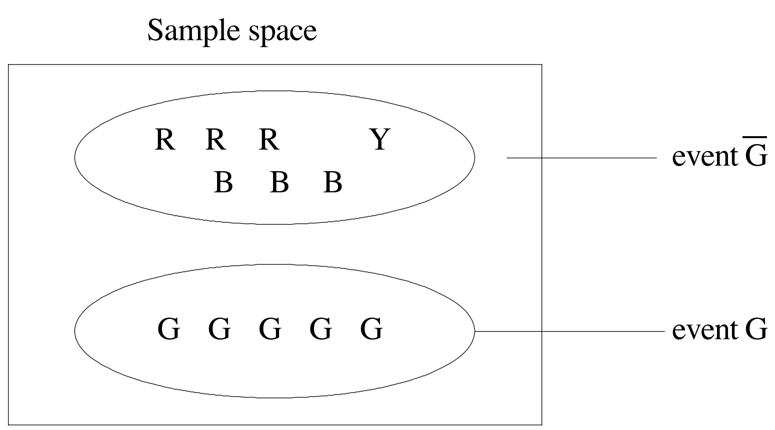
We have illustrated the law that if two events are **complementary**, then their **probabilities add up to 1**.

Example

A marble is drawn at random from a bag containing 3 red, 3 blue, 5 green and 1 yellow marbles. What is the probability that it is not green?

Solution

There are two ways of doing this problem.



$\overline{G}$ denotes not G. G and $\overline{G}$ are complementary events.

Method A:

We can work out the probability that the marble is green:
$P(G) = \frac{5}{12}$.

Since a marble is either green or not green, the probability that it is not green,
$P(\overline{G}) = 1 - \frac{5}{12} = \frac{7}{12}$.

Method B:

Alternatively, we can find the probability that the marble is red, blue or yellow which is $\frac{7}{12}$ (by counting from the diagram above).

Exercise 4

Three tulip bulbs are planted in a window box. Find the probability that at least one will flower if the probability that all will fail to flower is $\frac{1}{8}$.

Sometimes calculations are made easier by using complementary events. One such problem is included in the worksheet in Chapter 9 of this module.

5 Mutually Exclusive Events

Two events are **incompatible, disjoint** or **mutually exclusive** when the occurence of one precludes the occurrence of the other, i.e. they can never occur at the same time. For example, we can never have the head side and the tail side of a coin face up at the same time.

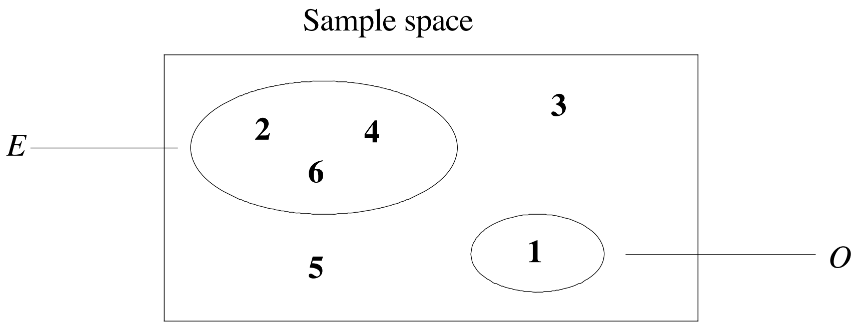
Example

Suppose a die is tossed.

Then the events E = 'obtaining an even number'

and O = 'obtaining a one'

are incompatible.



On our diagram the events do not intersect.

$$\begin{aligned} \text{Notice that } P(\text{throwing an even number or one}) &= P(1,2,4,6) \\ &= \frac{4}{6} \\ &= P(E)+P(O). \end{aligned}$$

Example

What is the probability of drawing a heart or spade from a pack of 52 cards when one card is drawn at random?

Solution

$P(\text{heart}) = \frac{13}{52}$.

$P(\text{spade}) = \frac{13}{52}$.

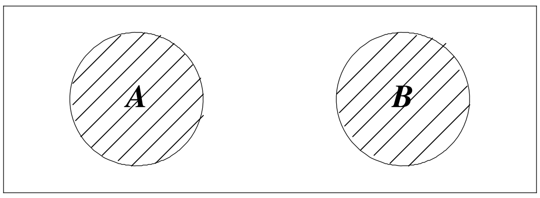
$P(\text{heart or spade}) = \frac{26}{52}$ since 26 of the cards are either hearts or spades.

