1 Exponents

1.1 Introduction

Whenever we use expressions like $7^3$ or $2^5$ we are using exponents. The symbol $2^5$ means $2 \times 2 \times 2 \times 2 \times 2$, with 5 factors. This symbol is spoken as ‘two raised to the power five’, ‘two to the power five’ or simply ‘two to the five’. The expression $2^5$ is just a shorthand way of writing ‘multiply 2 by itself 5 times’. The number 2 is called the base, and 5 the exponent.

Similarly, if $b$ is any real number then $b^3$ stands for $b \times b \times b$. Here $b$ is the base, and 3 the exponent.

If $n$ is a whole number, $b^n$ stands for $b \times b \times \cdots \times b$, with $n$ factors. We say that $b^n$ is written in exponential form, and we call $b$ the base and $n$ the exponent, power or index.

Special names are used when the exponent is 2 or 3. The expression $b^2$ is usually spoken as ‘$b$ squared’, and the expression $b^3$ as ‘$b$ cubed’. Thus ‘two cubed’ means $2^3 = 2 \times 2 \times 2 = 8$.

1.2 Exponents with the same base

We will begin with a very simple definition. If $b$ is any real number and $n$ is a positive integer then $b^n$ means $b$ multiplied by itself $n$ times. The rules for the behaviour of exponents follow naturally from this definition.

Rule 1: $b^n \times b^m = b^{n+m}$.

That is, to multiply two numbers in exponential form (with the same base), we add their exponents.

Rule 2: $\frac{b^n}{b^m} = b^{n-m}$.

In words, to divide two numbers in exponential form (with the same base), we subtract their exponents.

We have not yet given any meaning to negative exponents, so $n$ must be greater than $m$ for this rule to make sense. In a moment we will see what happens if $n$ is not greater than $m$.

Rule 3: $(b^m)^n = b^{mn}$

That is, to raise a number in exponential form to a power, we multiply the exponents.

Until now we have only considered exponents which are positive integers, such as 7 or 189. Our intention is to extend this notation to cover exponents which are not necessarily positive integers, for example $-5$, or $\frac{113}{31}$, or numbers such as $\pi \approx 3.14159$. 

Also, we have not attached any meaning to the expression $b^0$. It doesn’t make sense to talk about a number being multiplied by itself 0 times. However, if we want rule 2 to continue to be valid when $n = m$ then we must define the expression $b^0$ to mean the number 1.

If $b \neq 0$ then we define $b^0$ to be equal to 1. We do not attempt to give any meaning to the expression $0^0$. It remains undefined.

We initially had no idea of how to extend our notation to cover a zero exponent, but if we wish rules 1, 2 and 3 to remain valid for such an exponent then the definition $b^0 = 1$ is forced on us. We have no choice.

We have come up with a sensible definition of $b^0$ by taking $m = n$ in rule 2 and seeing what $b^0$ must be if rule 2 is to remain valid. To come up with a suitable meaning for negative exponents we can take $n < m$ in rule 2. For example, let’s try $n = 2$ and $m = 3$.

Rule 2 gives

$$\frac{b^2}{b^3} = b^{-1} \quad \text{or} \quad \frac{1}{b} = b^{-1}.$$ 

This suggests that we should define $b^{-1}$ to be equal to $\frac{1}{b}$. This definition, too, makes sense for all values of $b$ except $b = 0$.

In a similar way we can see that we should define $b^{-n}$ to mean $\frac{1}{b^n}$, except when $b = 0$, in which case it is undefined. You should convince yourself of this by showing that the requirement that rule 2 remains valid forces on us the definitions

$$b^{-2} = \frac{1}{b^2} \quad \text{and} \quad b^{-3} = \frac{1}{b^3}.$$ 

If $n$ is a positive integer (for example $n = 17$ or $n = 178$) then we define $b^{-n}$ to be equal to $\frac{1}{b^n}$. This definition makes sense for all values of $b$ except $b = 0$, in which case the expression $b^{-n}$ remains undefined.

Pause for a moment and look at what has been achieved. We have been able to give a meaning to $b^n$ for all integer values of $n$, positive, negative, and zero, and we have done it in such a way that all three of the rules above still hold. We can give meaning to expressions like $\left(\frac{35}{2}\right)^{13}$ and $\pi^{-7}$.

We have come quite a way, but there are a lot of exponents that we cannot yet handle. For example, what meaning would we give to an expression like $5^{\frac{1}{2}}$? Our next task is to give a suitable meaning to expressions involving fractional powers.

