Mathematics Learning Centre

The University of Sydney

Logarithmic Functions and the Log Laws

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1 Logarithms

1.1 Introduction

Taking logarithms is the reverse of taking exponents, so you must have a good grasp on exponents before you can hope to understand logarithms properly.

We begin the study of logarithms with a look at logarithms to base 10. It is important that you realise from the beginning that, as far as logarithms are concerned, there is nothing special about the number 10. Indeed, the most natural logarithms are logarithms to base e, and they are introduced in section 1.4. Logarithms to base 10 are in common use only because we use a decimal system of counting, and this is probably a result of the fact that humans have ten fingers. We have begun with logarithms to base 10 only to be definite, and we could just as easily have started with logarithms to any other convenient base.

1.2 Logarithms to base 10 (Common Logarithms)

We will begin by considering the function $y = 10^x$, graphed in Figure 1.

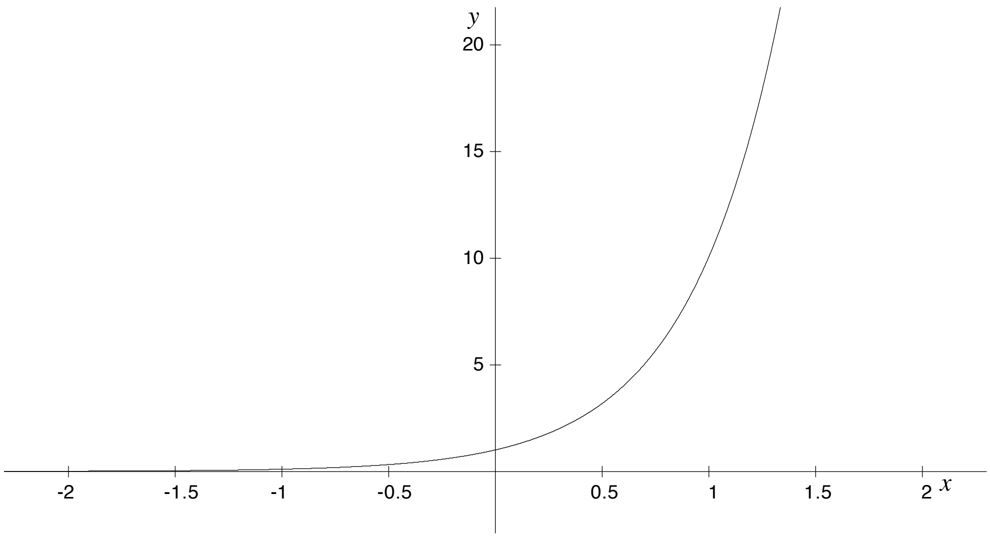

Figure 1: Graph of $f(x) = 10^x$

We know that, given any number x, we can raise 10 to the power of x to obtain another number which we write as 10^x.

What of the reverse procedure? Suppose we begin with a number and we wish to find the power to which 10 must be raised to obtain that number.

For example, suppose we begin with the number 7 and we wish to find the power to which 10 must be raised to obtain 7.

This number is called the *logarithm to the base* 10 *of* 7 and is written $\log_{10} 7$. Similarly, $\log_{10} 15$ is equal to the power to which 10 must be raised to obtain 15.

For a general number x, $\log_{10} x$ is equal to that power to which 10 must be raised to obtain the number x.

When we see an expresion like $\log_{10} 29$ we can think of it as a sort of a question.

The question we have in mind is this: to what power must we raise 10 to get 29? Or, $10^? = 29$.

The answer to this question is a number, and we call that number $\log_{10} 29$.

The definition of the logarithm to base 10 is the basis on which the remainder of this section rests, and it is extremely important that you understand it properly.

Again: $\log_{10} x$ *is equal to that power to which* 10 *must be raised to obtain the number* x.

As an example, let's calculate $\log_{10} 10^3$. According to the definition, $\log_{10} 10^3$ is equal to that power to which 10 must be raised to obtain 10^3. To what power must we raise 10 to obtain 10^3? Or, $10^? = 10^3$. Surely the answer is 3. Notice that $10^3 = 1000$, so we have worked out $\log_{10} 1000$, and without using a calculator! We have been able to work this out because we have understood the *meaning* of the logarithm of a number. We will need to use a calculator to work out the logarithms of most numbers, but it is very important that we understand what it is that the calculator is working out for us when we push the buttons.

Without a calculator we can work out the logarithms of many numbers.

