Mathematics Learning Centre

The University of Sydney

Functions and straight line graphs

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Functions and Straight Line Graphs

Functions

Consider the formula

$$d = \frac{w}{5}.$$

This formula (in fact) gives the dosage of a medication for threadworm where d is the dosage in ml and w is the body weight of a person in kg.

Thus a man with a body weight of 70kg would require a dose of 14ml of medication while a child of 20kg would require a dose of 4ml of medication. The formula gives the exact relationship between body weight (in kg) and dosage required (in ml) and can be used to calculate the dosage for any given weight. In this formula, d and w are called variables.

The formula

$$d = \frac{w}{5}$$

is a function as each value of w produces exactly one value of d.

Definition A *function* from a set X to a set Y is a rule or pairing that assigns to each element of the first set X exactly one element of the second set Y.

In the example above the sets will be sets of real numbers.

In the formula $d = \frac{w}{5}$, we say that d is a function of w which is written as $d = f(w)$. Therefore,

$$d = f(w) = \frac{w}{5}.$$

We also write $f(70)$ to mean the value of the function when $w = 70$. That is, $f(70) = \frac{70}{5} = 14$.

Another way of representing a function is by a table of values. For our dosage example we can draw up a table of values by evaluating $f(0)$, $f(20)$, $f(40)$, $f(60)$, $f(80)$ and $f(100)$.

w	0	20	40	60	80	100
$f(w)$	0	4	8	12	16	20

In fact, the formula gives us more information about the function than the table of values but we can use the table to represent the function in yet another way - its graph.

By plotting the points given in the table, we get the graph in Figure 1.

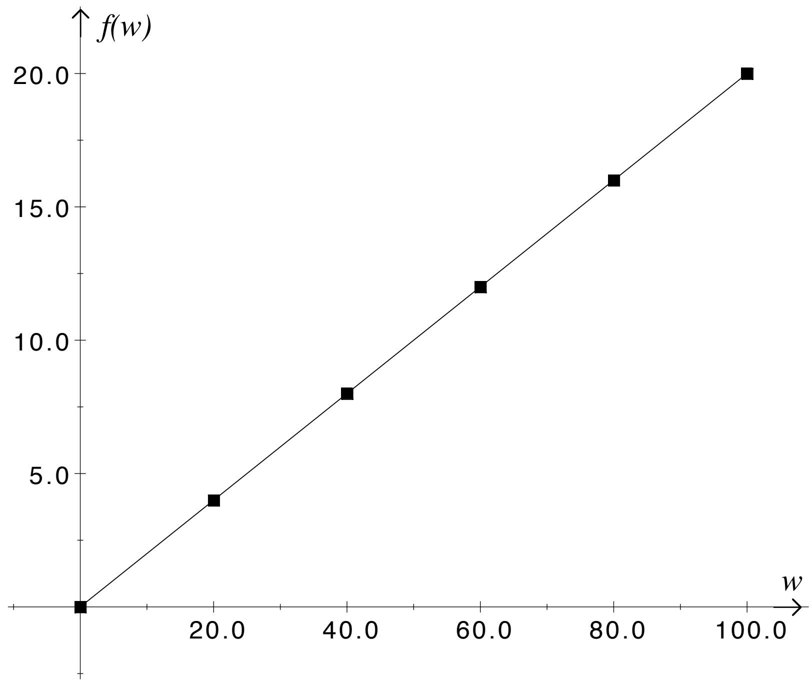

Figure 1: The graph of $d = f(w) = \frac{w}{5}$.

This is the graph of a straight line through $(0, 0)$.

Another formula, that gives us a way of converting temperature measured in degrees Celsius, x, to temperature measured in degrees Fahrenheit, y, is:

$$y = 32 + \frac{9}{5}x.$$

Here y and x are variables and the formula is function as each and every value of x gives an unique value of y. We write this as $y = f(x)$.

Again, we can draw up a table of values by evaluating $f(-40)$, $f(-20)$, $f(0)$, $f(20)$ and $f(40)$.

x	-40	-20	0	20	40
$y = f(x)$	-40	-4	32	68	104

If we plot these values in a graph we get a straight line through the point $(0, 32)$. This graph is shown in Figure 2.

