Introductory Algebra for Social Scientists

Sue Gordon

Mathematics Learning Centre
University of Sydney
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Acknowledgements

Contents

1 The Language of Mathematics

1.1 Mathematical Symbols

We can regard a mathematical statement as a sentence in a very simple foreign language.

We use numbers	such as	1.2	34	-7	$\frac{1}{2}$
symbols	such as	$+$	$-$	$/$	$\sqrt{}$
letters	such as	x	X	a	y.

In statistics you will need these special symbols: $\overline{X}$ (x bar) , x_i (x subscript i) eg, x_1 (read as x one) , x_2 (read as x two) , etc,

and some Greek letters eg, μ (mu), σ (small sigma), Σ (capital sigma), ρ (rho), α (alpha), χ (chi).

Suppose we have a list of scores on a test, say the first score is 80, the second is 60, the third is 77, the fourth is 83, and the fifth is 79.

We could write this as follows:

$$x_1 = 80, \; x_2 = 60, \; x_3 = 77, \; x_4 = 83, \; x_5 = 79.$$

The subscript tells us which score it is, and this notation has the advantage of not restricting us to 26 letters of the alphabet.

Suppose we want to find the mean or average of these scores by adding them up and dividing by 5. If we call our answer μ, we would write:

$$
\begin{aligned}
\mu &= \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} \\
&= \frac{80 + 60 + 77 + 83 + 79}{5} \\
&= 75.8.
\end{aligned}
$$

You may ask, why not write μ in terms of the numbers in the first place instead of introducing all that notation?

What makes algebra so useful and powerful, is that it gives us a concise way of expressing a general fact.

For example, suppose a group of newly born babies require a certain medication.

The amount needed is 2 ml plus half a ml for each kilogram of body weight.

Thus, a baby weighing 3 kg needs $2 + 1\frac{1}{2}$ ml, ie $3\frac{1}{2}$ ml.

We can express the formula mathematically as follows: Let W_i be the weight of the ith baby in kg so that W_1 is the weight of the first baby, W_2 is the weight of the second baby, and so on.

Let A_i be the amount of medication needed by the ith baby in ml.

Then

$$A_i = 2 + \frac{1}{2}W_i.$$

As W_i changes from baby to baby, so will A_i.

Try this one:

A magazine claims that for each hour of exercise a week a person does regularly (up to 10 hours a week), the life expectancy of that person increases by a year over 68 years.

Let N = number of years person expects to live,

and H = hours of exercise they do every week (up to 10 hours),

then $N = 68 + H$, for H a positive number less than or equal to 10. (Note: The magazine assumes that the average life expectancy of the group is 68.)

1.1.1 Exercises

Express each of the following as an algebraic formula:

1. The range (R) of a set of scores is the difference between the largest score (L) and the smallest score (S).

2. The relative frequency (R) of a score is the frequency of that score (F) divided by the total number of scores (N).

3. The weights of n children are W_1 , W_2 , ... ,W_n .
   a. Their total weight, T, is the sum of their weights.
   b. The mean of their weights, μ, is their total weight divided by the number of children.

4. A number of adults have their blood pressure measured. The percentage of people with high blood pressure (P) is equal to the number of people with high blood pressure (H) divided by the total number of people (N) multiplied by 100.

5. A z-score (z) is found by subtracting the mean (μ) from the raw score (x) and dividing the answer by the standard deviation (σ).

1.2   Some Conventions in Mathematics

If you are learning a foreign language, you need to know the basic vocabulary and grammar. For example, if you are reading a German sentence, you need to be aware that the verb may come at the end so you don't panic halfway. There are some conventions used in maths which you will find helpful to know.

Multiplication

We often leave out the multiplication sign and write ab, for example, instead of $a \times b$, or $10y$ instead of $10 \times y$.

(Does this mean mathematicians are inherently lazy? What do you think?)

$$3axy \quad \text{means} \quad 3 \times a \times x \times y.$$

This convention will obviously not work for numbers:

$$34 \text{ means thirty four, not 3 times 4.}$$

In this case we use brackets, so that: $3(4) + 2(6)$ means $3 \times 4 + 2 \times 6 = 24$.

Division

Instead of writing $a \div b$ we usually write a/b or $\dfrac{a}{b}$.

This convention does work for numbers as well as letters: For example, $1 \div 2$ is equal to $1/2$ or $\dfrac{1}{2}$.

Order of operations

The correct answer to $2 + 3 \times 5$ is 17.

(Try it on your calculator.)

If, however, what was required was to add $2 + 3$ and then multiply the answer by 5 (to get 25) we would write it like this: $(2 + 3)5$.

The order in which operations are carried out is:

> **brackets first, then multiplication or division, finally addition or subtraction.**

If operations are at the same level, (eg multiplication and division), work from left to right.

So, to find

$$15 - 6 \times \frac{2}{3},$$

we first evaluate $6 \times \frac{2}{3}$ (which is 4) then find $15 - 4$, to get 11.

Also,

$$2(3 + 1) = 2 \times 4 = 8$$

as we first evaluate what is inside the brackets.

Note that an expression like:

$$\frac{a + b}{c} \quad \text{means} \quad (a + b) \quad \text{divided by} \quad c.$$

To evaluate it we first add the values of a and b, then divide the answer by c.

For example:

$$\frac{6 + 1}{3} = \frac{7}{3} = 2\frac{1}{3}.$$

How would we find the value of

$$\frac{ac - bd}{a + b},$$

if $a = 2$, $b = 3$, $c = 1$, $d = 5$?

$$ac - bd = 2(1) - 3(5) = 2 - 15 = -13,$$

while

$$a + b = 2 + 3 = 5.$$

Hence the answer is

$$\frac{-13}{5} = -2.6.$$

Example: What is the difference between

$$\frac{1}{n}(x_1 + x_2 + x_3 + x_4 + x_5)$$

and

$$\frac{x_1 + x_2 + x_3 + x_4 + x_5}{n}?$$

Solution: There is no difference, they mean the same thing.

For instance

$$\frac{1}{3}(5 + 1) = \frac{1}{3}(6) = 2$$

and

$$\frac{5 + 1}{3} = \frac{6}{3} = 2.$$

Powers

a^m means $a \times a \times a \times \cdots \times a$, m times

so a^3 means $a \times a \times a$, eg, $2^3 = 2 \times 2 \times 2 = 8$. x^2 means $x \times x$, eg, $4^2 = 4 \times 4 = 16$.

Square roots

$\sqrt{a}$ is the positive square root of a, and means that positive number which when multiplied by itself gives a.

So that $\sqrt{4} = 2$ as $2 \times 2 = 4$.

$\sqrt{81} = 9$ as $9 \times 9 = 81$.

Inequalities

$a > 1$ means that a is greater than one.

$a < 3$ means that a is less than three.

$90 \leq a \leq 100$ means that a is greater than or equal to 90 but less than or equal to 100.

Brackets

Brackets may be used to mean multiplication or to separate one expression from another.

For example, $2(3) - 6$ means 2 multiplied by 3, then subtract 6, ie 0, while $3 + (-6)$ means 3 add the number -6, which equals -3.

Some Notation for Statistics

A commonly used convention in statistics is to use Greek letters for population values and Roman letters (English alphabet) for sample variables.

A population is the whole group we are interested in. For example, the population of all employees in Australia, or the population of students at Sydney University.

A sample is a portion of the population. You will be dealing mainly with simple random samples. Choosing a simple random sample from a population is like putting the names of everyone in the population in a hat, mixing well, and drawing out, say, 25 names.

1.2.1 Exercises

Substitute the values into the formulae given. Use a calculator by all means. Do not panic at the sight of strange letters. These are just names.

1. If $x = 3$, find the values of:

   a. $x(x + 2)$,

   b. $\dfrac{x}{x + 2}$,

   c. $\dfrac{(x + 2)x}{3}$.

2. Find the value of a^2b if $a = 3$ and $b = 4$.

3. Find the mean μ

   if $\mu = \dfrac{x_1 + x_2 + x_3}{n}$

   and $x_1 = 3$, $x_2 = 7$, $x_3 = 5$ and $n = 3$.

4. Find the power p if $p = 1 - \beta$ and $\beta = 0.05$.

5. If the expected value $E(X)$ is given by the formula

$$E(X) = np,$$

   find $E(X)$ when $n = 100$ and $p = 0.4$.

6. The standard error SE is equal to:

$$SE = \frac{\sigma}{\sqrt{n}}.$$

   Find the standard error if $\sigma = 15$ and $n = 100$.

7. Find the probability p if $p = 1 - \alpha$ and $\alpha = 0.1$.

8. $z = \dfrac{x - \mu}{\sigma}$.