Examples:

$$\begin{aligned}
\log_{10} 100 &= \log_{10} 10^2 = 2 \\
\log_{10} 0.1 &= \log_{10} 10^{-1} = -1 \\
\log_{10} 10\sqrt{10} &= \log_{10} 10^{1.5} = 1.5
\end{aligned}$$

Can we take the logarithm of any number? In other words, given any number x can we find a power to which 10 may be raised to obtain the number x?

Look at the graph of $y = 10^x$ in Figure 1. We see that 10^x is *never* negative and indeed never even takes the value 0. *There is no power to which we may raise* 10 *to obtain a number less than or equal to* 0. This means that we cannot take the logarithm of a number less than or equal to zero. We say that $\log_{10} x$ is *undefined* for $x \leq 0$.

The graph of 10^x gives us another important piece of information. If $x > 0$ then there is only one power to which we may raise 10 to get x. Our definition of $\log_{10} x$ is unambiguous.

The graph of $y = \log_{10} x$ is shown in Figure 2.

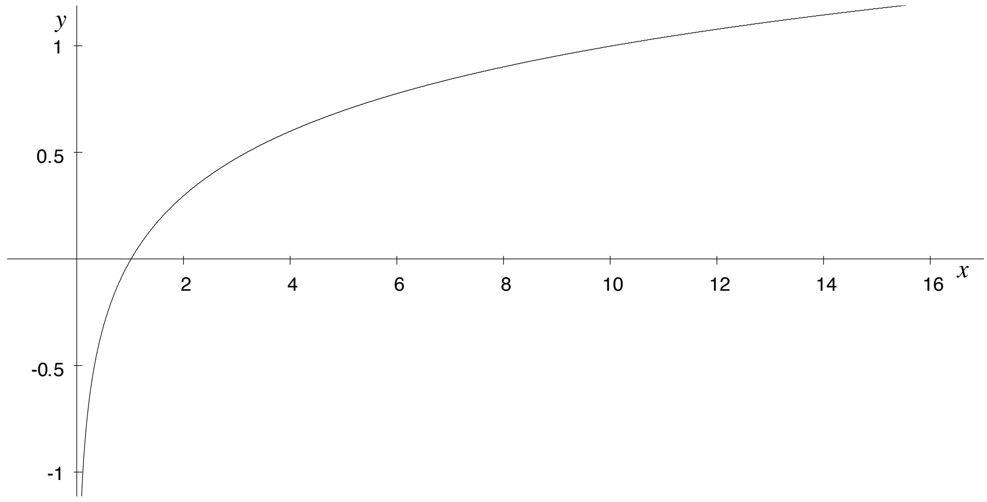

Figure 2: Graph of $f(x) = \log_{10} x$

You should pay attention to several important features of this graph.

The graph intercepts the x-axis at $x = 1$. In other words, $\log_{10} 1 = 0$. We should expect this because we know that $10^0 = 1$.

The graph does not extend to the left of the y-axis, and in fact never even intercepts the y-axis. We have already commented on the fact that the logarithm of a number less than or equal to zero is not defined.

The function $y = \log_{10} x$ gets as large as we like as x gets large. By this we mean that we can make $\log_{10} x$ as large as we choose by choosing x to be sufficiently large. The graph does not stay below a certain height as x gets large (it does not have a horizontal asymptote). However the function $y = \log_{10} x$ increases very slowly as x increases.

The fact that we bother to specify the base as being 10 suggests that we can take logarithms to other bases. We can, and we shall say more about this later, but for now let us stick with base 10.

You should be aware that many writers may not mention the base of the logarithms they are referring to if it is obvious from the context what that base is, or if it does not matter which base is used. They may just write 'the logarithm of x' or $\log x$.

Because logarithms to base 10 have been used so often they are called *common logarithms.* If you have a calculator it probably has a Log button on it. You could use it to find, for example, $\log_{10} 7$ and $\log_{10} 0.01$.

From the examples above you should be able to see that if we express a number as a power of 10 then we can read off the logarithm to base 10 of that number from the power. Let's try to make this precise.

Suppose that x is any real number. What is $\log_{10} 10^x$? Well, $\log_{10} 10^x$ is that power to which 10 must be raised to obtain the number 10^x. To what power must we raise 10 to obtain the number 10^x? Or, to put this question another way, $10^? = 10^x$. The answer must be x. Thus $\log_{10} 10^x = x$. This is our first rule of logarithms.

Rule A: For any real number x, $\log_{10} 10^x = x$.