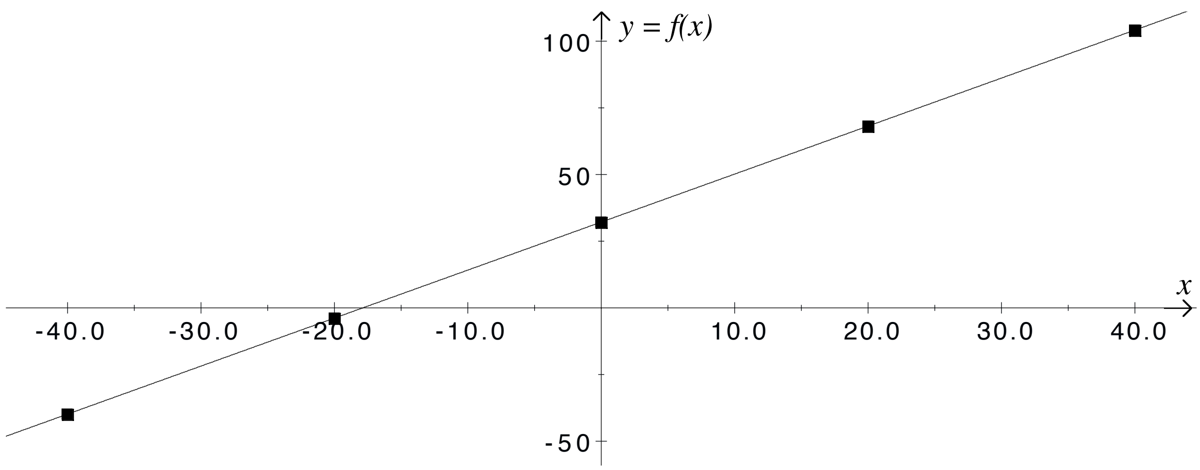

Figure 2: The graph of $y = f(x) = 32 + \frac{9}{5}x$.

Straight line graphs

The previous examples are both examples of linear functions; their graphs are straight lines.

Any function of the form,

$$y = mx + b \qquad \text{where } m \text{ and } b \text{ are constants}$$

will have a straight line as its graph.

Consider the function $y = 3x + 2$. Its graph is given in Figure 3.

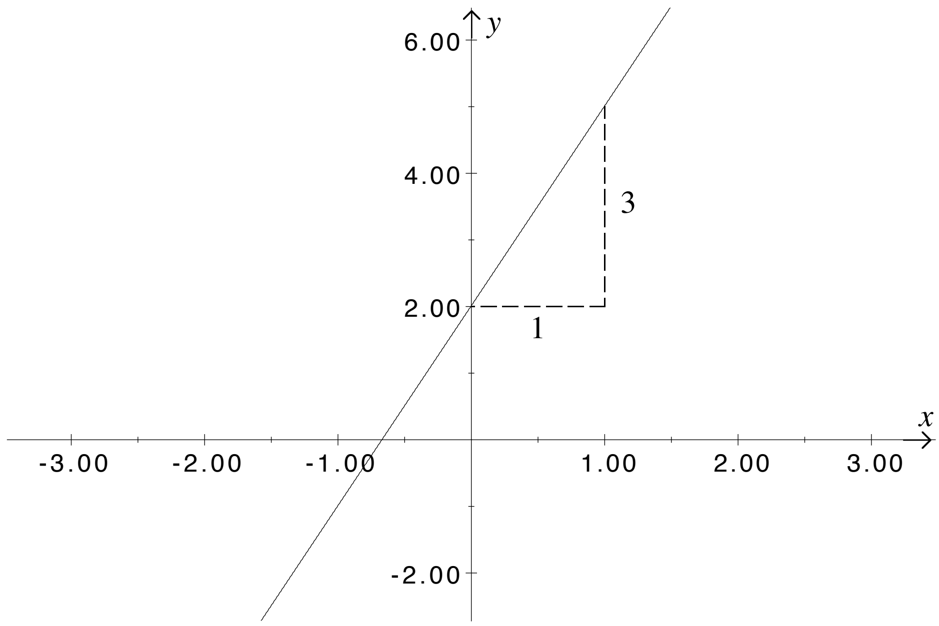

Figure 3: The graph of $y = 3x + 2$.

When $x = 0$ $y = 2$ and when $x = 1$ $y = 5$. That is, an increment of $1 - 0 = 1$ unit in the value of x will result in an increment of $5 - 2 = 3$ units in the value of y. This will always be the case for this function. (Pick a few values of x and try it.)

The ratio

$$\frac{\text{change in } y}{\text{change in } x}$$

is called the gradient (or slope) of the line.