Let us start with $b^{\frac{1}{2}}$. If rule 2 is to hold we must have

$$b^{\frac{1}{2}} \times b^{\frac{1}{2}} = b^{\frac{1}{2} + \frac{1}{2}} = b^1 = b.$$ 

So, $b^{\frac{1}{2}}$ is defined to be the positive square root of $b$, also written $\sqrt{b}$. So $b^{\frac{1}{2}} = \sqrt{b}$.

Of course, $b$ must be positive if $b^{\frac{1}{2}}$ is to have any meaning for us, because if we take any real number and multiply itself by itself then we get a positive number. (Actually there
is a way of giving meaning to the square root of a negative number. This leads to the
notion of complex numbers, a beautiful area of mathematics which is beyond the scope
of this booklet.)

That takes care of a meaning for $b^{\frac{1}{2}}$ if $b > 0$. Now have a look at $b^{\frac{1}{3}}$. If rule 2 is to remain
valid then we must have

$$b^{\frac{1}{3}} \times b^{\frac{1}{3}} \times b^{\frac{1}{3}} = b^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = b^1 = b.$$  

In general if we wish we wish to give meaning to expressions like $b^{\frac{1}{n}}$ in such a way that
rule 3 holds then we must have $(b^{\frac{1}{n}})^n = b^1 = b$.

If $b$ is positive, $b^{\frac{1}{n}}$ is defined to be a positive number, the $n^{th}$ root of $b$. That is, a number
whose $n^{th}$ power is equal to $b$. This number is sometimes written $\sqrt[n]{b}$.

If $b$ is negative we need to look at separately at the cases where $n$ is even and where $n$ is
odd.

If $n$ is even and $b$ is negative, $b^{\frac{1}{n}}$ cannot be defined, because raising any number to an
even power results in a positive number.

If $n$ is odd and $b$ is negative, $b^{\frac{1}{n}}$ can be defined. It is a negative number, the $n^{th}$ root of $b$. For example, $(-27)^{\frac{1}{3}} = -3$ because $(-3) \times (-3) \times (-3) = -27$.

Now we can see how to define $b^{\frac{p}{q}}$ for any number of the form $\frac{p}{q}$, where $p$ and $q$ are integers.
Such numbers are called rational numbers.

Notice that $\frac{p}{q} = p \times \frac{1}{q}$, so if rule 3 is to hold then $b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}$.

We know how to make sense of $(b^{\frac{1}{q}})^p$ and $(b^p)^{\frac{1}{q}}$, and they turn out to be equal, so this
tells us how to make sense of $b^{\frac{p}{q}}$. If we want rules 1, 2 and 3 to hold then we must define
$b^{\frac{p}{q}}$ to be either one of $(b^p)^{\frac{1}{q}}$ or $(b^{\frac{1}{q}})^p$.

This definition always makes sense when $b$ is positive, but we must take care when $b$ is
negative. If $q$ is even then we may have trouble in making sense of $b^{\frac{p}{q}}$ for negative $b$. For example we cannot make sense of $(-3)^{\frac{1}{2}}$. This is because we cannot even make sense of $(-3)^{\frac{1}{2}}$, let alone $((-3)^{\frac{1}{2}})^q$.

Trying to take the exponents in the other order does not help us because $(-3)^3 = -27$ and we cannot make sense of $(-27)^{\frac{1}{2}}$.

However it may be that the numerator and denominator of $\frac{p}{q}$ contain common factors
which, when cancelled, leave the denominator odd. For example we can make sense of $(-3)^{\frac{1}{2}}$, even though 6 is even, because $\frac{4}{6} = \frac{2}{3}$, and we can make sense of $(-3)^{\frac{2}{3}}$. A rational number $\frac{p}{q}$ is said to be expressed in its lowest form if $p$ and $q$ contain no common factors.

If $\frac{p}{q}$, when expressed in its lowest form, has $q$ odd then we can make sense of $b^{\frac{p}{q}}$ even for $b < 0$.

To recapitulate, we define

$$b^{\frac{p}{q}} = (b^p)^{\frac{1}{q}} = (b^{\frac{1}{q}})^p.$$  

This definition makes sense for all $\frac{p}{q}$ if $b > 0$. If $b < 0$ then this definition makes sense
providing that $\frac{p}{q}$ is expressed in its lowest form and $q$ is odd.
So far, if \( b > 0 \), we have been able to give a suitable meaning to \( b^x \) for all rational numbers \( x \). Not every number is a rational number. For example, \( \sqrt{2} \) is an irrational number: there do not exist integers \( p \) and \( q \) such that \( \sqrt{2} = \frac{p}{q} \). However for \( b > 0 \) it is possible to extend the definition of \( b^x \) to irrational exponents \( x \) so that rules 1, 2 and 3 remain valid. Thus if \( b > 0 \) then \( b^x \) is defined for all real numbers \( x \) and satisfies rules 1, 2 and 3. We will not show how \( b^x \) may be defined for irrational numbers \( x \).