Notice $P(\text{heart or spade}) = P(\text{heart}) + P\ (\text{spade})$.

We may now state the **addition law** for **mutually exclusive events**.

If two events A and B are mutually exclusive, the probability of A or B happening, denoted $P(A \cup B)$, is:

$$P(A \cup B) = P(A) + P(B).$$



shaded area $= A \cup B$

Exercise 5

A number is selected at random from the integers 2 to 25. Find the probability that it is:

(a) a perfect square;

(b) a prime number;

(c) a prime number or perfect square.

Example

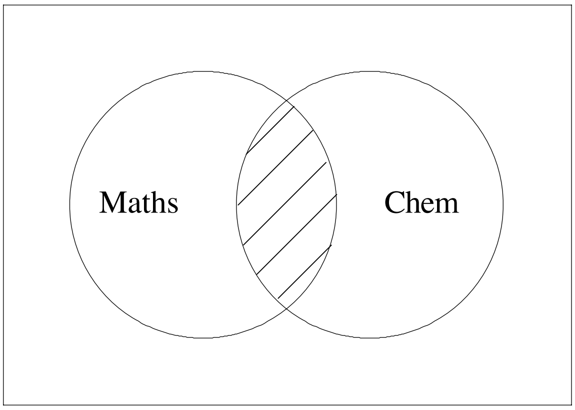
What is the flaw in the following argument?

'Seventy percent of first year science students study mathematics. Thirty percent of first year science students study chemistry. If a first year science student is selected at random, the probability that the student is taking maths is $\frac{70}{100}$, the probability that the student is taking chemistry is $\frac{30}{100}$, hence the probability that the student is taking maths or chemistry is $\frac{70}{100} + \frac{30}{100} = 1$ (i.e. a certainty).'

Solution

The two events are not mutually exclusive, therefore we cannot add the probabilities.

That is, to count all the students doing maths and/or chemistry, we need to count all the maths students, all the chemistry students, and subtract from this the number of students who were counted twice because they were in both classes.



M = 'student takes maths'
C = 'student takes chem'

$M \cup C$ means that at least one of M or C occurs. $M \cap C$ means that both M and C occur. Hence

$$P(M \cup C) = P(M) + P(C) - P(M \cap C).$$

To Summarise:

For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

This rule works even if A and B are mutually exclusive. If A and B are mutually exclusive then $P(A \cap B) = 0$, and A and B cannot happen together, so that $P(A \cup B) = P(A) + P(B)$.

Exercise 6

A maths class consists of 14 women and 16 men. Of these, 12 of the men and half of the women study computer science. A person is chosen at random from the class. Find the probability that the person selected is:

(a) a woman;

(b) studying computer science;

(c) a woman who is studying computer science;

(d) a woman or is taking computer science.

Exercise 7

A bag of marbles contains 23 Tiger's Eyes, 17 Rainbows and 5 Pearls.

One marble is drawn at random.

Denote by:

T the event 'a Tiger's Eye is drawn';

R the event 'a Rainbow is drawn';

P the event 'a Pearl is drawn'.

Describe the following events in words and find their probabilities:

(a) $R \cup T$,

(b) $R \cap T$,

(c) $T \cup \overline{P}$,

(d) $T \cup (R \cap P)$,

(e) $\overline{R} \cap (P \cup R \cup T)$.

6 Conditional Probability

A lecture on a topic of public health is held and 300 people attend. They are classified in the following way:

Gender	Doctors	Nurses	Total
Female	90	90	180
Male	100	20	120
Total	190	110	300

If one person is selected at random, find the following probabilities:

(a) P(a doctor is chosen);

(b) P(a female is chosen);

