The composite function rule
(the chain rule)

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1 The composite function rule (also known as the chain rule)

Have a look at the function \( f(x) = (x^2 + 1)^{17} \). We can think of this function as being the result of combining two functions. If \( g(x) = x^2 + 1 \) and \( h(t) = t^{17} \) then the result of substituting \( g(x) \) into the function \( h \) is

\[
h(g(x)) = (g(x))^{17} = (x^2 + 1)^{17}.
\]

Another way of representing this would be with a diagram like

\[
x \mapsto x^2 + 1 \mapsto (x^2 + 1)^{17}.
\]

We start off with \( x \). The function \( g \) takes \( x \) to \( x^2 + 1 \), and the function \( h \) then takes \( x^2 + 1 \) to \( (x^2 + 1)^{17} \). Combining two (or more) functions like this is called composing the functions, and the resulting function is called a composite function. For a more detailed discussion of composite functions you might wish to refer to the Mathematics Learning Centre booklet Functions.

Using the rules that we have introduced so far, the only way to differentiate the function \( f(x) = (x^2 + 1)^{17} \) would involve expanding the expression and then differentiating. If the function was \((x^2 + 1)^2 = (x^2 + 1)(x^2 + 1)\) then it would not take too long to expand these two sets of brackets. But to expand the seventeen sets of brackets involved in the function \( f(x) = (x^2 + 1)^{17} \) (or even to expand using the binomial theorem) would take a long time. The composite function rule shows us a quicker way.

**Rule 7 (The composite function rule (also known as the chain rule))**

If \( f(x) = h(g(x)) \) then \( f'(x) = h'(g(x)) \times g'(x) \).

In words: differentiate the ‘outside’ function, and then multiply by the derivative of the ‘inside’ function.

To apply this to \( f(x) = (x^2 + 1)^{17} \), the outside function is \( h(\cdot) = (\cdot)^{17} \) and its derivative is \( 17(\cdot)^{16} \). The inside function is \( g(x) = x^2 + 1 \) which has derivative \( 2x \). The composite function rule tells us that \( f'(x) = 17(x^2 + 1)^{16} \times 2x \).

As another example let us differentiate the function \( 1/(z^3 + 4z^2 - 3z - 3)^6 \). This can be rewritten as \((z^3 + 4z^2 - 3z - 3)^{-6}\). The outside function is \( (\cdot)^{-6} \) which has derivative \(-6(\cdot)^{-7}\). The inside function is \( z^3 + 4z^2 - 3z - 3 \) with derivative \( 3z^2 + 8z - 3 \). The chain rule says that

\[
\frac{d}{dz}(z^3 + 4z^2 - 3z - 3)^{-6} = -6(z^3 + 4z^2 - 3z - 3)^{-7} \times (3z^2 + 8z - 3).
\]

There is another way of writing down, and hence remembering, the composite function rule.
Rule 7 (The composite function rule (alternative formulation))

If \( y \) is a function of \( u \) and \( u \) is a function of \( x \) then

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.
\]

This makes the rule very easy to remember. The expressions \( \frac{dy}{du} \) and \( \frac{du}{dx} \) are not really fractions but rather they stand for the derivative of a function with respect to a variable. However for the purposes of remembering the chain rule we can think of them as fractions, so that the \( du \) cancels from the top and the bottom, leaving just \( \frac{dy}{dx} \).

To use this formulation of the rule in the examples above, to differentiate \( y = (x^2 + 1)^{17} \) put \( u = x^2 + 1 \), so that \( y = u^{17} \). The alternative formulation of the chain rules says that

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 17u^{16} \times 2x = 17(x^2 + 1)^{16} \times 2x.
\]

which is the same result as before. Again, if \( y = (z^3 + 4z^2 - 3z - 3)^{-6} \) then set \( u = z^3 + 4z^2 - 3z - 3 \) so that \( y = u^{-6} \) and

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -6u^{-7} \times (3z^2 + 8z - 3).
\]

You select the formulation of the chain rule that you find easiest to use. They are equivalent.

**Example**

Differentiate \((3x^2 - 5)^3\).

**Solution**

The first step is always to **recognise** that we are dealing with a composite function and then to split up the composite function into its components. In this case the outside function is \((\cdot)^3\) which has derivative \(3(\cdot)^2\), and the inside function is \(3x^2 - 5\) which has derivative \(6x\), and so by the composite function rule,

\[
\frac{d(3x^2 - 5)^3}{dx} = 3(3x^2 - 5)^2 \times 6x = 18x(3x^2 - 5)^2.
\]

Alternatively we could first let \( u = 3x^2 - 5 \) and then \( y = u^3 \). So

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 6x = 18x(3x^2 - 5)^2.
\]
Example

Find $\frac{dy}{dx}$ if $y = \sqrt{x^2 + 1}$.