**Examples**

\[
(\frac{1}{3})^{-1} = \frac{1}{(\frac{1}{3})} = 3
\]

\[
(0.2)^{-3} = \frac{1}{(0.2)^3} = \frac{1}{0.008} = 125
\]

\[
(-64)^{\frac{2}{3}} = [(-64)^{\frac{1}{3}}]^2 = (-4)^2 = 16 \text{ or,}
\]

\[
(-64)^{\frac{2}{3}} = [(-64)^{\frac{1}{2}}]^3 = (4096)^{\frac{1}{2}} = 16
\]

\[
16^{\frac{2}{3}} = (\sqrt[3]{16})^2 = 2^3 = 8
\]

\[
(-16)^{\frac{2}{3}} \text{ is not defined.}
\]

\[
5^{\frac{1}{2}} = 5^{1+\frac{1}{2}} = 5 \times 5^{\frac{1}{2}} = 5\sqrt{5}
\]

### 1.3 Exponents with different bases

From the definition of exponents we know that if \( n \) is a positive integer then

\[
(ab)^n = (ab) \times (ab) \times \cdots \times (ab)
\]

\[
= a \times a \times \cdots \times a \times b \times b \times \cdots \times b \quad \text{(switching the order around)}
\]

\[
= a^n b^n.
\]

Just as in section 1.2, we can show that this equation holds true for more general exponents than integers, and we can formulate the following rule:

**Rule 4:** \( (ab)^x = a^x b^x \) whenever both sides of this equation make sense, that is, when each of \( (ab)^x \), \( a^x \) and \( b^x \) make sense.

Again, from the definition of exponents we know that if \( n \) is a positive integer then

\[
\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b} \quad (b \neq 0)
\]

\[
= \frac{a \times a \times \cdots \times a}{b \times b \times \cdots \times b} \quad \text{(switching the order around)}
\]

\[
= \frac{a^n}{b^n}
\]

As in section 1.2, we can show that this equation remains valid if the integer \( n \) is replaced by a more general exponent \( x \). We can formulate the following rule:
Rule 5: \((\frac{a}{b})^x = \frac{a^x}{b^x}\) whenever both sides of this equation make sense, that is, whenever \((\frac{a}{b})^x\), \(a^x\) and \(b^x\) make sense.

An expression of the form \(a^x b^y\) cannot generally be simplified, though it can be written in the form \((ab)^{\frac{x}{y}}\) or \((a^\frac{x}{y} b^y)\) if necessary. For example, we cannot really make the expression \(a^2 b^5\) any simpler than it is, though we could write it in the form \((ab)^{2\frac{5}{2}}\) or \((a^2 b^\frac{5}{2})\).

Examples

\[(2 \times 3)^3 = 2^3 \times 3^3 = 8 \times 27 = 216 = 6^3\]

\[(4x)^{\frac{1}{2}} = 4^\frac{1}{2} x^\frac{1}{2} = 2x^\frac{1}{2} = 2\sqrt{x}\]

\[(-40)^{\frac{1}{2}} = (-8 \times 5)^{\frac{1}{2}} = (-8)^{\frac{1}{2}} \times (5)^{\frac{1}{2}} = -2 \times \sqrt{5}\]

\[(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}\]

\[(\frac{4}{7})^{-2} = \left(\frac{1}{(\frac{4}{7})^2}\right) = 1 \times \frac{7^2}{4^2} = \frac{49}{16}\]

\[(-\frac{27}{8})^{-\frac{1}{3}} = (-\frac{8}{27})^{\frac{1}{3}} = \frac{(-8)^{\frac{1}{3}}}{27^{\frac{1}{3}}} = -\frac{2}{3}\]

1.4 Summary

If \(b > 0\) then \(b^x\) is defined for all numbers \(x\). If \(b < 0\) then \(b^x\) is defined for all integers and all numbers of the form \(\frac{p}{q}\) where \(p\) and \(q\) are integers, \(\frac{p}{q}\) is expressed in its lowest form and \(q\) is odd. The number \(b\) is called the base and \(x\) is called the power, index or exponent. Exponents have the following properties:

1. If \(n\) is a positive integer and \(b\) is any real number then \(b^n = \underbrace{b \times b \times \cdots \times b}_{n\text{ factors}}\).

2. \(b^{\frac{1}{n}} = \sqrt[n]{b}\), and if \(n\) is even we take this to mean the positive \(n^{th}\) root of \(b\).

3. If \(b \neq 0\) then \(b^0 = 1\). \(b^0\) is undefined for \(b = 0\).

4. If \(p\) and \(q\) are integers then \(b^{\frac{p}{q}} = \left(b^{\frac{1}{q}}\right)^p = \left(b^p\right)^{\frac{1}{q}}\).