For this example, the gradient of the line $y = 3x + 2$ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{5-2}{1-0} = 3.$$

In general, the gradient of the line $y = mx + b$ is m.

If we are given two points on the line (x_1, y_1) and (x_2, y_2) the gradient m can be worked out as follows:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Thus take any two points on the line $y = 3x + 2$, $(-1, -1)$ and $(3, 11)$ say, then the gradient m is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - (-1)}{3 - (-1)} = \frac{12}{4} = 3.$$

For the function $y = 3x + 2$, when $x = 0$ $y = 2$, so the point $(0, 2)$ lies on the line. Thus 2 is the y-intercept; the y value where the line crosses the y-axis.

In the general equation of a line $y = mx + b$, b is the y-intercept since it is the value of y when $x = 0$.

The equation of a straight line can be determined completely from any two points on the line.

For example, let $(-2, 4)$ and $(2, 2)$ be points on a straight line.

Then the gradient of the line is:

$$m = \frac{4-2}{-2-2} = \frac{2}{-4} = -\frac{1}{2}.$$

The equation of the line is $y = -\frac{1}{2}x + b$. We can determine b by substituting either point into the equation.

$$4 = -\frac{1}{2}(-2) + b = 1 + b \qquad \text{which gives} \qquad b = 3.$$

Therefore, the equation of the straight line going through the points $(-2, 4)$ and $(2, 2)$ is

$$y = -\frac{1}{2}x + 3.$$

The graph is given in Figure 4.

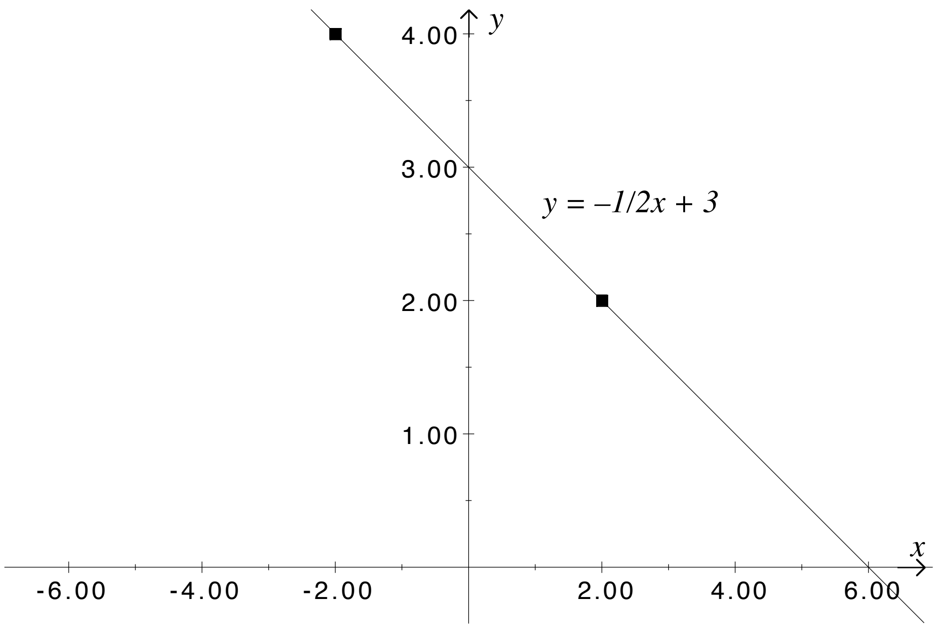

Figure 4: The graph of $y = -\frac{1}{2}x + 3$.

Notice the effect of a negative gradient; the line slopes downward.

Exercises 1

1. Find the gradient and y-intercept of the line $2x + 3y - 6 = 0$.

2. Find the equation of each of the following lines:

i. **a.** the line through the point $(0, 4)$ with gradient -3,
b. the line throught the point $(-1, -3)$ with gradient 2;
c. the line through the point $(2, -2)$ with gradient $\frac{1}{2}$,
d. the line through the point (c, d) with gradient -2.

ii. Find the points at which the line in c) cuts the x and y axes.

iii. Sketch the line in c).

3. Find the gradient of each of the following lines.

i. **a.** the line through $(1, 3)$ and $(6, 4)$,
b. the line through $(-3, 1)$ and $(2, 4)$,
c. the line through $(1, 2)$ and $(6, -8)$,
d. the line through $(-2, -4)$ and $(-6, -2)$.

ii. Find the equation of each line.

iii. Sketch the line in d).