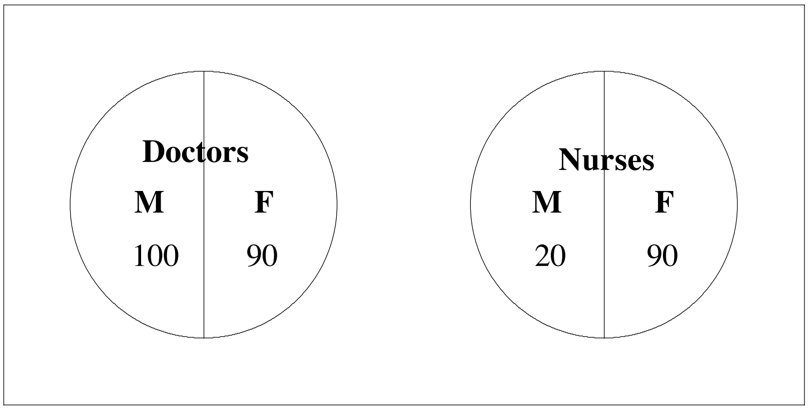
(c) P(a nurse is chosen);

(d) P(a male is chosen);

(e) P(a female nurse is chosen);

(f) P(a male doctor is chosen).

Solution:



(a) The number of doctors is 190 and the total number of people is 300, so $P(\text{doctor}) = \frac{190}{300}$.

(b) $P(\text{female}) = \frac{180}{300}$.

(c) $P(\text{male}) = \frac{120}{300}$.

(d) $P(\text{nurse}) = \frac{110}{300}$.

(e) There are 90 female nurses, therefore $P(\text{female} \cap \text{nurse}) = \frac{90}{300}$.

(f) $P(\text{male doctor}) = P(\text{male} \cap \text{doctor}) = \frac{100}{300}$.

Now suppose you are given the information that a female is chosen and you wish to find the probability that she is a nurse. This is what we call **conditional probability**. We want the probability that the person chosen is a nurse, subject to the condition that we know she is female. The notation used for this is:

$$P(\text{nurse} \mid \text{female})$$

Read this as 'the probability of the person chosen being a nurse, **given** that she is female'.

Since there are 180 females and of these 90 are nurses, the required probability is $\frac{90}{180} = \frac{1}{2}$.

We can see that
$$\begin{aligned} P(\text{nurse} \mid \text{female}) &= \tfrac{90}{180} \\ &= \tfrac{90/300}{180/300} \\ &= \frac{P(\text{nurse} \cap \text{female})}{P(\text{female})} \end{aligned}$$

Definition

The conditional probability of A given B is defined by

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)}, \text{ provided that } P(B) \neq 0.$$

Hence if A and B are any two events with probabilities greater than 0, then

$P(A \cap B) = P(A \,|\, B).P(B)$

or $P(B \,|\, A).P(A)$, as this is $P(B \cap A)$ which is equal to $P(A \cap B)$.

Exercise 8

In the example above, find

$P(\text{female} \mid \text{nurse})$, $P(\text{doctor} \mid \text{male})$, $P(\text{male} \mid \text{doctor})$.

Note that the order matters here: $P(A \mid B)$ is not the same as $P(B \mid A)$.

Example

Suppose a pair of fair dice is tossed; one white and one yellow.

Let A be the event 'the sum of their scores is 4', and let B be the event 'exactly one die shows the score 1'.

Find $P(A)$, $P(B)$, $P(B \mid A)$, $P(A \mid B)$, $P(A \cap B)$.

Solution:

$A = \{ (1,3)\ (3,1)\ (2,2) \}$.
$B = \{ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)\ (2,1)\ (3,1)\ (4,1)\ (5,1)\ (6,1) \}$.
The sample space has 36 elements in it.

White Die						
6	1, 6	2, 6	3, 6	4, 6	5, 6	6, 6
5	1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
4	1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
3	1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
2	1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1	1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
	1	**2**	**3**	**4**	**5**	**6**

Yellow Die

$P(A) = \frac{3}{36}$, and $P(B) = \frac{10}{36}$.
Now, to find $P(B \mid A)$, that is $P(\text{one die shows 1} \mid \text{sum is 4})$, we need only consider the set $A = \{ (1,3)\ (3,1)\ (2,2) \}$.
Two of the three cases show 'exactly one die shows the score 1'.
Therefore, $P(B \mid A) = \frac{2}{3}$.
$P(A \mid B) = P(\text{sum is 4} \mid \text{one die shows 1})$.

