Determinants

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Determinants

Recall, we defined the determinant of the 2×2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as $\det A = ad - bc$.

We also write this as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

We can define the determinant of any square matrix.

Let's start with a 1×1 matrix.

If $B = \begin{bmatrix} b_{11} \end{bmatrix}$, then

$$\det B = \begin{vmatrix} B \end{vmatrix} = b_{11}.$$

Determinant of a 3×3 matrix

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ then

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Notice the negative sign before b (to be discussed later).

The 2×2 determinant after each coefficient is the determinant you get by deleting the row and column that coefficient is in.

Click to see how this works.

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Determinant of a 3×3 matrix continued

Recall

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

We have expanded the determinant along the first row; our coefficients of the 2×2 matrices are a, b and c.

But the coefficient of b is negative and so to see why we need the following "matrix of signs".

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The "matrix of signs" tells us whether to multiply our coefficient by $+1$ or -1 according to its position. a and c are multiplied by $+1$ while b is multiplied by -1.

Expanding a determinant

We can use the "matrix of signs" to expand the determinant along any row or column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Suppose now we want to expand the determinant down the second column.

$$\det A = |A| = -b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + e \begin{vmatrix} a & c \\ g & k \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix}.$$

Click to see how we get the correct 2×2 determinants.

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Definition and an example

Expansion along any row or any column of a determinant always gives us the same answer, which is the value of the determinant.

Let $A = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & -4 \\ 4 & 1 & -1 \end{bmatrix}$ then expanding along the first row gives

$$\begin{aligned} |A| &= -1\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} - 5\begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ &= -1(0+4) - 5(-1+16) + 2(1-0) = -77. \end{aligned}$$

Expanding down the second column gives

$$\begin{aligned} |A| &= -5\begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 0 - 1\begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} \\ &= -5(-1+16) - 1(4-2) = -77. \end{aligned}$$

Expanding larger determinants: a 4×4 example

Example: For 4×4 matrices, the "matrix of signs" is

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\begin{aligned} \begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ -4 & -1 & 3 & 1 \\ 0 & 2 & -1 & 1 \end{vmatrix} &= 0 + 0 - 1 \begin{vmatrix} 3 & -1 & 2 \\ -4 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix} + 0 \\ &= (-1) \left[-2 \begin{vmatrix} 3 & 2 \\ -4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ -4 & -1 \end{vmatrix} \right] \\ &= (-1)[(-2)(3+8) + (-3-4)] = 29. \end{aligned}$$

We expanded the 4×4 determinant along the second row , and the 3×3 determinant along the third row.