Examples

$$\begin{aligned}
\log_{10} 10^{3.7} &= 3.7 \\
\log_{10} 0.0001 &= \log_{10} 10^{-4} = -4 \\
\log_{10} 10^4 \sqrt[5]{10^3} &= \log_{10} 10^4 \times (10^3)^{\frac{1}{5}} = \log_{10} 10^{4+\frac{3}{5}} = \frac{23}{5}
\end{aligned}$$

Rule A tells us what happens if we first raise 10 to the power x and then take the logarithm to base 10 of the result. We end up with what we started with.

What happens if we do things in the reverse order?

Consider the number $\log_{10} 7$. If you have a calculator with a Log button on it you can see that this number is approximately 0.8451. Now suppose we raise 10 to the power $\log_{10} 7$. What do you think the result is? In symbols, what is $10^{\log_{10} 7}$?

Well, remember that $\log_{10} 7$ is equal to that power to which 10 must be raised to give the number 7. So if we raise 10 to that power then we must get 7.

The same reasoning applies to show that if $x > 0$ then $10^{\log_{10} x} = x$. The number $\log_{10} x$ is that power to which 10 must be raised to obtain x. So if we raise 10 to this power we must get x. We will write this down as the second of our rules of logarithms.

Rule B: For any real number $x > 0$, $10^{\log_{10} x} = x$.

Examples

$$\begin{aligned}
10^{\log_{10} \pi} &= \pi \\
10^{\log_{10}(x^2+y^2)} &= x^2 + y^2 \\
10^{\log_{10} 10^{3x^3}} &= 10^{3x^3}
\end{aligned}$$

Rules A and B express the fact that the functions $y = 10^x$ and $y = \log_{10} x$ are *inverse functions* of one another. If you have not come across the concept of inverse functions before then do not worry about what this means. If you have, then you will probably remember that the graph of an inverse function is obtained by reflecting the graph of the original function in the line $y = x$, that is the line which runs in the north-east and south-west direction. Take another look at Figures 1 and 2.

We can use the rules of exponents to work out more rules for logarithms.

If x and y are numbers greater than zero then, by rule B, $x = 10^{\log_{10} x}$ and $y = 10^{\log_{10} y}$, so

$$\begin{aligned}
xy &= 10^{\log_{10} x} \times 10^{\log_{10} y} \\
&= 10^{\log_{10} x + \log_{10} y} \qquad \text{(by the rules for exponents).}
\end{aligned}$$

This equation tell us that if we raise 10 to the power $\log_{10} x + \log_{10} y$ then we get the number xy. In other words it tells us that $\log_{10} x + \log_{10} y$ is the answer to the question $10^? = xy$. But the answer to this question is also $\log_{10} xy$. Thus $\log_{10} xy = \log_{10} x + \log_{10} y$. This we will call our third rule of logarithms.

Rule C: For any real numbers $x > 0$ and $y > 0$, $\log_{10} xy = \log_{10} x + \log_{10} y$.

So much for multiplication. What of division? If $x > 0$ and $y > 0$ then

$$\begin{aligned}\frac{x}{y} &= \frac{10^{\log_{10} x}}{10^{\log_{10} y}} \qquad \text{(by rule B)} \\ &= 10^{\log_{10} x - \log_{10} y} \qquad \text{(by the rules for exponents).}\end{aligned}$$

This equation tells us that if we raise 10 to the power $\log_{10} x - \log_{10} y$ then we get the number $\frac{x}{y}$. In other words, $\log_{10} \frac{x}{y} = \log_{10} x - \log_{10} y$. This is our fourth rule of logarithms.

Rule D: For any real numbers $x > 0$ and $y > 0$, $\log_{10}(\frac{x}{y}) = \log_{10} x - \log_{10} y$.

If x is a number, $x > 0$, and n is any number at all then:

$$\begin{aligned}x^n &= (10^{\log_{10} x})^n \qquad \text{(by rule B)} \\ &= 10^{n \times \log_{10} x} \qquad \text{(by the rules for exponents).}\end{aligned}$$

This equation tells us that if we raise 10 to the power $n \log_{10} x$ then we get the number x^n. In other words, $\log_{10} x^n = n \log_{10} x$. This is our fifth rule of logarithms.

Rule E: For real numbers x and n, with $x > 0$, $\log_{10} x^n = n \log_{10} x$

Examples

$$\begin{aligned}\log_{10} \frac{xy}{z} &= \log_{10} x + \log_{10} y - \log_{10} z \\ \log_{10} x^3 y^{-2} &= 3 \log_{10} x - 2 \log_{10} y \\ 2 \log_{10} y - 4 \log_{10}(x^2 - z^3) &= \log_{10} \frac{y^2}{(x^2 - z^3)^4}\end{aligned}$$

1.3 Logarithms to Base b

As we mentioned above, we can take logarithms to other bases. If b is a real number, $b > 1$, and if x is a real number, $x > 0$, then we define the logarithm to base b of x to be that power to which b must be raised to obtain the number x.