Consider the set $B = \{ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)\ (2,1)\ (3,1)\ (4,1)\ (5,1)\ (6,1) \}$. Of these, there are two which add to 4, $(1,3)$ and $(3,1)$.
Therefore, $P(A \mid B) = P(\text{sum is 4} \mid \text{one die shows 1}) = \frac{2}{10}$.

$P(A \cap B)$ can be found by directly:

$A \cap B = \{ (1,3)\ (3,1) \}$
$P(A \cap B) = \frac{2}{36}$,
or $P(A \cap B) = P(A \mid B).P(B) = \frac{2}{10}.\frac{10}{36} = \frac{2}{36}$,
or $P(A \cap B) = P(B \mid A).P(A) = \frac{2}{3}.\frac{3}{36} = \frac{2}{36}$.

Exercise 9

A pair of dice is tossed. If the numbers appearing are different, find the probability that the sum of their scores is 6.

7 Independence

Here is a game of chance. A friend tosses a coin and you bet on the outcome. Suppose she has tossed the coin 3 times and obtained 'heads' all three times. What would you bet on the fourth trial?

You might be inclined to guess 'tails' but would still only have probability $\frac{1}{2}$ of being right.

The chance of getting 'tails' on any one throw is $\frac{1}{2}$. The outcome of one throw is not affected by previous ones - the coin has no memory! (You might, of course, check that the coin does have a tail.)

When the chance of a given outcome remains the same, irrespective of whether or not another event has occurred, the events are said to be **independent**.

Definition:

Two events A and B are said to be independent if and only if $P(A \mid B) = P(A)$, that is, when the conditional probability of A **given** B is the same as the probability of A.

Example

In the problem of nurses and doctors given on page 15, define A to be the event 'a nurse is chosen' and B to be the event 'a female is chosen'.

Are the events A and B independent?

Solution

$$\begin{aligned} P(A \mid B) &= \tfrac{90}{180} = \tfrac{1}{2} \\ P(A) &= \tfrac{110}{300} \neq P(A \mid B) \end{aligned}$$

So A and B are not independent.

Note: From the definition of conditional probability we have

$$P(A \cap B) = P(B \mid A).P(A)$$

Now if A and B are independent, then $P(A \mid B) = P(B)$, so $P(A \cap B) = P(A).P(B)$.

When two events are **independent**, the chance that **both** will happen is found by **multiplying** their individual chances.

This gives us a simple way of checking whether or not events are independent:

A and B are independent events if and only if $P(A \cap B) = P(A).P(B)$.

Note: It is possible to define independence in this way without referring to conditional probability.

Example

What is the probability of obtaining 'six' and 'six' on two successive rolls of a die?

Solution

P(obtaining 6 on a roll of a die) $= \frac{1}{6}$.
The two rolls are independent - the die cannot remember what happened first.
So $P(6 \text{ and } 6) = \frac{1}{6}.\frac{1}{6} = \frac{1}{36}$.
Notice that this is equal to the probability of '2' followed by '3', or indeed any ordered sequence of two numbers.

Example

A box contains three white cards and three black cards numbered as follows:

White			Black		
1	2	2	1	1	2

One card is picked out of the box at random. If A is the event 'the card is black' and B is the event 'the card is marked 2', are A and B independent?

Solution

$P(A) = \frac{1}{2}$ since 3 of the cards are black.
$P(B) = \frac{1}{2}$ since 3 of the cards have '2'.
$P(A \cap B) = P$(card is black and marked 2) $= \frac{1}{6}$ since only one card satisfies this condition.
Now $\frac{1}{6} \neq \frac{1}{2}.\frac{1}{2}$, so A and B are not independent.
If we know the card is black then the chance of it being a 'two' is changed:- it is now $\frac{1}{3}$.
Hence the outcome of one event **does** affect the outcome of the other, which again shows that A and B are not independent.

Exercise 10

A couple has two children. Let A be the event 'they have one boy and one girl' and B the event 'they have at most one boy'. Are A and B independent?

Exercise 11

Two different missiles are shot simultaneously at a practice target. If the probability of the first one hitting the target is $\frac{1}{4}$ and of the second one hitting is $\frac{2}{5}$, what is the probability that

(a) both missiles will hit,

(b) at least one will hit?