4. Find the equation of the line which is parallel to the line $y = 2x - 5$ and passes through the point $(\frac{1}{2}, -1)$.

Solutions to exercises 1

1. First we will rewrite the equation $2x + 3y - 6 = 0$ as:

$$\begin{aligned} 2x + 3y - 6 &= 0 \\ 3y &= -2x + 6 \\ y &= -\frac{2}{3}x + 2. \end{aligned}$$

Now the equation is written in this form we can see that the gradient of the line is $-\frac{2}{3}$ and the y-intercept is 2.

2. i. a. The gradient of the line is -3 so $y = -3x + b$.
The point $(0, 4)$ tells us that when $x = 0$, $y = 4$, so 4 is the y-intercept giving us the equation of the line:

$$y = -3x + 4.$$

b. The gradient of the line is 2 so $y = 2x + b$. The point $(-1, -3)$ lies on the line so

$$\begin{aligned} -3 &= 2(-1) + b \\ -3 + 2 &= +b \\ b &= -1. \end{aligned}$$

So, the equation of the line is $y = 2x - 1$.

c. Let's start with $y = \frac{1}{2}x + b$. The point $(2, -2)$ lies on the line so

$$\begin{aligned} -2 &= \frac{1}{2}(2) + b \\ -2 &= 1 + b \\ b &= -3. \end{aligned}$$

The equation of the line is $y = \frac{1}{2}x - 3$.

d. Let's start with $y = -2x + b$. The point (c, d) lies on the line so

$$\begin{aligned} d &= -2(c) + b \\ b &= 2c + d. \end{aligned}$$

The equation of the line is $y = -2x + 2c + d$.

ii. A line cuts the x-axis when $y = 0$, so

$$\begin{aligned} 0 &= \frac{1}{2}x - 3 \\ \frac{1}{2}x &= 3 \\ x &= 6. \end{aligned}$$

The line in c) cuts the x-axis at $x = 6$.

A line cuts the y-axis when $x = 0$. This is the y-intercept. The equation tells us that the y-intercept is -3. So, the line in c) cuts the y-axis at $y = -3$.

iii.

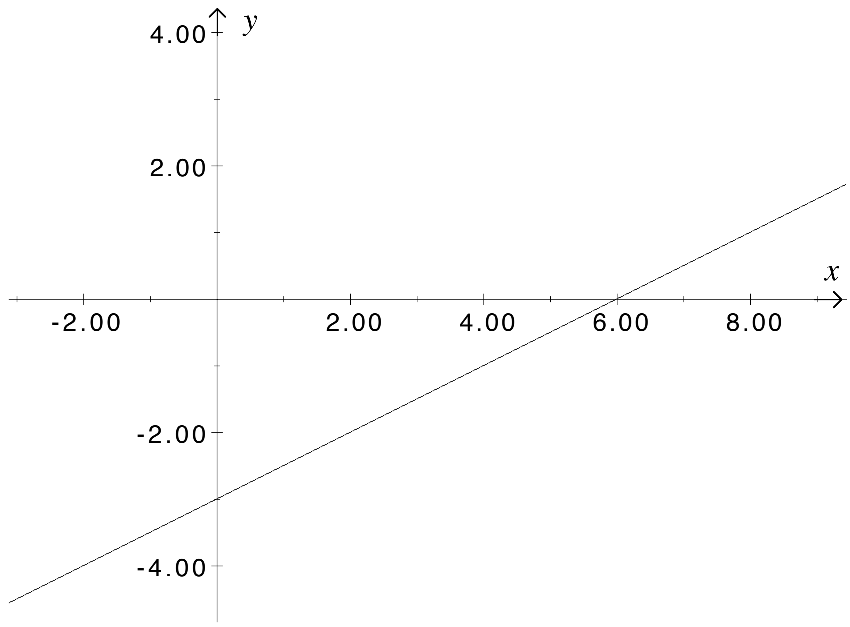

Figure 5: The graph of $y = \frac{1}{2}x - 3$.

