Mathematics Learning Centre

The University of Sydney

Mathematical Induction

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Contents

1 Mathematical Induction

Mathematical Induction is a powerful and elegant technique for proving certain types of mathematical statements: general propositions which assert that something is true for all positive integers or for all positive integers from some point on.

Let us look at some examples of the type of result that can be proved by induction.

Proposition 1. The sum of the first n positive integers $(1, 2, 3, \ldots)$ is $\frac{1}{2}n(n + 1)$.

Proposition 2. In a convex polygon with n vertices, the greatest number of diagonal that can be drawn is $\frac{1}{2}n(n - 3)$.

Note, we give an example of a convex polygon together with one that is not convex in Figure 1.

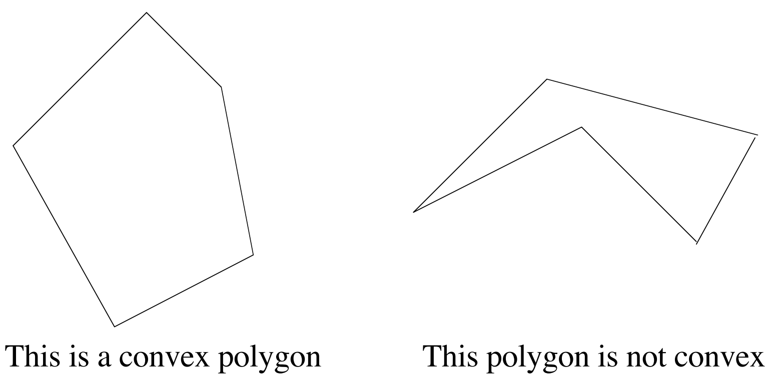

Figure 1: Examples of polygons

A polygon is said to be convex if any line joining two vertices lies within the polygon or on its boundary.

By a diagonal, we mean a line joining any two non-adjacent vertices.

As you see, the subject matter of the statements can vary widely. It can include algebra, geometry and many other topics. What is common to all the examples is the number n that appears in the statement. In all cases it is either stated, or implicitly assumed, that n can be any positive integer.

1.1 Why do we need proof by induction?

A natural starting point for proving many mathematical results is to look at a few simple cases. This helps us understand what is being claimed and may even give us some pointers for finding a proof.

Let's do this with Proposition 1. The results are recorded in the following table.

n	1	2	3	4
sum of first n numbers	1	$1+2=3$	$1+2+3=6$	$1+2+3+4=10$
$\frac{1}{2}n(n+1)$	$\frac{1}{2}\times 1\times 2=1$	$\frac{1}{2}\times 2\times 3=3$	$\frac{1}{2}\times 3\times 4=6$	$\frac{1}{2}\times 4\times 5=10$

You may find this quite convincing and wonder whether any further proof is needed. If a statement is true for all numbers we have tested, can we conclude that it is true for all values of n?

To answer this, let's look at another example.

Proposition 3. If p is any prime number, 2^p-1 is also a prime. Let us try some special cases here too.

p	2	3	5	7
2^p-1	3	7	31	127

Since 3, 7, 31, 127 are all primes, we may be satisfied the result is always true. But if we try the next prime, 11, we find that

$$2^{11}-1=2047=23\times 89.$$

So it is not a prime, and our general assertion is therefore FALSE.

Let us turn again to Proposition 1, and ask how many cases we would need to check, before we could say for certain that it is true. Imagine getting a computer on the job and setting it the task. No matter how many values of n we found the proposition to be true for, we could never be sure that there was not an ever bigger value for which it was false. This explains the need for a general proof which covers all values of n. Mathematical induction is one way of doing this.

1.2 What is proof by induction?

One way of thinking about mathematical induction is to regard the statement we are trying to prove as not *one* proposition, but a whole sequence of propositions, one for each n. The trick used in mathematical induction is to prove the first statement in the sequence, and then prove that if any particular statement is true, then the one after it is also true. This enables us to conclude that *all* the statements are true.

Let's state these two steps in more formal language.

The initial step

Prove the proposition is true for $n=1$.

(Or, if the assertion is that the proposition is true for $n\geq a$, prove it for $n=a$.)

Inductive step

Prove that *if* the proposition is true for $n = k$, then it *must* also be true for $n = k + 1$.

This step is the difficult part, and it may help you if we break it up into several stages.

Stage 1 Write down what the proposition asserts for the case $n = k$. This is what you are going to assume. It is often called the *inductive hypothesis*.