8 Summary

1. If there are a finite number of equally likely outcomes of an experiment, the probability of an event A is

$$P(A) = \frac{\text{Number of possible outcomes in } A}{\text{Total number of possible outcomes}}$$

2. The probability of an event happening lies between zero and one. If the event cannot happen, its probability is zero and if it is certain to happen, its probability is one.

3. If two events are **complementary**, i.e. they are mutually exclusive (can't happen together) and make up the whole sample space, then their probabilities add up to 1.

4. For two events A and B the probability of A or B (or both) happening is

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   In particular if A and B are **mutually exclusive**, $P(A \cup B) = P(A) + P(B)$. That is, the chance that **at least one of them** will happen equals the sum of their probabilities.

5. The **conditional probability** of A given B is $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided that $P(B) \neq 0$.

   So $P(A \cap B) = P(A \mid B).P(B)$.

6. A and B are defined to be independent events if $P(A) = P(A \mid B)$. That is, knowing the outcome of one event does not change the probability of the outcome of the other. From (5.) above we see that in this case $P(A \cap B) = P(A).P(B)$, that is the probability of **both** events happening is the **product** of the individual probabilities.

Exercise 12

Can two events A and B, ever be both mutually exclusive and independent?

9 Self Assessment

Try these problems to see if you have understood the module. Wherever possible, use diagrams to aid calculations.

1. Suppose a four sided die is rolled (i.e. a tetrahedron) with sides labelled {1,2,3,4} and the number facing the table is noted.

   Let A be the event 'the number is 3',

   and B the event 'the number is even'.

   Find $P(A), P(B)$.

2. A letter is chosen at random from the word 'probability'. What is the chance that the letter is not the letter 'y'?

3. Two coins are tossed. Find the probability of obtaining:

   (a) at least one head;

   (b) at most two heads.

4. In a raffle there is one first prize of \$100, one second prize of \$50 and one third prize of \$10. There are 100 tickets sold and first prize is drawn first, then second, then third, without replacement. Find the probability that a person buying one ticket in the raffle wins

   (a) first prize;

   (b) a prize;

   (c) at least \$50.

5. Consider the experiment of rolling two dice and adding the numbers obtained to get a total score.

   (a) Calculate the possible values which this total will take and their corresponding probabilities.

   (b) What is the probability of obtaining a score of at least 3?

   (Check that you can do this in two ways - by counting directly and by using complementary events.)

6. Find the probability that a horse 'Mercury' will win the Melbourne Cup if the odds on it winning are 2:3. This means that a horse is likely to win twice for every three losses. If the odds on a second horse 'Alex' winning are 1:4, find the probability that Alex or Mercury wins. Assume a draw is impossible.

7. A couple decide they will have three children. What is the probability that all three will be girls, assuming the probability of producing a daughter is always $\frac{1}{2}$?

8. In a group of 850 first and second year university students, 500 of them take medicine and 350 study engineering. Four hundred of the medical students are in first year and the remaining 100 in second year, while 300 of the engineers are in first year and 50 in second year.

   (a) Summarise this information in a table, assuming no one can simultaneously be a medical and engineering student.

   (b) A student is selected at random from the group.
   Define:
   M to be the event 'the student takes medicine';
   E to be the event 'the student is in engineering';
   F to be the event 'the student is in first year';
   S to be the event 'the student is in second year'.
   Find $P(M \mid F)$, $P(F \mid M)$, $P(E \mid S)$, $P(F)$, $P(S)$, $P(E \cap S)$.

9. Complete each of the following:

   (a) If an event cannot happen, its probability is ....

   (b) If a coin is tossed six times: $P(H\,H\,H\,H\,H\,H) =$ ....

   (c) A die is rolled 4 times the probability of obtaining at least one 6 = ...

10. In a first year mathematics class, 72% of the students are right handed while the remaining 28% are left handed. Among those students known to be left handed, 23% are female while 53% of the right handers are female.

    Let R = 'student is right handed' Let L = 'student is left handed' Let F = 'student is female'.

    A student is selected at random from the class.