5. \(b^x \times b^y = b^{x+y}\) whenever both sides of this equation are defined.

6. \(\frac{b^x}{b^y} = b^{x-y}\) whenever both sides of this equation are defined.

7. \(b^{-x} = \frac{1}{b^x}\) whenever both sides of this equation are defined.

8. \((ab)^x = a^x b^y\) whenever both sides of this equation are defined.

9. \((\frac{a}{b})^x = \frac{a^x}{b^x}\) whenever both sides of this equation are defined.
1.5 Exercises

The following expressions evaluate to quite a ‘simple’ number. If you leave some of your answers in fractional form you won’t need a calculator.

1. $9^{\frac{1}{2}}$  
2. $16^{\frac{3}{4}}$  
3. $(\frac{1}{5})^{-1}$  
4. $(3^{-1})^2$  
5. $(\frac{5}{2})^{-2}$

6. $(-8)^{\frac{3}{2}}$  
7. $(-\frac{27}{8})^{\frac{2}{3}}$  
8. $5^{27}5^{-24}$  
9. $8^{\frac{1}{2}}2^{\frac{1}{3}}$  
10. $(-125)^{\frac{2}{3}}$

These look a little complicated but are equivalent to simpler ones. ‘Simplify’ them. Again, you won’t need a calculator.

11. $\frac{3^{n+2}}{3^n-2}$  
12. $\sqrt[6]{\frac{16}{x^6}}$  
13. $(a^{\frac{1}{2}} + b^{\frac{1}{3}})^2$

14. $(x^2 + y^2)^{\frac{1}{2}} - x^2(x^2 + y^2)^{-\frac{1}{2}}$  
15. $\frac{x^2 + x}{x^{\frac{1}{3}}}$  
16. $(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + (uv)^{\frac{1}{3}} + v^{\frac{2}{3}})$
1.6 Solutions to exercises

1. \(9^{\frac{1}{2}} = \sqrt{9} = 3\)

2. \(16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8\)

3. \((\frac{1}{5})^{-1} = \frac{1}{\frac{1}{5}} = 5\)

4. \((3^{-1})^2 = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}\)

5. \((\frac{5}{2})^{-2} = (\frac{2}{5})^2 = \frac{4}{25}\)

6. \((-8)\frac{3}{2}\) is not defined.

7. \((-\frac{27}{8})^{\frac{3}{2}} = ((-\frac{27}{8})^{\frac{1}{2}})^2 = (-\frac{3}{2})^2 = \frac{9}{4}\)

8. \(5^{275-24} = 5^{27-24} = 5^3 = 125\)

9. \(8^{\frac{1}{2}} \cdot 2^\frac{1}{2} = (8 \times 2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4\)

10. \((-125)^\frac{2}{3} = ((-125)^{\frac{1}{3}})^2 = (-5)^2 = 25\)

11. \(\frac{3^{n+2}}{2^{n+2}} = 3^{n+2-(n-2)} = 3^4 = 81\)

12. \(\sqrt{\left(\frac{16}{x^2}\right)^\frac{3}{2}} = \frac{16^{\frac{3}{2}}}{x^6} = \frac{4}{x^3}\)

13. \((a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 = (a^{\frac{1}{2}})^2 + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2 = a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b\)

14.

\[
(x^2 + y^2)^\frac{1}{2} - x^2(x^2 + y^2)^{-\frac{1}{2}} = \frac{x^2}{(x^2 + y^2)^\frac{1}{2}} - \frac{x^2}{x^2 + y^2} \\
= \frac{(x^2 + y^2)^\frac{1}{2}(x^2 + y^2)^\frac{1}{2} - x^2}{(x^2 + y^2)^\frac{1}{2}} \\
= \frac{x^2 + y^2 - x^2}{(x^2 + y^2)^\frac{1}{2}} \\
= \frac{y^2}{(x^2 + y^2)^\frac{1}{2}}
\]

15. \(\frac{x^{\frac{1}{2}} + x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = 1 + x^{\frac{1}{2}}\)

16.

\[
(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + u^{\frac{1}{3}}v^{\frac{1}{3}} + v^{\frac{2}{3}}) = u^{\frac{1}{3}}u^{\frac{2}{3}} + u^{\frac{1}{3}}u^{\frac{1}{3}}v^{\frac{1}{3}} + u^{\frac{1}{3}}v^{\frac{2}{3}} - v^{\frac{1}{3}}u^{\frac{2}{3}} - v^{\frac{1}{3}}u^{\frac{1}{3}}v^{\frac{1}{3}} - v^{\frac{2}{3}}v^{\frac{1}{3}} \\
= u - v
\]