   Find z if $x = 3.5$, $\mu = 6$ and $\sigma = 2.7$.

9. $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$.

   Find z if $\overline{x} = 96$, $\mu = 80$, $\sigma = 5$ and $n = 25$.

10. $x = z\sigma + \mu$.

   Find x if z is 1.85, σ is 15 and $\mu = 100$.

11. A straight line has the equation $y = 1.67x - 1.29$.

   Find the value of y when $x = 4.3$.

2 Sigma Notation

2.1 Understanding Sigma Notation

The symbol Σ (capital sigma) is often used as shorthand notation to indicate the sum of a number of similar terms. Sigma notation is used extensively in statistics.

For example, suppose we weigh five children. We will denote their weights by x_1, x_2, x_3, x_4 and x_5.

The sum of their weights $x_1 + x_2 + x_3 + x_4 + x_5$ is written more compactly as $\sum\limits_{j=1}^{5} x_j$.

The symbol Σ means 'add up'. Underneath Σ we see $j = 1$ and on top of it 5. This means that j is replaced by whole numbers starting at the bottom number, 1, until the top number,5, is reached.

Thus

$$\sum_{j=2}^{5} x_j = x_2 + x_3 + x_4 + x_5,$$

and

$$\sum_{j=2}^{4} x_j = x_2 + x_3 + x_4.$$

So the notation $\sum\limits_{j=1}^{n} x_j$ tells us:

a. to add the scores x_j,

b. where to start: x_1,

c. where to stop: x_n (where n is some number).

Now take the weights of the children to be $x_1 = 10$kg, $x_2 = 12$kg, $x_3 = 14$kg, $x_4 = 8$kg and $x_5 = 11$kg. Then the total weight (in kilograms) is

$$\begin{aligned}
\sum_{i=1}^{5} x_i &= x_1 + x_2 + x_3 + x_4 + x_5 \\
&= 10 + 12 + 14 + 8 + 11 \\
&= 55.
\end{aligned}$$

Notice that we have used i instead of j in the formula above. The j is what we call a dummy variable - any letter can be used, ie,

$$\sum_{j=1}^{n} x_j = \sum_{i=1}^{n} x_i.$$

Now let us find $\sum\limits_{i=1}^{4} 2x_i$ where $x_1 = 2$, $x_2 = 3$, $x_3 = -2$ and $x_4 = 1$.

Again, starting with $i = 1$ we replace the expression $2x_i$ with its value and add up the terms until $i = 4$ is reached. So,

$$\begin{aligned}
\sum_{i=1}^{4} 2x_i &= 2x_1 + 2x_2 + 2x_3 + 2x_4 \\
&= 2(2) + 2(3) + 2(-2) + 2(1) \\
&= 4 + 6 - 4 + 2 \\
&= 8.
\end{aligned}$$

Similarly, let us find $\sum_{k=1}^{3}(x_k - 4)$ where $x_1 = 7$, $x_2 = 4$, $x_3 = 1$.

Here,

$$\begin{aligned}
\sum_{k=1}^{3}(x_k - 4) &= (x_1 - 4) + (x_2 - 4) + (x_3 - 4) \\
&= (7 - 4) + (4 - 4) + (1 - 4) \\
&= 3 + 0 + (-3) \\
&= 0.
\end{aligned}$$

Example: Write out in full: $\sum_{k=1}^{5} x^k$.

Solution: $x^1 + x^2 + x^3 + x^4 + x^5$.

We also use sigma notation in the following way:

$$\sum_{j=1}^{4} j^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

This is the same principle: replace j in the expression (this time j^2) by whole numbers starting with 1 and ending with 4 , and add.

2.1.1 Exercises

1. Evaluate $\sum_{i=1}^{4} x_i$ where $x_1 = 5$, $x_2 = 2$, $x_3 = 3$, $x_4 = 8$.

2. Evaluate $\sum_{k=1}^{n} 5x_k$ where $x_1 = 10$, $x_2 = 14$, $x_3 = -2$, and $n = 3$.

3. Find $\mu = \dfrac{1}{5}\sum_{j=1}^{5} x_j$ where the $x_1 = 10\text{kg}$, $x_2 = 12\text{kg}$, $x_3 = 14\text{kg}$, $x_4 = 8\text{kg}$ and $x_5 = 11\text{kg}$ are the weights of 5 children. (μ is the mean weight of the children.)

4. Find the value of $\sum_{i=1}^{3}(x_i - \mu)^2$ where $x_1 = 105$, $x_2 = 100$, $x_3 = 95$, and $\mu = 100$.

2.2   Using Sigma Notation in Statistics

Here are some examples of how sigma notation is used in statistics:

The formula for a mean of a group of N scores, is

$$\mu = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

A measure of how spread out the scores are, called the variance, has the following formula:

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2.$$

For example, the number of customers having lunch at a certain restaurant on 7 weekdays were $x_1 = 92$, $x_2 = 84$, $x_3 = 70$, $x_4 = 76$, $x_5 = 66$, $x_6 = 80$, $x_7 = 71$.

The mean is

$$
\begin{aligned}
\mu &= \frac{1}{N} \sum_{i=1}^{N} x_i \\
&= \frac{1}{7}(92 + 84 + 70 + 76 + 66 + 80 + 71) \\
&= \frac{539}{7} \\
&= 77.
\end{aligned}
$$

This means that, on average, there were 77 people having lunch at the restaurant on those weekdays.

The variance is

$$
\begin{aligned}
\sigma^2 &= \frac{1}{7} \sum_{i=1}^{7} (x_i - 77)^2 \\
&= \frac{1}{7}[(15)^2 + (7)^2 + (-7)^2 + (-1)^2 + (-11)^2 + (3)^2 + (-6)^2] \\
&= \frac{1}{7}[225 + 49 + 49 + 1 + 121 + 9 + 36] \\
&= \frac{1}{7}[490] \\
&= 70.
\end{aligned}
$$

This value is difficult to interpret but its square root, called the standard deviation, may be thought of as the 'give or take' number. Now $\sqrt{70} = 8.37$ (approximately 8).

Thus, on average 77 people, give or take 8 people, had lunch at the restaurant on those 7 weekdays. So, on average, the restaurant had between $68 = 77 - 9$ and $86 = 77 + 9$ customers for lunch on those weekdays.

An alternative formula for variance is

$$\sigma^2 = \frac{1}{N}(\sum_{i=1}^{N} x_i^2 - N\mu^2)$$

For the above example we get:

$$
\begin{aligned}
\sigma^2 &= \frac{1}{N}[x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 - N\mu^2] \\
&= \frac{1}{7}[92^2 + 84^2 + 70^2 + 76^2 + 66^2 + 80^2 + 71^2 - 7(77)^2] \\
&= \frac{1}{7}[8464 + 7056 + 4900 + 5776 + 4356 + 6400 + 5041 - 7(5929)] \\
&= \frac{1}{7}[490] \\
&= 70
\end{aligned}
$$

as before.

2.2.1 Exercises

1. **a.** Find the variance of the weights of the five children (in Exercise 2.1.1 number 3), using each of the above formulae for σ^2.

    b. Find the standard deviation of their weights, σ.

2. During a 5 week period, a salespersons weekly income (in dollars) was $x_1 = 400$, $x_2 = 250$, $x_3 = 175$, $x_4 = 300$, $x_5 = 375$.

    Calculate $\mu = \frac{1}{5}\sum_{i=1}^{5} x_i$ and $\sigma^2 = \frac{1}{5}(\sum_{i=1}^{5} x_i^2 - 5\mu^2)$.

3. An insurance company is concerned about the length of time required to process claims. The length of time, measured in days, taken to process 7 claims produced the data $x_1 = 23$, $x_2 = 20$, $x_3 = 22$, $x_4 = 25$, $x_5 = 24$, $x_6 = 23$, $x_7 = 21$.

    Evaluate the mean μ and variance σ^2 for these data.

4. **(Hard)** A certain statistic called χ^2 (read chi square) has the following formula:

    $$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}.$$

    For example, if $n = 3$ and $O_1 = 10$, $O_2 = 8$, $O_3 = 12$ and $E_1 = 8$, $E_2 = 10$, $E_3 = 15$,

then

$$\begin{aligned}
\chi^2 &= \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \\
&= \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \frac{(O_3 - E_3)^2}{E_3} \\
&= \frac{(10 - 8)^2}{8} + \frac{(8 - 10)^2}{10} + \frac{(12 - 15)^2}{15} \\
&= 0.5 + 0.4 + 0.6 \\
&= 1.5.
\end{aligned}$$

Calculate χ^2 for the following data:

a.

$O_1 = 31, \quad O_2 = 64, \quad O_3 = 72, \quad O_4 = 28, \quad O_5 = 5,$

$E_1 = 30, \quad E_2 = 40, \quad E_3 = 70, \quad E_4 = 40, \quad E_5 = 20.$

b.