    (a) Find the probability that the student is female.
    Hint: $P(F) = P(F \cap L) + P(F \cap R)$

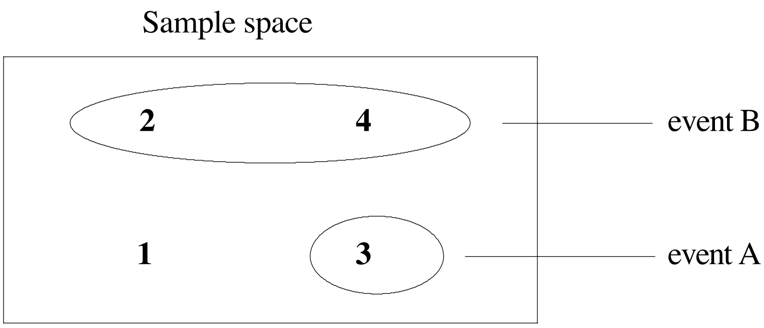
    (b) In this class, is being left handed independent of gender?

    (c) *Given that the student selected is female, find the probability that she is right handed.

    *This is an example of Bayes' Theorem, which you may encounter in your statistics course.

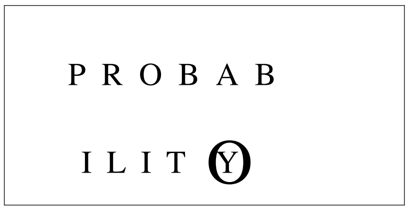
10 Solutions to Self-Assessment

1. From the diagram



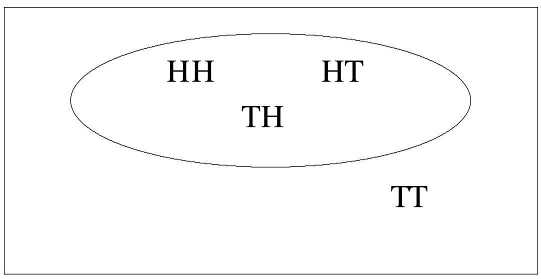
$P(A) = \frac{1}{4}$ $\qquad$ $P(B) = \frac{2}{4} = \frac{1}{2}$.

2. Let Y be the event 'the letter y is chosen'



$$\begin{aligned} P(\text{not the letter y}) &= P(\overline{Y}) = 1 - P(Y) \\ &= 1 - \tfrac{1}{11} \\ &= \tfrac{10}{11}. \end{aligned}$$

3. Let the event $L =$ 'at least one head", and $M =$ 'at most two heads'



   (a) $P(L) = \frac{3}{4}$.

   (b) $P(M) = 1$.

4. (a) $P(\text{winning first prize}) = \frac{1}{100}$.

(b) $P(\text{winning first or second or third prize}) = \frac{1}{100} + \frac{1}{99} + \frac{1}{98}$.
These are mutually exclusive events since once a ticket is drawn it is not replaced.

(c) $P(\text{winning at least \$50}) = P(\text{winning first or second prize}) = \frac{1}{100} + \frac{1}{99}$.

5.

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	sum = 2	3	4	5	6	7
	1	**2**	**3**	**4**	**5**	**6**

Sum of Scores	Probability
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

$$\begin{aligned} P(\text{obtaining at least 3}) &= 1 - P(\text{obtaining less than three}) \\ &= 1 - P(\text{obtaining 2}) \end{aligned}$$

$$= 1 - \frac{1}{36} = \frac{35}{36}$$

(or add $P(3) + P(4) + \cdots + P(12)$).

6. Odds on Mercury winning are 2:3.

   Probability of Mercury winning is $\frac{2}{5}$.

   Probability of Alex winning is $\frac{1}{5}$.

   $P(\text{Mercury or Alex wins}) = P(\text{Mercury wins}) + P(\text{Alex wins})$

   (since these events are mutually exclusive events)

   $$= \frac{2}{5} + \frac{1}{5}$$
   $$= \frac{3}{5}.$$

7. $P(\text{girl and girl and girl}) = P(G) \times P(G) \times P(G)$ assuming the events are independent.

   $$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$
   $$= \frac{1}{8}.$$

   (or write out the sample space and find the number of elements in it).