3. i. We will find the gradient of each of the following lines using the formula:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

a. Let $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (6, 4)$. Then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 3}{6 - 1} \\ &= \frac{1}{5}. \end{aligned}$$

So, the gradient of the line is $\frac{1}{5}$.

b. Let $(x_1, y_1) = (-3, 1)$ and $(x_2, y_2) = (2, 4)$. Then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 1}{2 - (-3)} \\ &= \frac{3}{5}. \end{aligned}$$

So, the gradient of the line is $\frac{3}{5}$.

c. Let $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (6, -8)$. Then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-8 - 2}{6 - 1} \\ &= \frac{-10}{5} \\ &= -2. \end{aligned}$$

So, the gradient of the line is -2.

d. Let $(x_1, y_1) = (-2, -4)$ and $(x_2, y_2) = (-6, -2)$. Then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - (-4)}{-6 - (-2)} \\ &= \frac{2}{-4} \\ &= -\frac{1}{2}. \end{aligned}$$

So, the gradient of the line is $-\frac{1}{2}$.

ii. a. Let's start with the equation $y = \frac{1}{5}x + b$. The point $(1, 3)$ lies on the line so

$$\begin{aligned} 3 &= \frac{1}{5}(1) + b \\ b &= 3 - \frac{1}{5} \\ &= \frac{14}{15}. \end{aligned}$$

So, the equation of the line is $y = \frac{1}{5}x + \frac{14}{15}$.

b. The point $(-3, 1)$ lies on the line, so starting with $y = \frac{3}{5}x + b$ we get

$$\begin{aligned} 1 &= \frac{3}{5}(-3) + b \\ 1 &= -\frac{9}{5} + b \\ b &= 1 + \frac{9}{5} \\ &= \frac{14}{5}. \end{aligned}$$

So the equation of the line is $y = \frac{3}{5}x + \frac{14}{5}$.

c. The point $(1, 2)$ lies on the line, so starting with $y = -2x + b$ we get

$$\begin{aligned} 2 &= -2(1) + b \\ b &= 4. \end{aligned}$$

So the equation of the line is $y = -2x + 4$.

d. The point $(-2, -3)$ lies on the line so

$$\begin{aligned} -3 &= -\frac{1}{2}(-2) + b \\ -3 &= 1 + b \\ b &= -4. \end{aligned}$$

So the equation of the line is $y = -\frac{1}{2}x - 4$.

iii.

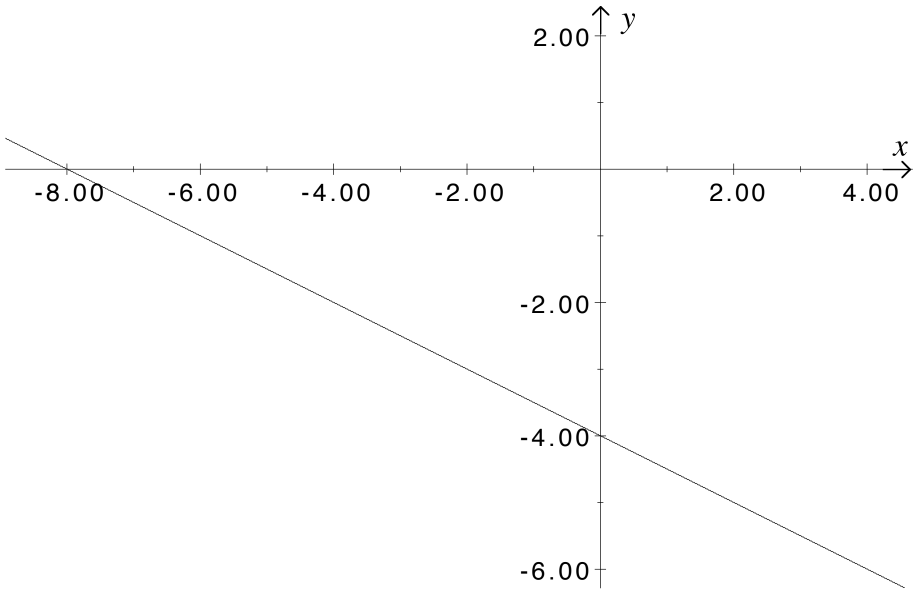

Figure 6: The graph of $y = -\frac{1}{2}x - 4$.

4. The gradient of the line $y = 2x - 5$ is 2, so a line parallel to this line also has gradient 2. The point $(\frac{1}{2}, -1)$ lies on the line so starting with $y = 2x + b$ we get

$$\begin{aligned} -1 &= 2(\frac{1}{2}) + b \\ b &= -2. \end{aligned}$$

So the equation of the line parallel to $y = 2x - 5$ that passes throught the point $(\frac{1}{2}, -1)$ is $y = 2x - 2$.