$O_1 = 16, \quad O_2 = 22, \quad O_3 = 15, \quad O_4 = 23, \quad O_5 = 24, \quad O_6 = 20,$

$E_1 = 20, \quad E_2 = 20, \quad E_3 = 20, \quad E_4 = 20, \quad E_5 = 20, \quad E_6 = 20.$

3 Solving Linear Equations

3.1 Solving Linear Equations Using Backtracking

An equation may be thought of as a set of directions. You are familiar with using directions in your everyday life.

The following is a set of directions for getting from the Transient Building (on Fisher Road opposite Physics Road) to the Carslaw Building:

1) Exit the Transient Building;

2) Turn left;

3) Walk from the Transient Building door to the first alley;

4) Turn left;

5) Walk to the end of the alley;

6) Turn right;

7) Walk up the stairs;

8) Cross the road;

9) Turn right;

10) Walk into the Carslaw entrance.

Now let us look at an equation:
$$3x + 4 = 7.$$

We can think of this equation as a set of directions as follows:

1) Start with a number which we will represent by x;

2) Multiply by 3;

3) Add 4;

4) You have 7.

The process of finding the number that x represents is called solving the equation.

It is done in the same way as finding the starting point (Transient Building) from the Carslaw in the first set of directions. How would you do that? Try writing a set of directions to guide me from Carslaw to the Transient Building.

This process of reversing the order of the directions and reversing each step is what we do to solve the equation. Hence, to solve the equation $3x + 4 = 7$ we would:

1) Start with 7;

2) Subtract 4;

3) Divide by 3;

4) We have x.

Mathematically we would write the above 4 steps like this:

1) The value is 7; $\qquad\qquad$ $3x + 4 = 7$.

2) Subtract 4; $\qquad\qquad$ $3x + 4 - 4 = 7 - 4$, ie, $3x = 3$.

3) Divide by 3; $\qquad\qquad$ $\frac{3x}{3} = \frac{3}{3}$.

4) We have x; $\qquad\qquad$ $x = 1$.

Note:

1) You should always check your answer by substituting back into the equation.

In this case $3x + 4 = 3(1) + 4 = 7$, which is correct.

2) Notice that each step of the instructions is carried out on both sides of the equation. In the above solution the left-hand side of each line shows the 'undoing' or 'reversing' of the original set of directions while the right hand side shows us the current value.

At each stage we use an inverse operation between numbers, ie, a subtraction or a division, to reverse or 'undo' a corresponding operation of addition or multiplication, using properties of real numbers such as:

$$a - a = 0$$

and

$$a \times \frac{1}{a} = 1 \qquad \text{for} \qquad a \neq 0$$

as in steps 1 and 2 above and in steps 3 and 4 above.

The strategy is to isolate the unknown or required variable (eg x above) in terms of the other given variables or numbers. Thus if

$$x + a = b$$

then

$$x + a - a = b - a$$

and so

$$x = x + 0 = b - a.$$

This process, called backtracking, makes x the subject of the expression. It requires a reverse sequence of 'undoing type' operations as outlined in the above examples, to isolate the subject variable (x or 'Transient building').

Example: Write out the series of directions represented by the equation

$$(4y - 1) \times 3 = 21.$$

Solve the equation by backtracking.

Solution: The equation is the following set of directions:

1) Let y represent some number;

2) Multiply by 4;

3) Subtract 1;

4) Multiply by 3;

5) The result is 21.

Backtracking and writing the mathematical step alongside:

1) The final value is 21; $\quad\quad\quad\quad (4y - 1)3 = 21.$

2) Divide by 3; $\quad\quad\quad\quad\quad\quad 4y - 1 = 7.$

3) Add 1; $\quad\quad\quad\quad\quad\quad\quad\quad 4y = 8.$

4) Divide by 4; $\quad\quad\quad\quad\quad\quad\quad y = 2.$

5) We have y. $\quad\quad\quad\quad$ Check that $y = 2$ works, by substituting in the equation.

Example: Write out the series of directions represented by the equation

$$12 = \frac{20n - 10}{3} + 2$$

and solve the equation by backtracking.

Solution: Remembering that $\dfrac{20n - 10}{3}$ means $\dfrac{(20n - 10)}{3}$, the equation says:

1) Let n represent a number;

2) Multiply by 20;

3) Subtract 10;

4) Divide by 3;

5) Add 2;

6) The answer is 12.

Backtracking,

1) The answer is 12; $\quad\quad\quad 12 = \frac{20n-10}{3} + 2.$

2) Subtract 2; $\quad\quad\quad\quad\quad 10 = \frac{20n-10}{3}.$

3) Multiply by 3; $\quad\quad\quad\quad 30 = 20n - 10.$

4) Add 10; $\quad\quad\quad\quad\quad\quad 40 = 20n.$

5) Divide by 20; $\quad\quad\quad\quad\quad 2 = n.$

6) We have n. $\quad\quad\quad\quad\quad\quad n = 2.$

Notice that whether n is on the right or the left does not matter to the process of solving the equation, as we always carry out each step on both sides of the equation.

We do not need numbers in our equation to use this method.

Example: The formula for a $z-$score is

$$z = \frac{x - \mu}{\sigma}$$

where x is the raw score, μ is the population mean and σ is the population standard deviation.

Solve the equation to find x in terms of z, μ and σ.

Solution: The equation says:

1) Let x represent the raw score;

2) Subtract μ;

3) Divide by σ;

4) z is the $z-$score.

Backtracking,

   1) z is the $z-$score; $z = \dfrac{x - \mu}{\sigma}$.

   2) Multiply by σ; $z\sigma = x - \mu$.

   3) Add μ; $z\sigma + \mu = x$.

   4) x is the raw score. $x = z\sigma + \mu$.

3.1.1 Exercises

Write down the series of instructions represented by these equations and solve them by 'backtracking', writing, next to each step, the new algebraic equation you get by performing the operation. Check your answers by substituting back into the equation.

1. $\dfrac{r}{4} + 5 = 8$, find r.

2. $5(n - 2) - 3 = 42$, find n.

3. $\dfrac{6y - 1}{5} = 1$, find y.

4. $\dfrac{x - 15}{3} = 0.5$, find x.

5. $t = \dfrac{x - \mu}{s}$, find x in terms of t, μ and s.

   If you feel comfortable with this process, try solving these equations by doing the backtracking part in algebra only, without writing out the steps in words.

6. $x - 0.5 = 7.5$, solve for x.

7. $3(t - 2) = 6$, solve for t.

8. $\dfrac{\overline{x} - 100}{\frac{50}{\sqrt{25}}} = 2$ solve for $\overline{x}$.

9. $\dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = z$ solve for $\overline{x}$ in terms of z, μ and $\frac{\sigma}{\sqrt{n}}$.

The graphs of linear equations are straight lines. In the next chapter we will see how to sketch such graphs.

4   Sketching Linear Equations

Before we can graph linear equations that is, straight lines, we need some basic concepts
.

4.1   Plotting Points

We represent points on a pair of axes. Figure 1 shows a horizontal axis, or X axis, and a
vertical axis, the Y axis. (Some people call the $x-$coordinate of a point the abscissa and
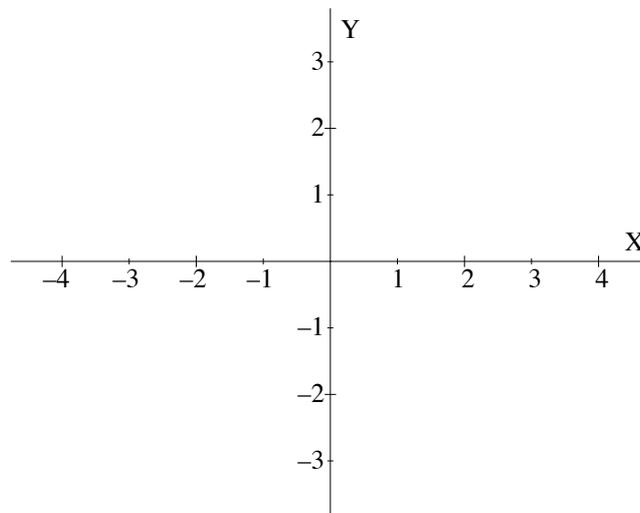the $y-$coordinate the ordinate.)

Figure 1: The coordinate axes.