8. (a)

	Medical Faculty	Engineers	Total
First Year	400	300	700
Second Year	100	50	150
Total	500	350	850

   (b) $P(M \mid F) = \frac{400}{700} = \frac{4}{7}$,
   as we can see from the above table there are 700 first year students and of these 400 are medical students.
   $P(F \mid M) = \frac{400}{500} = \frac{4}{5}$, since there are a total of 500 medical students, 400 of whom are in first year.
   $P(E \mid S) = \frac{50}{150} = \frac{1}{3}$,
   as there are 150 second year students, 50 of whom are engineers.
   $P(F) = \frac{700}{850} = \frac{14}{17}$.
   $P(S) = \frac{150}{850} = \frac{3}{17}$.
   $P(E \cap S) = \frac{50}{850} = \frac{1}{17}$.

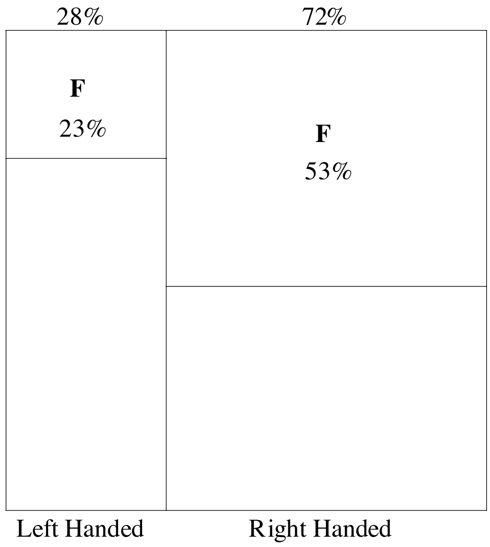
9. (a) Zero.

   (b) $(\frac{1}{2})^6$, since $P(H) = \frac{1}{2}$ and the events are independent.

   (c) $P(\text{at least one six}) = 1 - P(\text{no sixes}) = 1 - (\frac{5}{6})^4$, since $P(\text{not a six}) = \frac{5}{6}$ and the 4 trials are independent.

10. We have: $P(R) = 0.72$ $\quad P(L) = 0.28$

    $P(F \mid L) = 0.23$ $\quad P(F \mid R) = 0.53$



(a) $P(F) = P(F \cap L) + P(F \cap R)$

Now $P(F \cap L) = P(F \mid L) \times P(L)$

$= (0.23)(0.28)$

(or from the diagram 23% of 28%)

$= 0.0644$

And $P(F \cap R) = P(F \mid F) \times P(R)$

$= (0.53)(0.72)$

(or from the diagram 53% of 72%)

$= 0.3816$

Therefore $P(F) = 0.0644 + 0.3816$

$= 0.446 \quad$ (or 44.6%).

(b) Gender and handedness are independent if
$P(F \cap L) = P(F) \times P(L)$.
Now $P(F \cap L) = 0.0644$
and $P(F) \times P(L) = 0.446 \times 0.28 = 0.12488$.
So they are not independent.

(c) We want to find the conditional probability $P(R \mid F)$.

$$P(R \mid F) = \frac{P(F \cap R)}{P(F)}$$

$= \frac{0.3816}{0.446}$ from (a)

$= 0.8556$ (about 86%).

11 Solutions to Exercises

1. If a raffle is conducted in such a way that each ticket has an equal chance of being drawn then

   $P(\text{winning}) = \frac{1}{325}$.

2. $P(A) = \frac{1}{5}$,

   $P(B) = \frac{2}{5}$.

3. From the grid on p.5 it can be seen that 10 of the boxes have exactly one '3' showing and one is the box (3,3).

   Therefore $P(\text{at least one 3 shows}) = \frac{11}{36}$.

4. $P(\text{at least one bulb flowers}) = 1 - P(\text{no bulbs flower})$

   $= 1 - \frac{1}{8}$

   $= \frac{7}{8}$.

5. $S = \{2, 3, 4, ..., 25\}$

   Let Q be the event 'a perfect square is chosen'. Then $Q = \{4, 9, 16, 25\}$.

   Let R be the event 'a prime number is chosen'.

   Then $R = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$.

   (a) $P(Q) = \frac{4}{24} = \frac{1}{6}$.
   (b) $P(R) = \frac{9}{24} = \frac{3}{8}$.
   (c) $P(Q \cup R) = P(Q) + P(R) = \frac{13}{24}$.
   (since the events Q and R are mutually exclusive, i.e. disjoint)

6.

Classification	Women	Men	Total
Computer Science	7	12	19
No Computer Science	7	4	11
Total	14	16	30

   Let W be the event 'the chosen person is a woman'.