You may also think of $\log_b x$ as the answer to the question $b^? = x$. You should notice that if $b = 10$ then this definition agrees with the one given earlier for $\log_{10} x$.

Again: *the logarithm to base b of a number $x > 0$* (written $\log_b x$) *is that power to which b must be raised to obtain the number x.*

Examples:

$$\begin{aligned}\log_5 125 &= \log_5 5^3 = 3 \\ \log_{16} 2 &= \log_{16} 16^{\frac{1}{4}} = \frac{1}{4} \\ \log_7 \frac{1}{49} &= \log_7 7^{-2} = -2\end{aligned}$$

We have required the base of our logarithms, b, to be greater than 1. In fact we can take logarithms to any base b provided $b > 0$ and $b \neq 1$. It is more usual though to use $b > 1$, and in this booklet we will always use a base $b > 1$.

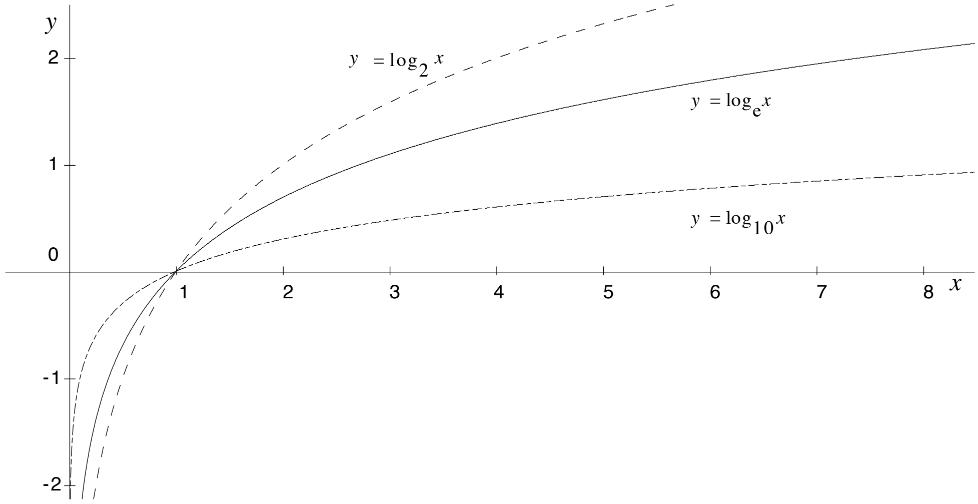

Figure 3: Graph of $f(x) = \log_b x$ for various values of b.

Figure 3 shows graphs of the functions $y = \log_b x$ for various values of b. As you can see from these graphs, the logarithm functions behave in a similar fashion for different bases b, providing $b > 1$.

All of what we said earlier remains true for $\log_b x$ if 10 is replaced by b. In particular the five rules of logarithms remain true. Let us restate these to be applicable to $\log_b x$.

For a real number $b > 1$:

Rule 1: For any real number x, $\log_b b^x = x$

Rule 2: For any real number $x > 0$, $b^{\log_b x} = x$

Rule 3: For any real numbers $x > 0$ and $y > 0$, $\log_b xy = \log_b x + \log_b y$

Rule 4: For any real numbers $x > 0$ and $y > 0$, $\log_b \frac{x}{y} = \log_b x - \log_b y$

Rule 5: For real numbers x and n, with $x > 0$, $\log_b x^n = n \log_b x$

Now that we have shown how to define logarithms to any base $b > 1$, let us see how these logarithms are related to each other. We will consider logarithms to two bases $a > 1$ and $b > 1$. By rule 2,

$$x = a^{\log_a x}.$$

Taking logarithms to base b of both sides of this equation yields

$$\begin{aligned} \log_b x &= \log_b(a^{\log_a x}) \\ &= \log_a x \times \log_b a \qquad \text{(by rule 5).} \end{aligned}$$

This, our sixth rule of logarithms, tells us how logarithms to different bases are related.

Rule 6: For numbers $x > 0$, $a > 1$ and $b > 1$, $\log_b x = \log_b a \times \log_a x$.

From this rule we see that $\log_b a \times \log_a b = \log_b b = 1$, and so

$$\log_b a = \frac{1}{\log_a b}.$$

This fact enables us to calculate the logarithm of a number to any base from a calculator which calculates logarithms to one base only.

Example: If your calculator only has logarithms to base 10 on it, how can you find $\log_7 9$?