Figure 2 shows the coordinate axes and the point $(1, 2)$. The point has an $x-$coordinate
of 1 and a $y-$coordinate of 2.
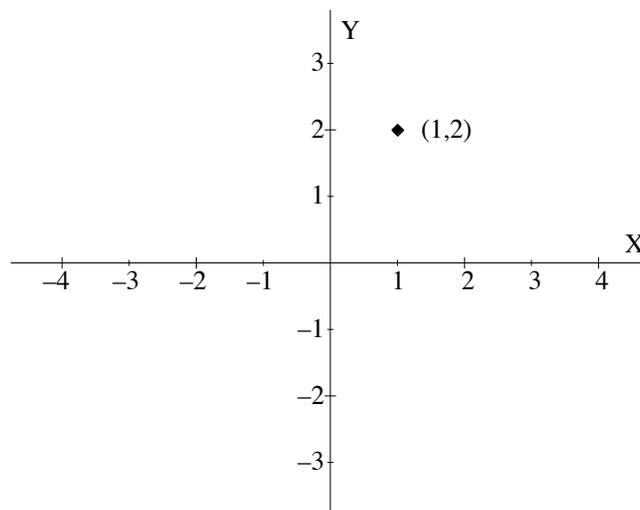
Figure 2: The coordinate axes and the point $(1, 2)$.

Now try plotting the point $(2, 1)$, ie, $x = 2$, $y = 1$, on a pair of axes. To do this we go 2 units along the X axis and draw an imaginary vertical line. Then we go one unit up the Y axis and draw an (imaginary) horizontal line. The intersection of the two lines is the required point. This is illustrated in Figure 3.
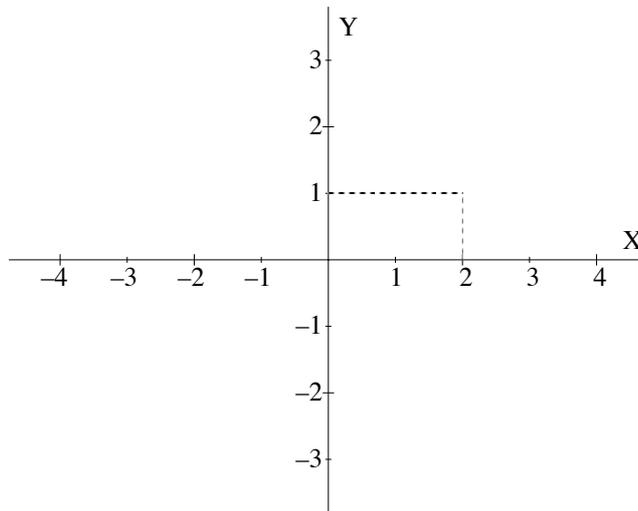
Figure 3: Plotting the point $(2, 1)$.

Example: For which of the shaded regions in Figures 4 and 5 is $x \geq 1$ and for which is $y \geq 1$?
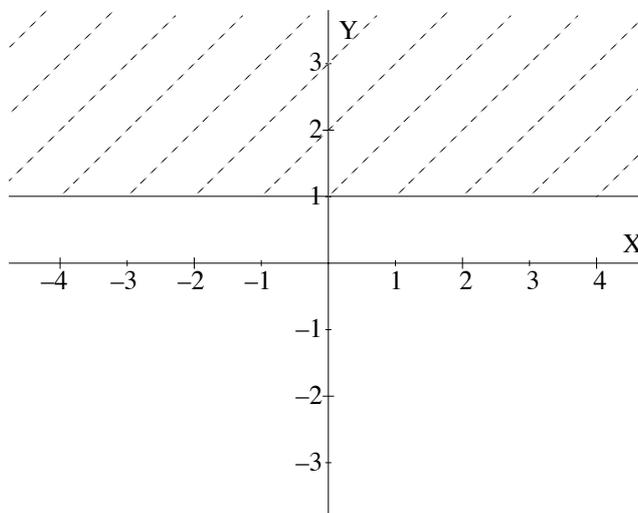
Figure 4: Is the shaded region $x \geq 1$ or $y \geq 1$?
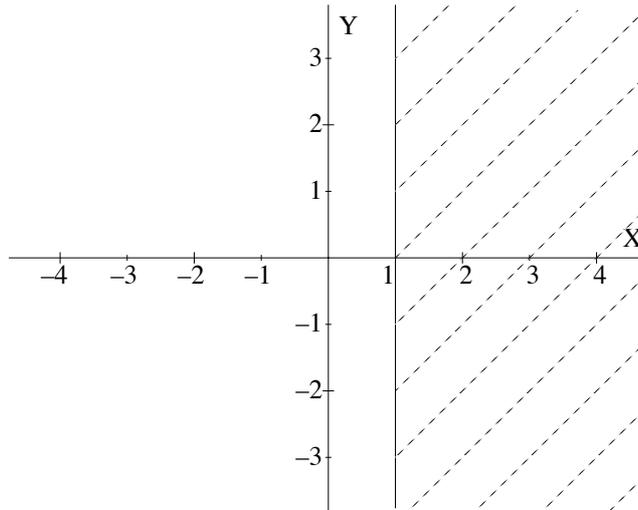
Figure 5: Is the shaded region $x \geq 1$ or $y \geq 1$?

Solution: The shaded region in Figure 4 is $y \geq 1$, while the shaded region in Figure 5 is $x \geq 1$.

4.1.1 Exercises

1. Write down the coordinates of points labelled in Figure 6.
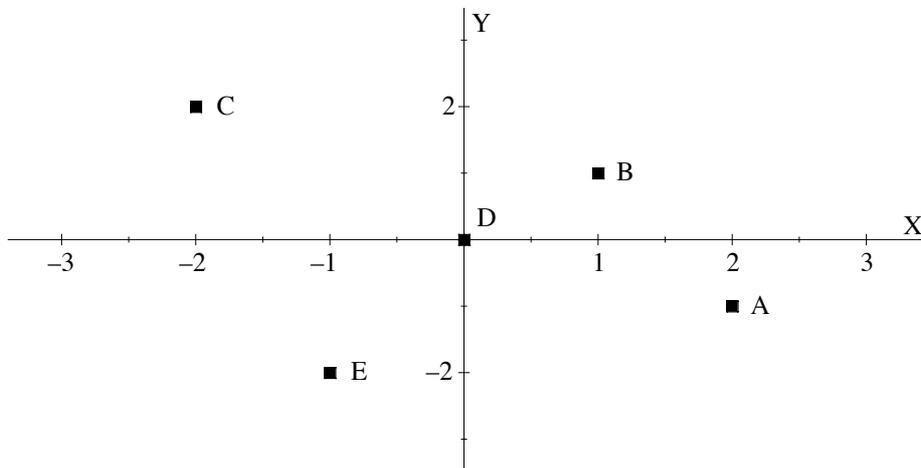
Figure 6: Graph with points A to E.

In statistics, the term 'scattergram' is used to denote a graph showing the positions of a number of points, as in Figure 6.

2. The animated spot.

The spot is at position $A = (3, 8)$.

a. If it falls straight down onto the X axis, what will the coordinates of its position be? This is illustrated in Figure 7.
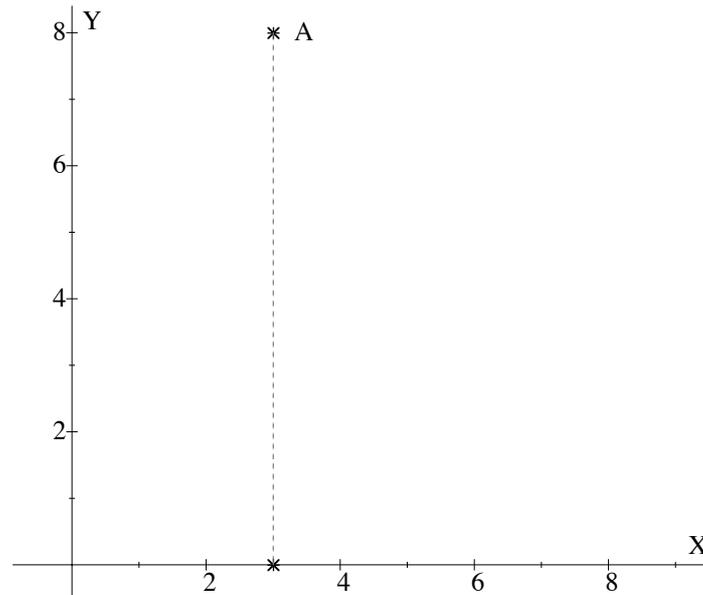
Figure 7: Graph with spot at A= (3, 8).