   Let C be the event 'the chosen person takes computer science'.

   (a) $P(W) = \frac{14}{30}$.
   (b) $P(C) = \frac{19}{30}$.

(c) $P(W \cap C) = \frac{7}{30}$, as from the table the number of women taking computer science is 7.

(d) $P(W \cup C) = P(W) + P(C) - P(W \cap C) = \frac{14}{30} + \frac{19}{30} - \frac{7}{30} = \frac{26}{30}$.

7. (a) $R \cup T$ is the event 'a Rainbow or a Tiger's Eye is drawn'.
$P(R \cup T) = P(R) + P(T) = \frac{40}{45} = \frac{8}{9}$, as these are mutually exclusive.

(b) $P(R \cap T)$ is the event 'a Rainbow and a Tiger's Eye is drawn'.
$P(R \cap T) = 0$.

(c) $T \cup \overline{P}$ is the event 'a Tiger's Eye or not a Pearl is drawn'. This is equivalent to the event 'a Tiger's Eye or a Rainbow is drawn'.
$P(T \cup \overline{P}) = \frac{40}{45} = \frac{8}{9}$.

(d) $T \cup (R \cap P)$ is the event 'a Tiger's Eye is drawn or a Rainbow and a Pearl is drawn'. Since $(R \cap P) = \{\}$, the empty set,
$P(T \cup (R \cap P)) = P(T) = \frac{23}{45}$.

(e) $\overline{R} \cap (P \cup R \cup T)$ is the event 'not a Rainbow and either a Pearl, a Rainbow or a Tiger's Eye is drawn'.
Since $P \cup R \cup T$ is the whole sample space,
$\overline{R} \cap (P \cup R \cup T) = \overline{R}$ and $P(\overline{R}) = \frac{28}{45}$.

8. $P(\text{female} \mid \text{nurse}) = \frac{90}{110} = \frac{9}{11}$, since there are 110 nurses and of these 90 are female.
$P(\text{doctor} \mid \text{male}) = \frac{100}{120} = \frac{5}{6}$, since there are 120 males of whom 100 are doctors.
$P(\text{male} \mid \text{doctor}) = \frac{100}{190} = \frac{10}{19}$, since there are 190 doctors and of these 100 are male.

9. Let A be the event 'the sum of scores is 6'.
Let B be the event 'the numbers on the dice are different'.
Then $A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$ so A has 5 elements
B has 30 elements (list them and count). The sample space has 36 elements.
Hence $P(\text{the sum is 6} \mid \text{the numbers are different})$
$= P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
$= \frac{4}{36} / \frac{30}{36} = \frac{2}{15}$.

10. Sample space = {GG, BG, GB, BB}
Note that a 'girl followed by a boy' is **not** the same event as 'a boy followed by a girl'.
$A = \{\text{BG, GB}\}$, $B = \{\text{GG, BG, GB}\}$, $A \cap B = \{\text{BG,GB}\}$,
$P(A \cap B) = \frac{2}{4} = \frac{1}{2}$ and $P(A).P(B) = \frac{2}{4}.\frac{3}{4} = \frac{3}{8}$.
Since $P(A \cap B) \neq P(A).P(B)$, A and B are not independent.

11. Let 'missile one hits target' be denoted by M_1 and 'missile two hits target' by M_2

    Then $P(M_1) = \frac{1}{4}$ and $P(M_2) = \frac{2}{5}$,

    (a) $P(M_1 \cap M_2) = \frac{1}{4} \times \frac{2}{5}$, since the events are independent
    $= \frac{1}{10}$.

    (b) $P(M_1 \cup M_2) = P(M_1) + P(M_2) - P(M_1 \cap M_2)$ since these are not mutually exclusive
    $= \frac{1}{4} + \frac{2}{5} - \frac{1}{10}$
    $= \frac{11}{20}$.

12. If events A and B are mutually exclusive, then $P(A \cap B) = 0$. If events A and B are independent, then $P(A \cap B) = P(A).P(B)$.

    So they can be mutually exclusive **and** independent only if $P(A).P(B) = 0$.

    This can only happen if either $P(A) = 0$ or $P(B) = 0$, that is, either the event A or event B is impossible.

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