Solution: By rule 6,

$$\begin{aligned}\log_7 9 &= \log_7 10 \times \log_{10} 9 \\ &= \frac{1}{\log_{10} 7} \times \log_{10} 9\end{aligned}$$

and the last expression can be evaluated by any calculator which can evaluate logarithms to base 10.

1.4 Logarithms to base e (Natural Logarithms)

Logarithms to the base 10 are commonly used, because we use a decimal number system and not a base 8 system, or a base 2 system. If humans were born with 3 toes (or if sloths could count) then logarithms to base 3 might be in common use. Apart from the fact that we use a decimal number system, there is no reason for us to prefer logarithms to base 10 over logarithms to any other base. Indeed, the function $y = e^x$ is a very important function in mathematics, and it is therefore reasonable to expect that logarithms to base e will also assume special importance.

They do, and are given the name 'Natural Logarithms' or 'Napierian Logarithms'. They are even given a special symbol, ln, so that $\ln x = \log_e x$. One of the graphs in Figure 3 is a plot of the function $y = \log_e x = \ln x$. Notice that the function $y = \ln x$ behaves in a similar fashion to the function $y = \log_{10} x$. This comes as no surprise to us since that the functions e^x and 10^x are very similar to each other.

1.5 Exponential functions revisited

In a previous lecture, we saw how much the exponential functions resemble each other. If $b > 1$ then the exponential function b^x looks very much like any of the other exponential functions with base greater than 1, and if $b < 1$ then b^x looks a lot like any of the exponential functions with base less than one. We will now be able to see more clearly what is going on here.

Consider the function $y = 2^x$. Now $2 = e^{\log_e 2}$, so we can write

$$2^x = (e^{\log_e 2})^x = e^{x \log_e 2}.$$

We have been able to write the function 2^x as a function involving the base e, though the exponent is now not simply x, but x multiplied by some fixed number, namely $\log_e 2$.

Similarly, we could write

$$\begin{aligned} 5^x &= e^{x \log_e 5} \\ 19^{-x} &= e^{-x \log_e 19} \\ 7^{-x} &= 4^{-x \log_4 7} \\ e^x &= 13^{x \log_{13} e} \end{aligned}$$

We can write all exponential functions in the form $y = e^{kx}$, where k is some constant which may be negative.

1.6 Summary

For any real number $b > 1$ and any $x > 0$, $\log_b x$ is equal to that number to which b must be raised to obtain the number x. One can think of $\log_b x$ as the answer to the question $b^? = x$. The number $\log_b x$ is called the *logarithm to base b of x.*

The function $\log_b x$ satisfies the following rules:

Rule 1: For any real number x, $\log_b b^x = x$

Rule 2: For any real number $x > 0$, $b^{\log_b x} = x$

Rule 3: For any real numbers $x > 0$ and $y > 0$, $\log_b xy = \log_b x + \log_b y$

Rule 4: For any real numbers $x > 0$ and $y > 0$, $\log_b \frac{x}{y} = \log_b x - \log_b y$

Rule 5: For real numbers x and n, with $x > 0$, $\log_b x^n = n \log_b x$

Rule 6: For numbers $x > 0$, $a > 1$ and $b > 1$, $\log_b x = \log_b a \times \log_a x$.

Logarithms to base 10 are in common use and for this reason they are called Common Logarithms.

Logarithms to base e are of special importance. They are often called natural logarithms or Napierian logarithms, and the symbol $\ln x$ is often used for them. Thus $\ln x = \log_e x$.

Any exponential function may be written in the form e^{kx}, where the constant k may be negative.

1.7 Exercises

1. By expressing these numbers as powers of 10, and without using a calculator, calculate the logarithms to base 10 of the following numbers.

a. 10000 **b.** $\frac{1}{100}$ **c.** 0.001 **d.** $10^{12.3}$

e. $\sqrt{10}$ **f.** $10\sqrt[4]{10}$ **g.** $(\frac{1}{1000})\sqrt[3]{10}$ **h.** $\frac{1}{0.001}$

2. Simplify the following expressions.

a. $10^{\log_{10} 37.23}$ **b.** $\log_{10} 10^{x^2 y}$ **c.** $10^{\log_{10}(10^x)}$ **d.** $\log_{10} \sqrt{10}^{\frac{x}{2y}}$

e. $10^{10^{\log_{10} x}}$ **f.** $\log_{10} 10^{\frac{x+y}{z}}$ **g.** $10^{\log_{10}(\frac{3xy}{z})}$ **h.** $\log_{10} 10^{10^{2x}}$