b. Suppose the spot falls straight down from A and bounces straight up again. Each time it bounces it reaches half its previous height. Find the coordinates of its highest position after: one bounce, two bounces, three bounces.
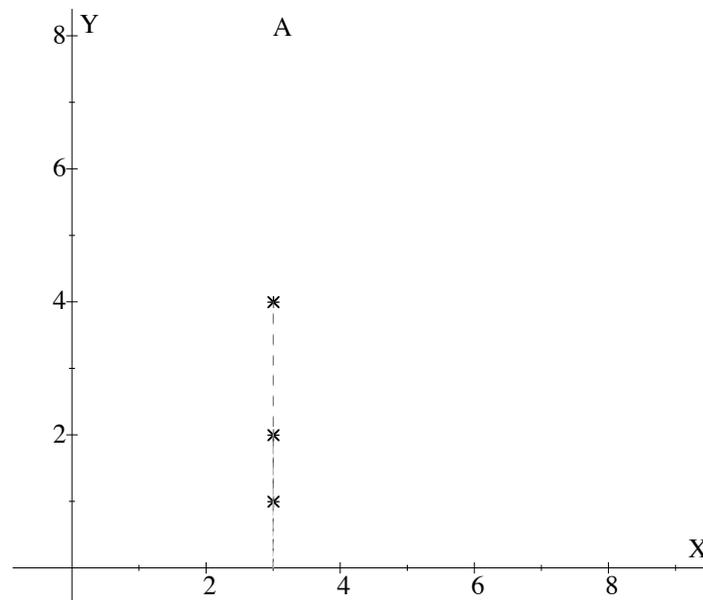
Figure 8: Graph with spot showing position after each of three bounces.

c. If after each bounce on the X axis, the spot's new position is one unit to the right and one unit up, what will its new coordinates be after 1 bounce, 2 bounces, 3 bounces? (Start from A and assume it falls straight down each time.)
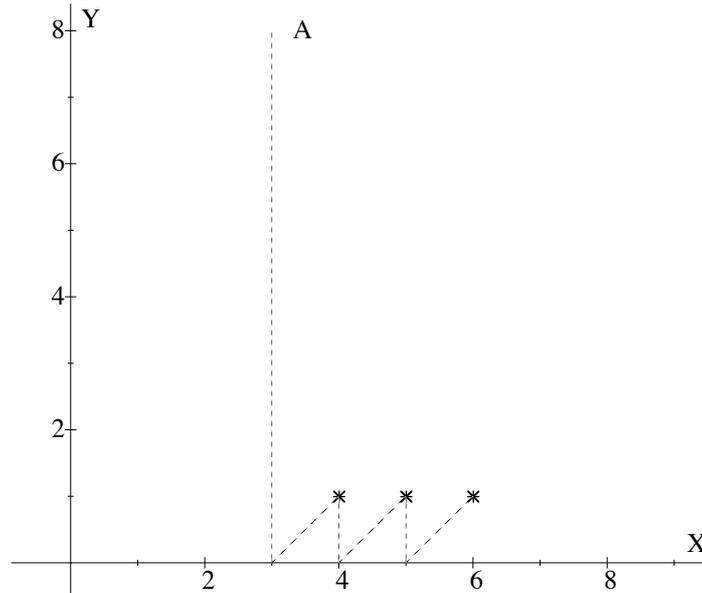
Figure 9: Graph with position of spot after each of three bounces to the right.

d. If the spot moves in a straight line from A to the origin O, how many units has it fallen vertically and how many units has it moved horizontally to the left?
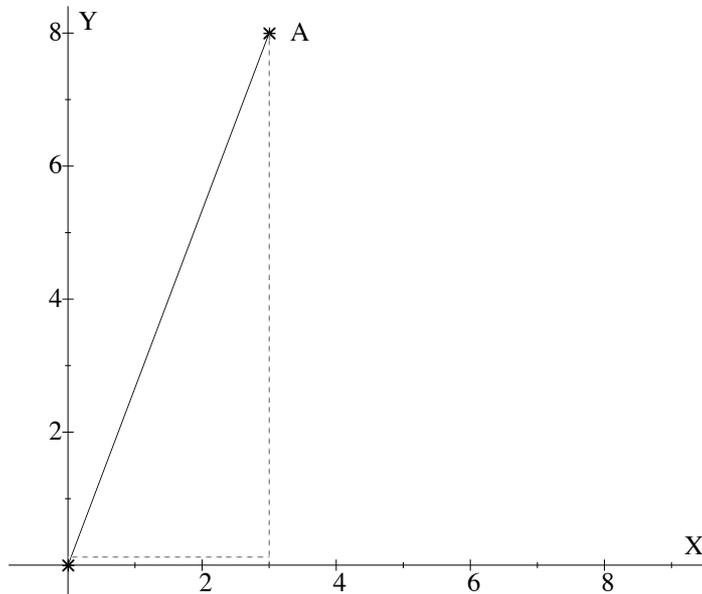
Figure 10: Graph of spot moving in a straight line to the origin.

This exercise indicates the idea behind instructions to a computer to move animations about a screen.

4.2 Finding the Equation of a Line Using the Gradient and Y Intercept

A straight line is described as an equation of the form

$$y = bx + a \qquad \text{where} \qquad a \text{ and } b \text{ are constants, ie, fixed values.}$$

Every equation of this form can be graphed as a straight line.

4.2.1 Exercises

Which of the following equations are linear (ie, describe straight lines)?

1. $y = -1 + 2x$;

2. $y = x^2 + 1$;

3. $y = \sqrt{3}x + 1$;

4. $y = 3\sqrt{x} + 1$.

Consider the equation

$$y = 3x + 2.$$

To sketch the graph of this line, we need only two points on the line. Choose any two values of x, say $x = -1$ and $x = 4$, then find the corresponding values of y:

When $x = -1$, $y = 3(-1) + 2 = -1$.

When $x = 4$, $y = 3(4) + 2 = 14$.

These points are illustrated in Figure 11.
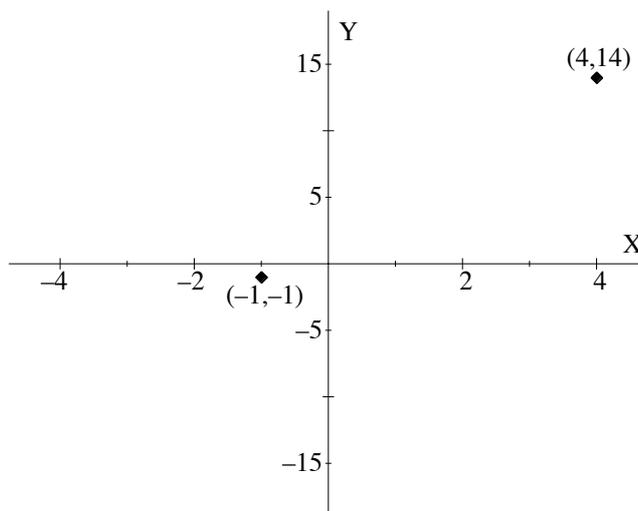
Figure 11: Graph with points at $(-1, -1)$ and $(4, 14)$.

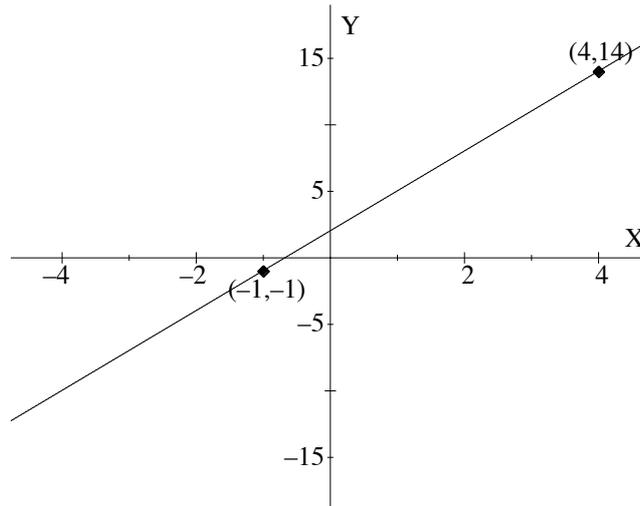Clearly if we join these points we will have a straight line.

Figure 12: Graph with line drawn throught points $(-1, -1)$ and $(4, 14)$.

Can you see why two points are sufficient to describe a unique line?

A line joining these two points is drawn in Figure 12.

Now choose any other value of x, and find the corresponding value of y by substituting for x in the equation.

Plot this point and check that it lies on the line.

Now look more closely at this line. How can we describe it?

Two useful attributes of the line are its 'steepness' or slope and its 'position' or where it cuts the Y axis. This is called the intercept on the Y axis.

The slope or gradient of the line describes how steep it is. Imagine yourself standing on a point on the line and walking 'up' it. The steeper it is the more you will have to walk up for the same amount along. Thus we can define slope as the ratio

$$\frac{\text{amount uphill}}{\text{amount along}}$$

from any point to any other on the line which is to the right of the first point.