Solution

The outside function is $\sqrt{\cdot} = (\cdot)^{\frac{1}{2}}$ which has derivative $\frac{1}{2}(\cdot)^{-\frac{1}{2}}$, and the inside function is $x^2 + 1$ so that

$$y' = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \times 2x.$$

Alternatively, if $u = x^2 + 1$, we have $y = \sqrt{u} = u^{\frac{1}{2}}$. So

$$\frac{dy}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \times 2x = \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \times 2x.$$

Exercise 1.1

Differentiate the following functions using the composite function rule.

- a. $(2x + 3)^2$
- b. $(x^2 + 2x + 1)^{12}$
- c. $(3 - x)^{21}$
- d. $(x^3 - 1)^5$
- e. $f(t) = \sqrt{t^2 - 5t + 7}$
- f. $g(z) = \frac{1}{\sqrt{2 - z^4}}$
- g. $y = (t^3 - \sqrt{t})^{-3.8}$
- h. $z = (x + \frac{1}{x})^{\frac{2}{7}}$

Exercise 1.2

Differentiate the functions below. You will need to use both the composite function rule and the product or quotient rule.

- a. $(x + 2)(x + 3)^2$
- b. $(2x - 1)^2(x + 3)^3$
- c. $x\sqrt{1 - x}$
- d. $x^\frac{1}{3}(1 - x)^\frac{2}{3}$
- e. $\frac{x}{\sqrt{1 - x^2}}$
Solutions to exercises

Exercise 1.1

a. \( \frac{d}{dx} (2x + 3)^2 = 8x + 12 \)

b. \( \frac{d}{dx} (x^2 + 2x + 1)^{12} = 12(x^2 + 2x + 1)^{11}(2x + 2) \)

c. \( \frac{d}{dx} (3 - x)^21 = -21(3 - x)^{20} \)

d. \( \frac{d}{dx} (x^3 - 1)^5 = 5(x^3 - 1)^4 3x^2 = 15x^2(x^3 - 1)^4 \)

e. \( \frac{d}{dt} \sqrt{t^2 - 5t + 7} = \frac{d}{dt} (t^2 - 5t + 7)^{\frac{1}{2}} = \frac{1}{2} (t^2 - 5t + 7)^{-\frac{1}{2}} (2t - 5) \)

f. \( \frac{d}{dz} \left( \frac{1}{\sqrt{2 - z^4}} \right) = \frac{d}{dz} \left( (2 - z^4)^{-\frac{1}{2}} \right) = 2z^3(2 - z^4)^{-\frac{3}{2}} \)

g. \( \frac{d}{dt} (t^3 - \sqrt{t})^{-3.8} = -3.8(t^3 - \sqrt{t})^{-4.8} (3t^2 - \frac{1}{2\sqrt{t}}) \)

h. \( \frac{d}{dx} \left( x + \frac{1}{x} \right)^{\frac{3}{2}} = \frac{3}{2} (x + \frac{1}{x})^{-\frac{1}{2}} (1 - \frac{1}{x^2}) \)

Exercise 1.2

a. \( \frac{d}{dx} \left( (x + 2)(x + 3)^2 \right) = (x + 3)^2 + 2(x + 2)(x + 3) \)

b. \( \frac{d}{dx} \left( 2x - 1 \right)^2 (x + 3)^3 = 4(2x - 1)(x + 3)^3 + 3(2x - 1)^2(x + 3)^2 \)

c. \( \frac{d}{dx} \left( x\sqrt{1 - x} \right) = \sqrt{1 - x} - \frac{x}{2\sqrt{1 - x}} \)

d. \( \frac{d}{dx} \left( x^{\frac{1}{2}} (1 - x)^{\frac{3}{2}} \right) = \frac{1}{3} x^{-\frac{3}{2}} (1 - x)^{\frac{5}{2}} - \frac{2}{3} x^{\frac{1}{2}} (1 - x)^{-\frac{1}{2}} \)

e. \( \frac{d}{dx} \left( \frac{x}{\sqrt{1 - x^2}} \right) = \frac{\sqrt{1 - x^2} + x^2(1 - x^2)^{-\frac{1}{2}}}{1 - x^2} \)