3. Rewrite the following expressions so that they involve just one logarithm.

a. $\log_{10} x^3 - 2.5\log_{10} y$ **b.** $\log_{10} 6 + \log_{10} x^{-2}$

c. $5\log_{10} 3x - 4\log_{10}(xy + z^2)$ **d.** $2\log_{10} xy + 3\log_{10}(z^2 - y^2)$

e. $\log_{10}(x + y) - 3\log_{10} 4$ **f.** $\log_{10} xy - 1.7\log_{10} y^2$

4. Simplify these expressions.

a. $\log_2 2^x 2^{x+y}$ **b.** $5^{\log_5 \frac{x+y}{3}}$ **c.** $\log_7 49^{uv}$ **d.** $3^{\log_9 \frac{u}{3w}}$

5. Rewrite the following expressions so that they involve only one logarithm.

a. $2\log_3(x + y) - 3\log_3(xy) + \log_3 x^2$ **b.** $\log_6 xy - 4\log_6(x + y)$

c. $4\log_{17} xy^2 + \log_{17}(x^2 + y^2) - 2.5\log_{17} x$

6. Using a calculator, find the following logarithms.

a. $\log_3 17$ **b.** $\log_5 2$ **c.** $\log_{22} 14$ **d.** $\log_4 8$

7. Write each of the following functions in the form $y = e^{kx}$ for a suitable constant k.

a. $y = 10^x$ **b.** $y = 7.5^{-x}$ **c.** $y = 4^{-x}$ **d.** $y = (\frac{1}{4})^x$

Without using a calculator, find the following numbers.

8. $\log_{10} 10^{-19}$ **9.** $\log_e e\sqrt[5]{e}$ **10.** $\log_2 16$

11. $\log_{17} \frac{17^3}{\sqrt{17}}$ **12.** $\ln \frac{e^2}{e^{21}}$ **13.** $\frac{\ln e^7}{\log_{11} 121}$

14. $5^{\log_5 32.7}$ **15.** $e^{\ln \frac{9}{2}}$ **16.** $e^{\ln \sqrt[3]{27}}$

Rewrite the following expressions using the rules of logarithms, and simplify where possible.

17. $\log_{10} \frac{100x^2}{9y}$ **18.** $\ln \frac{xy^{-3}}{e^{1.37}}$ **19.** $\log_4 \frac{4^{-1.3}z^7}{x^2y^3}$

20. $\log_3 \frac{x^3y^2}{27z^{\frac{1}{2}}}$ **21.** $\ln(e^{-2.4}x^6)$ **22.** $\log_5 \frac{125x^3}{0.2y^2}$

Using the rules of logarithms, rewrite the following expressions so that just one logarithm appears in each.

23. $3\log_2 x + \log_2 30 + \log_2 y - \log_2 w$ **24.** $2\ln x - \ln y + a\ln w$

25. $12(\ln x + \ln y)$ **26.** $\log_3 e \times \ln 81 + \log_3 5 \times \log_5 w$

27. $\log_7 10 \times \log_{10} x^2 - \log_7 49x$ **28.** $\log_{10} 0.1 \times \log_6 x - 2\log_6 y + \log_6 4 \times \log_4 e$

Given that $\log_e 5 \approx 1.6094$, and $\log_e 7 \approx 1.9459$, find the following numbers without using a calculator except to perform multiplication or division.

29. $\log_5 e$ **30.** $\log_5 7$ **31.** $\log_5 7^2$ **32.** $\log_{49} 5$ **33.** $\log_{49} 25$ **34.** $\log_e 25$

1.8 Solutions to exercises

1. **a.** $\log_{10} 10000 = \log_{10} 10^4 = 4$

b. $\log_{10} \frac{1}{100} = \log_{10} 10^{-2} = -2$

c. $\log_{10} 0.001 = \log_{10} 10^{-3} = -3$

d. $\log_{10} 10^{12.3} = 12.3$

e. $\log_{10} \sqrt{10} = \log_{10} 10^{\frac{1}{2}} = \frac{1}{2}$

f. $\log_{10} 10\sqrt[4]{10} = \log_{10} 10 \times 10^{\frac{1}{4}} = \log_{10} 10^{1+\frac{1}{4}} = \frac{5}{4}$

g. $\log_{10}(\frac{1}{1000})\sqrt[3]{10} = \log_{10} 10^{-3+\frac{1}{3}} = -\frac{8}{3}$

h. $\log_{10} \frac{1}{0.001} = \log_{10} 10^3 = 3$

2. **a.** $10^{\log_{10} 37.23} = 37.23$ by rule B.

b. $\log_{10} 10^{x^2 y} = x^2 y$ by Rule A.