For example, suppose you are at the point $(-1, -1)$ on the line $y = 3x + 2$. To get to the point $(4, 14)$ you have to walk 15 units up for 5 along, therefore the slope is $\frac{15}{5} = 3$.

Notice that if the equation of the line is $y = bx + a$ the 'b' tells us the slope.

Example: Graph the line

$$y = -3x + 2$$

by choosing any two points on it. Then find the slope of the line.

Solution: The graph of $y = -3x + 2$ is given in Figure 13.

The point A is $(0, 2)$ and the point B is $(1, -1)$. To get from A to B you would have to walk 3 units downhill and one along to the right. Hence, if we adopt the convention that 'downhill' is negative, the slope is $\frac{-3}{1} = -3$.

Notice that we can also think of -3 as $-3 = \dfrac{-3}{1} = \dfrac{3}{-1}$ which tells us that if we go one step to the left -1, we must climb $+3$ units.
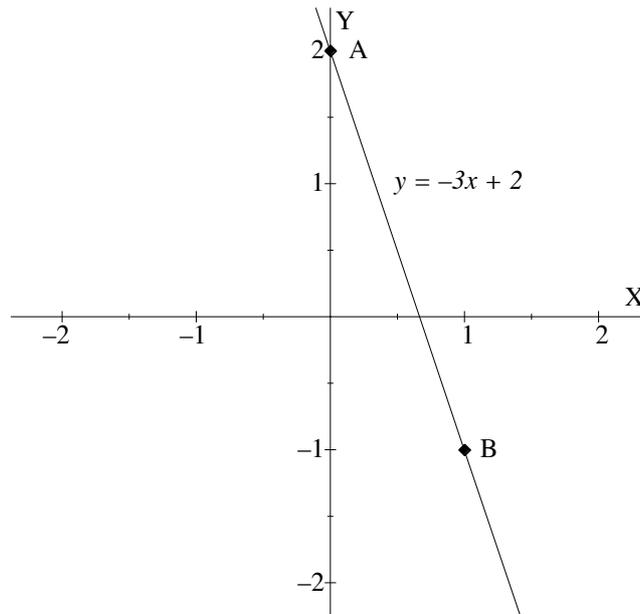
Figure 13: Graph of $y = -3x + 2$.

The position of the line or intercept on the Y axis is given by 'a'.

This is because when $x = 0$, $y = b(0) + a = a$.

This means that the line $y = 3x + 1$ will have the same slope but a different intercept on the Y axis to $y = 3x + 2$.

We check this by sketching the line $y = 3x + 1$ on the same axes as $y = 3x + 2$ in Figure 14.
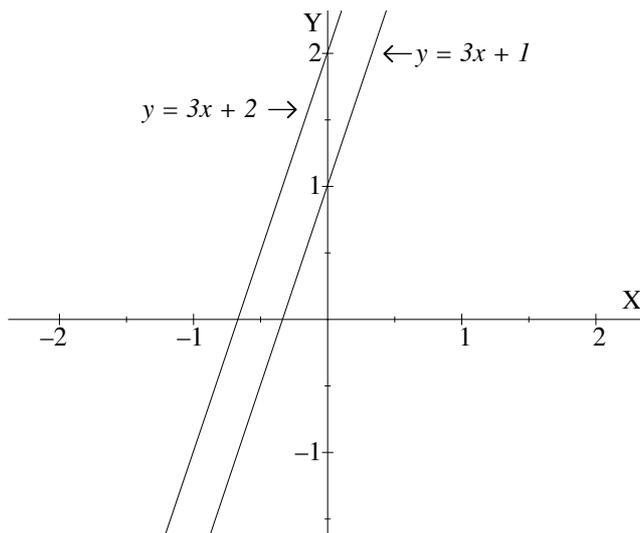

Figure 14: Graphs of $y = 3x + 2$ and $y = 3x + 1$.

Example: What is the slope of $y = -3x + 1$? In what way will this line differ from that of $y = 3x + 1$?

Solution: The slope of $y = -3x + 1$ is -3, while the slope of $y = 3x + 1$ is $+3$. Therefore the former has a downward slope while the latter has an upward slope. They both have the same intercept on the Y axis, namely $+1$. This is illustrated in Figure 15.
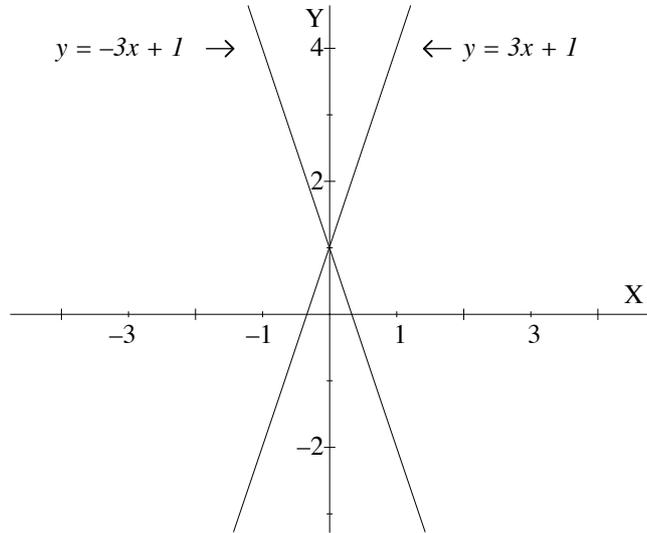
Figure 15: Graphs of $y = -3x + 1$ and $y = 3x + 1$.

Example: Find the equation of a line which has Y intercept 7 and gradient -3.

Solution: $y = bx + a$ where $a = 7$ and $b = -3$. So the equation of the line is $y = -3x + 7$.

4.2.2 Lines with 'funny' slopes.

There are two special cases of lines which we will consider.

Case 1: $y = 4$ is the equation of the straight line given in Figure 16.
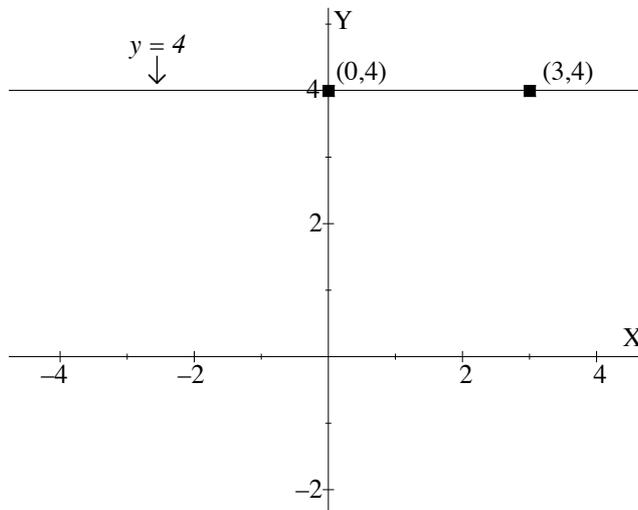
Figure 16: Graphs of $y = 4$.

Let us find its slope. Choose any two x values, say $x = 0$ and $x = 3$. For both of these, $y = 4$ gives a point on the line. Join the points $(0, 4)$ and $(3, 4)$. We have a horizontal line. Hence the slope of this line is zero There is no amount up, for three units along, or any number of units along.

Case 2: Find the slope of the line through the points $(1, 3)$ and $(1, 6)$.

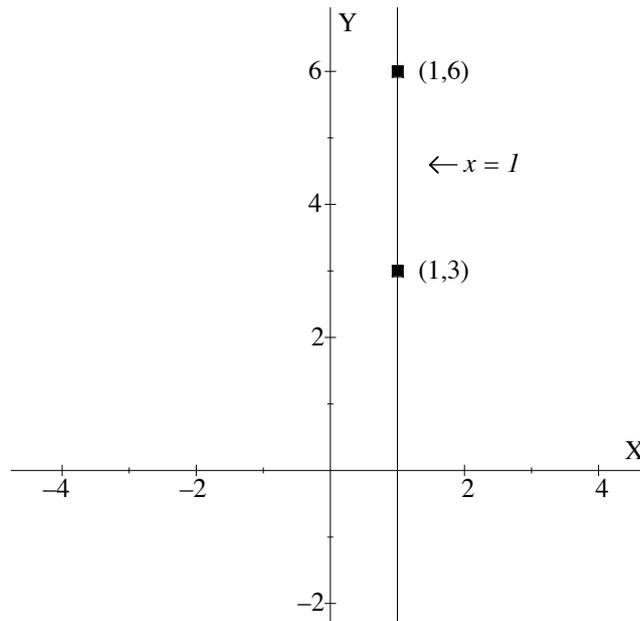The line through the points $(1, 3)$ and $(1, 6)$ is shown is Figure 17.

Figure 17: Graphs of $x = 4$.

Clearly we have to go 3 units up, but no distance at all along. According to our rule, the slope should be $\frac{3}{0}$. But division by 0 is not possible, so we say the slope is undefined.

A line which is parallel to the vertical axis has equation $x = a$, for some number a. In this case the equation is $x = 1$ as x is always 1, no matter what the value of y is.

4.2.3   Exercises

1.   Find the equations of the following lines:

   a. The gradient is -4 and the Y intercept is 2.

   b. The line has the same slope as (is parallel to) $y = 3x - 2$ and has Y intercept 5.

   c. The line is horizontal and passes through $(1, 2)$.

   d. The line is vertical and passes through $(-1, 3)$.

2. Match up the following statements in the table below:

The line $y = 3x + 7$,

 a. is parallel to the line   **i.** $2y = 6x + 14$

 b. has the same Y intercept but different gradient to the line **ii.** $y = 3x - 7$

 c. has gradient of opposite sign to the line   **iii.** $y = 5x + 7$

 d. is identical to the line   **iv.** $y = -3x - 7$

3. Use your knowledge of gradient and intercept to sketch the graphs of $y = x$ and $y = 2x$.

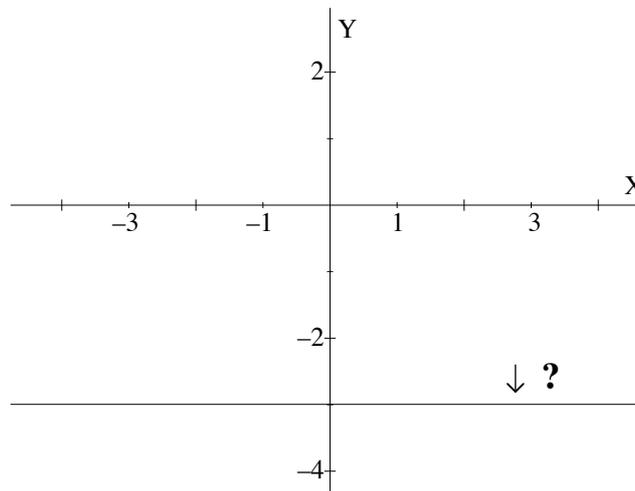**4.** What are the equations of the following lines?
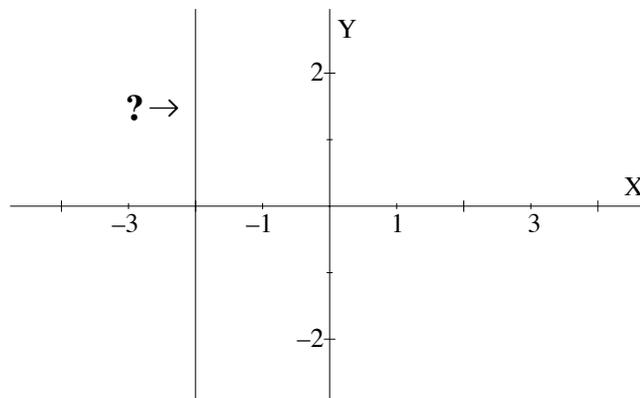
 a.

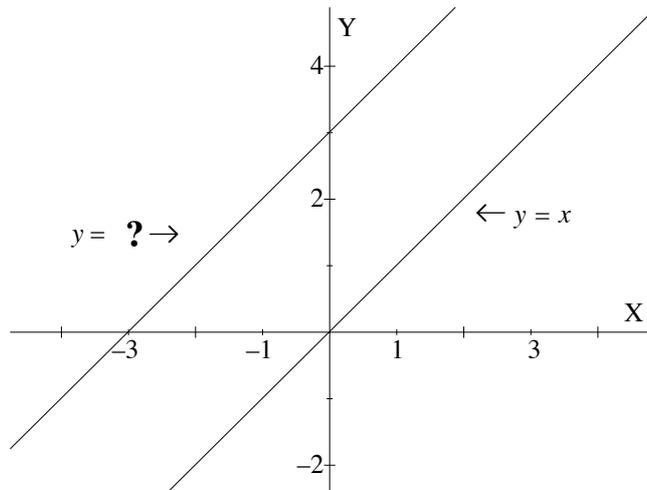Figure 18:

 b.

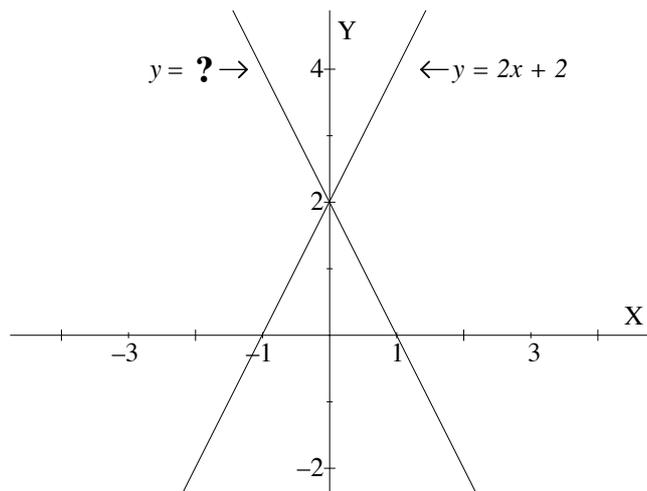Figure 19:

c.

Figure 20:

d.

Figure 21:

4.3   Finding the Equation of a Line from Two Points on the Line
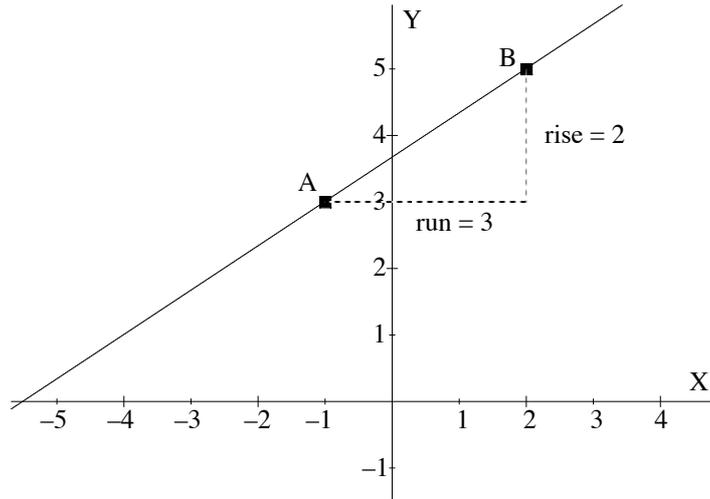
If a line passes through two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, the gradient of the line is given by

$$b = \frac{\text{amount uphill}}{\text{amount along}} = \frac{\text{rise}}{\text{run}}.$$

The gradient b equals the ratio

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Example: Find the gradient of the line passing through $A = (-1, 3)$ and $B = (2, 5)$ as shown in Figure 22.

Figure 22: Line passing through points $(-1, 3)$ and $(2, 5)$.

Solution:

$$
\begin{aligned}
\text{Gradient} \quad &= \quad \frac{y_2 - y_1}{x_2 - x_1} \\
&= \quad \frac{5 - 3}{2 - (-1)} \\
&= \quad \frac{2}{3}.
\end{aligned}
$$

That is, in the equation $y = a + bx$, $b = \frac{2}{3}$.

Now, to find the equation of the line, substitute the coordinates of one of the points, say the point $(-1, 3)$, into the equation $y = a + bx$.

When $x = -1$, $y = 3$.

Thus,

$$
\begin{aligned}
3 \quad &= \quad a + b(-1) \\
&= \quad a + \frac{2}{3}(-1) \\
&= \quad a - \frac{2}{3}.
\end{aligned}
$$

We can solve this last equation for 'a' by the backtracking method.

The equation says:

1) Let a represent some number;

2) Subtract $\frac{2}{3}$;

3) The result is 3.

Backtracking:

1) The final value is 3, $\qquad\qquad 3 = a - \frac{2}{3}$.

2) Add $\frac{2}{3}$, $\qquad\qquad\qquad\quad 3 + \frac{2}{3} = a$.

3) We have a. $\qquad\qquad\qquad a = 3\frac{2}{3}$.

Hence, the equation of the line is

$$y = \frac{2}{3}x + 3\frac{2}{3}.$$

Let us check that this line does indeed pass through the second point $(2, 5)$.

When $x = 2$, $y = \dfrac{2}{3}(2) + 3\dfrac{2}{3} = 5$, so the equation is correct.

4.3.1 Exercise

1. Suppose marks on a psychology test are scaled according to the formula $y = bx + a$.

   Let x be the mark (out of 100) actually received on the test and let y be the scaled mark.