c. $10^{\log_{10}(10^x)} = 10^x$, since $\log_{10}(10^x) = x$ by rule A.

d. $\log_{10} \sqrt{10}^{\frac{x}{2y}} = \log_{10}(10^{\frac{1}{2}})^{\frac{x}{2y}} = \log_{10} 10^{\frac{1}{2}\times\frac{x}{2y}} = \frac{x}{4y}$ by rule A.

e. $10^{10^{\log_{10} x}} = 10^x$, since $10^{\log_{10} x} = x$ by rule B.

f. $\log_{10} 10^{\frac{x+y}{z}} = \frac{x+y}{z}$ by rule A.

g. $10^{\log_{10} \frac{3xy}{z}} = \frac{3xy}{z}$ by rule B.

h. $\log_{10} 10^{10^{2x}} = 10^{2x}$ by rule A.

3. **a.** $\log_{10} x^3 - 2.5 \log_{10} y = \log_{10} x^3 - \log_{10} y^{2.5} = \log_{10} \frac{x^3}{y^{2.5}}$

b. $\log_{10} 6 + \log_{10} x^{-2} = \log_{10} 6x^{-2}$

c. $5 \log_{10} 3x - 4 \log_{10}(xy + z^2) = \log_{10}(3x)^5 - \log_{10}(xy + z^2)^4 = \log_{10} \frac{(3x)^5}{(xy+z^2)^4}$

d. $2 \log_{10} xy + 3 \log_{10}(z^2 - y^2) = \log_{10}(xy)^2 + \log_{10}(z^2 - y^2)^3 = \log_{10}(xy)^2(z^2 - y^2)^3$

e. $\log_{10}(x + y) - 3 \log_{10} 4 = \log_{10}(x + y) - \log_{10} 4^3 = \log_{10} \frac{x+y}{64}$

f. $\log_{10} xy - 1.7 \log_{10} y^2 = \log_{10} xy - \log_{10}(y^2)^{1.7} = \log_{10} \frac{xy}{y^{3.4}} = \log_{10} xy^{-2.4}$

4. **a.** $\log_2 2^x 2^{x+y} = \log_2 2^{x+x+y} = 2x + y$

b. $5^{\log_5 \frac{x+y}{3}} = \frac{x+y}{3}$

c. $\log_7 49^{uv} = \log_7(7^2)^{uv} = \log_7 7^{2uv} = 2uv$

d. $3^{\log_9 \frac{u}{3w}} = (9^{\frac{1}{2}})^{\log_9 \frac{u}{3w}} = 9^{\frac{1}{2}\times\log_9 \frac{u}{3w}} = 9^{\log_9(\frac{u}{3w})^{\frac{1}{2}}} = (\frac{u}{3w})^{\frac{1}{2}}$

5. **a.**

$$\begin{aligned} 2\log_3(x+y) - 3\log_3(xy) + \log_3 x^2 &= \log_3(x+y)^2 - \log_3(xy)^3 + \log_3 x^2 \\ &= \log_3 \frac{((x+y)^2 x^2)}{(xy)^3} \end{aligned}$$

b.

$$\begin{aligned} \log_6 xy - 4\log_6(x+y) &= \log_6 xy - \log_6(x+y)^4 \\ &= \log_6 \frac{xy}{(x+y)^4} \end{aligned}$$

c.

$$\begin{aligned} 4\log_{17} xy^2 + \log_{17}(x^2+y^2) - 2.5\log_{17} x &= \log_{17}(xy^2)^4 + \log_{17}(x^2+y^2) - \log_{17} x^{2.5} \\ &= \log_{17} \frac{x^4y^8(x^2+y^2)}{x^{2.5}} \end{aligned}$$

6. **a.** $\log_3 17 = \log_3 10 \times \log_{10} 17 = \frac{\log_{10} 17}{\log_{10} 3} \approx \frac{1.2304}{0.4771} \approx 2.5789$

b. $\log_5 2 = \log_5 10 \times \log_{10} 2 = \frac{\log_{10} 2}{\log_{10} 5} \approx \frac{0.3010}{0.6990} \approx 0.4306$

c. $\log_{22} 14 = \log_{22} 10 \times \log_{10} 14 = \frac{\log_{10} 14}{\log_{10} 22} \approx \frac{1.1461}{1.3424} \approx 0.8538$

d. $\log_4 8 = \log_4 10 \times \log_{10} 8 = \frac{\log_{10} 8}{\log_{10} 4} \approx \frac{0.9031}{0.6021} \approx 1.5$ (in fact $\log_4 8$ is exactly 1.5 because $8 = 4^{\frac{3}{2}}$)