   Anne's actual mark is 60 and her scaled mark is 73, so that A $= (60, 73)$.

   David's actual mark is 48 and his scaled mark is 65, so that $D = (48, 65)$.

   a. Find the scaling formula.

   b. What will Bill's scaled mark be if he received an actual mark of 75?

5 Additional Reading

Algebra is based on arithmetic. If you are not confident about how to manipulate numbers, including positve and negative numbers, read the Mathematics Learning Centre booklet: *Numerical Skills: An Intuitive Approach.*

A revision of fractions, percentages and decimals as well as the rules governing the order of operations is also covered in this publication.

If you wish to have more practice and increase your understanding of algebra, work through Module 2 of *Countdown to Mathematics.* An alternative approach to algebra can be found in *The Math Workshop: Algebra*, by Deborah Hughes-Hallett, Chapters 8 and 12.

More about graphing straight lines can be studied from *The Math Workshop: Elementary Functions* by Deborah Hughes-Hallett, Chapter 2, or *Countdown to Mathematics*, Module 3.

Finally, an excellent book for revising elementary statistics, or for preparing for a first statistics course, is *Statistics Without Tears A Primer For Non-Mathematicians*, by Derek Rowntree. This book explains many of the basic concepts of statistics without using much mathematics.

All the books mentioned above are available for reading in the Mathematics Learning Centre at the University of Sydney.

6 Answers to Exercises

6.1 Answers to Exercises in Chapter 1

Answers to Exercise 1.1.1

1.	$R = L - S$	**2.**	$R = \frac{F}{N}$
3a.	$T = W_1 + W_2 + \cdots + W_n$	**3b.**	$\mu = \frac{T}{n}$ or $\mu = \frac{W_1+W_2+\cdots+W_n}{n}$
4.	$P = \frac{H}{N} \times 100 = \frac{100H}{N}$	**5.**	$z = \frac{X-\mu}{\sigma}$

Answers to Exercise 1.2.1

1a.	15	**1b.**	$\frac{3}{5}$ or 0.6
1c.	5		
2.	36	**3.**	$\mu = 5$
4.	$p = 0.95$	**5.**	$E(X) = 40$
6.	$SE = 1.5$	**7.**	$p = 0.9$
8.	$z = -0.9259$ or $z = -0.93$ to 2 dec. places	**9.**	$z = 16$
10.	$x = 127.75$	**11.**	$y = 5.891$

6.2 Answers to Exercises in Chapter 2

Answers to Exercise 2.1.1

1. 18   **2.** 110   **3.** 11kg   **4.** 50

Answers to Exercise 2.2.1

1.   **a.** $\sigma^2 = 4$
    b. $\sigma = \sqrt{4} = 2$

2. $\mu = 300$, $\sigma^2 = 6750$

3. $\mu = 22.57$ to two decimal places, $\sigma^2 = 2.53$

taking the mean as 22.57 and using the formula $\sigma^2 = \frac{1}{N}\Sigma_{i=1}^{N}(x_i - \mu)^2$.

If, however the formula $\sigma^2 = \frac{1}{N}[\Sigma_{i=1}^{N}x_i^2 - N\mu^2]$ is used, then the answer $\sigma^2 = 2.60$ is obtained. This discrepancy is due to round off error and can be avoided by using $\mu = 22.571429$ in the above formula.

4.   **a.** 29.34
    b. 3.5

6.3 Answers to Exercises in Chapter 3

Answers to Exercise 3.1.1

1. $r = 12$
2. $n = 11$
3. $y = 1$
4. $x = 16.5$
5. $x = \mu + ts$
6. $x = 8$
7. $t = 4$
8. $\bar{x} = 120$
9. $\bar{x} = \frac{z\sigma}{\sqrt{n}} + \mu$ or $\mu + \frac{\sigma z}{\sqrt{n}}$

6.4 Answers to Exercises in Chapter 4

Answers to Exercise 4.1.1

1. A $= (2, -1)$ B $= (1, 1)$ C $= (-2, 2)$ D $= (0, 0)$ E $= (-1, -2)$
2a. $(3, 0)$ **2b.** $(3, 4), (3, 2)$ and $(3, 1)$ **2c.** $(4, 1), (5, 1)$ and $(6, 1)$ **2d.** 8 and 3

Answers to Exercise 4.2.1

1, 3.

Answers to Exercise 4.2.3

1a. $y = -4x + 2$ **1b.** $y = 3x + 5$ **1c.** $y = 2$ **1d.** $x = -1$

2a. ii **2b.** iii **2c.** iv **2d.** i

3. The graph of $y = x$ is shown in Figure 23.
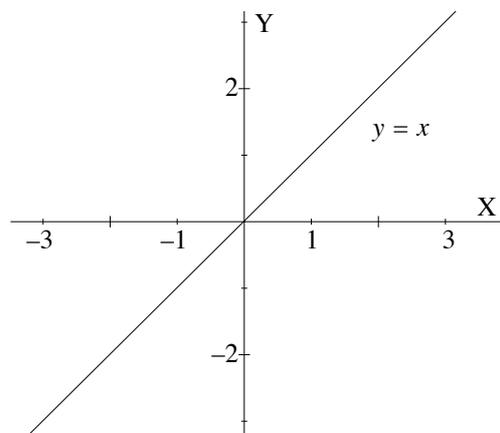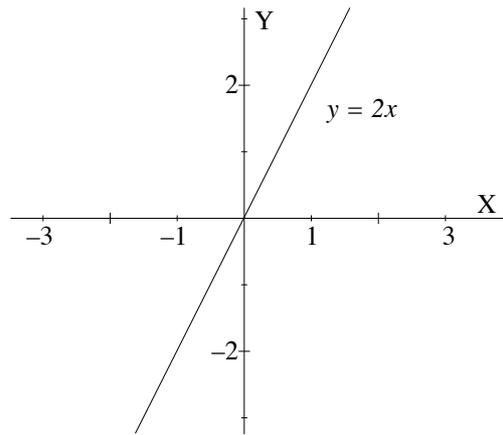Its gradient is $b = 1$ and Y intercept is 0.

Figure 23: Graph of $y = x$.

The graph of $y = 2x$ is shown in Figure 24. Its gradient is $b = 2$ and Y intercept is 0.

Figure 24: Graph of $y = 2x$.

4a. $y = -3$ **4b.** $x = -2$ **4c.** $y = x + 3$ **4d.** $y = -2x + 2$

Answer to Exercise 4.3.1

1. **a.** A $= (60, 73)$ and D $= (48, 65)$. Thus the gradient,

$$
\begin{aligned}
b &= \frac{73 - 65}{60 - 48} \\
&= \frac{8}{12} \\
&= \frac{2}{3}.
\end{aligned}
$$

Therefore, the equation is,

$$y = \frac{2}{3}x + a.$$

Substituting $x = 60$ and $y = 73$ into this equation, and solving for a, we obtain $a = 33$.

Hence the scaling formula is $y = \frac{2}{3}x + 33$.

b. Bill's actual mark is 75, thus his scaled mark is

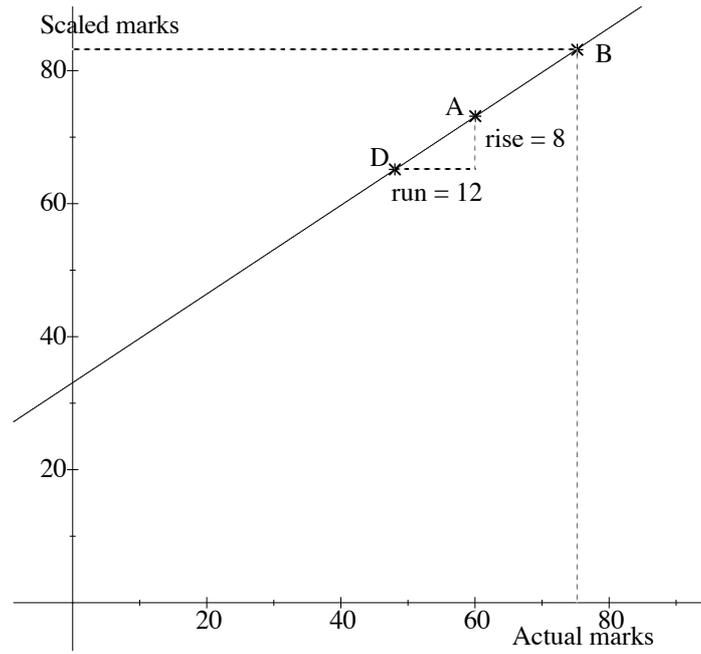$$y = \frac{2}{3}(75) + 33 = 83.$$

This is illustrated in Figure 25.

Figure 25: Graph of $y = \frac{2}{3}x + 3\frac{2}{3}$.

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