7. **a.** $y = 10^x = (e^{\ln 10})^x = e^{(\ln 10)x}$

b. $7.5^{-x} = (e^{\log_e 7.5})^{-x} = e^{-(\log_e 7.5)x}$

c. $4^{-x} = (e^{\log_e 4})^{-x} = e^{-(\log_e 4)x}$

d. $(\frac{1}{4})^x = (e^{\log_e \frac{1}{4}})^x = e^{-(\log_e 4)x}$

8. $\log_{10} 10^{-19} = -19$

9. $\log_e e\sqrt[5]{e} = \log_e e^{\frac{6}{5}} = \frac{6}{5}$

10. $\log_2 16 = \log_2 2^4 = 4$

11. $\log_{17} \frac{17^3}{\sqrt{17}} = \log_{17} 17^{3-\frac{1}{2}} = \frac{5}{2}$

12. $\ln \frac{e^2}{e^{21}} = \ln e^{2-21} = -19$

13. $\frac{\ln e^7}{\log_{11} 121} = \frac{7}{\log_{11} 11^2} = \frac{7}{2}$

14. $5^{\log_5 32.7} = 32.7$

15. $e^{\ln \frac{9}{2}} = \frac{9}{2}$

16. $e^{\ln \sqrt[3]{27}} = \sqrt[3]{27} = 3$

17. $\log_{10} \frac{100x^2}{9y} = \log_{10} 100 + \log_{10} x^2 - \log_{10} 9y = 2 + 2\log_{10} x - \log_{10} 9 - \log_{10} y$

18. $\ln(\frac{xy^{-3}}{e^{1.37}}) = \ln x - 3\ln y - \ln e^{1.37} = \ln x - 3\ln y - 1.37$

19. $\log_4(\frac{4^{-1.3}z^7}{x^2y^3}) = -1.3 + 7\log_4 z - 2\log_4 x - 3\log_4 y$

20. $\log_3 \frac{x^3y^2}{27z^{\frac{1}{2}}} = 3\log_3 x + 2\log_3 y - \log_3 27 - \frac{1}{2}\log_3 z = 3\log_3 x + 2\log_3 y - 3 - \frac{1}{2}\log_3 z$

21. $\ln(e^{-2.4}x^6) = -2.4 + 6\ln x$

22. $\log_5 \frac{125x^3}{0.2y^2} = \log_5 5^3 + 3\log_5 x - \log_5 \frac{1}{5} - 2\log_5 y = 4 + 3\log_5 x - 2\log_5 y$

23. $3\log_2 x + \log_2 30 + \log_2 y - \log_2 w = \log_2 \frac{30x^3y}{w}$

24. $2\ln x - \ln y + a\ln w = \ln x^2 - \ln y + \ln w^a = \ln \frac{x^2w^a}{y}$

25. $12(\ln x + \ln y) = \ln(xy)^{12}$

26. $\log_3 e \times \ln 81 + \log_3 5 \times \log_5 w = \log_3 81 + \log_3 w = 4 + \log_3 w$

27. $\log_7 10 \times \log_{10} x^2 - \log_7 49x = \log_7 x^2 - \log_7 49 - \log_7 x = -2 + \log_7 \frac{x^2}{x} = -2 + \log_7 x$

28. $\log_{10} 0.1 \times \log_6 x - 2\log_6 y + \log_6 4 \times \log_4 e = -1 \times \log_6 x - \log_6 y^2 + \log_6 e = \log_6 \frac{e}{xy^2}$

29. $\log_5 e = \frac{1}{\log_e 5} \approx \frac{1}{1.6094} \approx 0.6213$

30. $\log_5 7 = \log_5 e \times \log_e 7 = \frac{\log_e 7}{\log_e 5} \approx \frac{1.9459}{1.6094} \approx 1.2091$

31. $\log_5 7^2 = \log_5 e \times 2\log_e 7 = \frac{2\log_e 7}{\log_e 5} \approx 2 \times \frac{1.9459}{1.6094} \approx 2.4182$

32. $\log_{49} 5 = \log_{49} e \times \log_e 5 = \frac{\log_e 5}{\log_e 49} = \frac{\log_e 5}{\log_e 7^2} \approx \frac{1.6094}{2\times 0.9459} \approx 0.4135$

33. $\log_{49} 25 = \log_{49} e \times \log_e 5^2 = \frac{\log_e 5^2}{\log_e 7^2} \approx \frac{2\times 1.6094}{2\times 1.9459} \approx 0.8271$

34. $\log_e 25 = \log_e 5^2 \approx 2 \times \log_e 5 \approx 2 \times 1.6094 \approx 3.2188$