Stage 2 Write down what the proposition asserts for the case $n = k + 1$. This is what you have to prove. Keep this clearly in mind as you go.

Stage 3 Prove the statement in Stage 2, using the assumption in Stage 1. We can't give you any recipe for how to do this. It varies from problem to problem, depending on the mathematical content. You have to use your ingenuity, common sense and knowledge of mathematics here. The question to ask is "how can I get from Stage 1 to Stage 2?"

Once the initial and the inductive step have been carried out, we can conclude immediately that the proposition is true for all $n \geq 1$ (or for all $n \geq a$ if we started at $n = a$.)

To explain this, it may help to think of mathematical induction as an authomatic "statement proving" machine.

We have proved the proposition for $n = 1$.

By the inductive step, since it is true for $n = 1$, it is also true for $n = 2$. Again, by the inductive step, since it is true for $n = 2$, it is also true for $n = 3$. And since it is true for $n = 3$, it is also true for $n = 4$, and so on.

Because we have proved the inductive step, this process will never come to an end. We could set the "machine" running, and it would keep going forever, eventually reaching any n, no matter how big it may be.

Suppose there was a number N for which the statement was false. Then when we get to the number $N - 1$, we would have the following situation:

The statement is true for $n = N - 1$, but false for $n = N$.

This contradicts the inductive step, so it cannot possible happen. Hence the statement must be true for *all* positive integers n.

If you are familiar with computer programming, it may be helpful for you to compare this argument with a looping process, in which a computation is carried out, an indexing variable is advanced by one, and the computation is repeated.

The two processes have much in common. In a computer program you must begin by setting the initial value of your variables (this is analogous to our initial step). Then you must set up the loop, calling on the previous values of your variables to calculate new values (this is analogous to our inductive step).

There is one other thing necessary in a computer program: you must set up a "stop" condition otherwise your program will run forever. That has no analogy in our process – our theoretical machine *will* run forever! That is why we can be certain our result is true for all positive integers.

Now let's see how this works in practice, by proving Proposition 1.

Proposition 1. The sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

Initial step: If $n = 1$, the sum is simply 1.

Now, for $n = 1$, $\frac{1}{2}n(n+1) = \frac{1}{2} \times 1 \times 2 = 1$. So the result is true for $n = 1$.

Inductive step:

Stage 1: Our assumption (the inductive hypothesis) asserts that

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1).$$

Stage 2: We want to prove that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{1}{2}(k+1)[(k+1)+1] = \frac{1}{2}(k+1)(k+2).$$

Stage 3: How can we get to stage 2 from stage 1?

The answer here is that we get the left hand side of stage 2 from the left hand side of stage 1 by adding $(k+1)$.

So,

$$\begin{aligned}
1 + 2 + 3 + \cdots + (k+1) &= 1 + 2 + 3 + \cdots + k + (k+1) \\
&= \frac{1}{2}k(k+1) + (k+1) && \text{using the inductive hypothesis} \\
&= (k+1)(\frac{1}{2}k + 1) && \text{factorising} \\
&= \frac{1}{2}(k+1)(k+2) && \text{which is what we wanted.}
\end{aligned}$$

This completes the inductive step.

Hence, the result is true for all $n \geq 1$.

As a further example, let's try proving Proposition 2.

Proposition 2. The number of diagonals of a convex polygon with n vertices is $\frac{1}{2}n(n-3)$, for $n \geq 4$.

Initial step: If $n = 4$, the polygon is a quadrilateral, which has two diagonals as shown in Figure 2.

Also, for $n = 4$, $\frac{1}{2}n(n-3) = \frac{1}{2}(4)(1) = 2$.

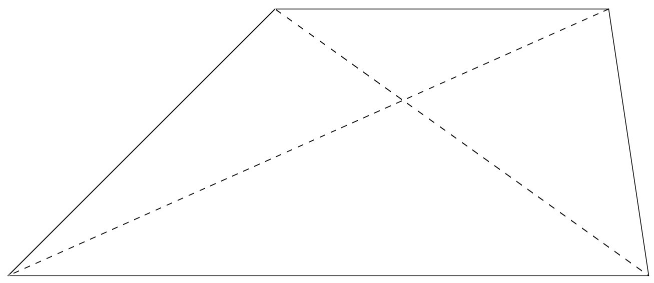

Figure 2: A quadrilateral showing 2 diagonals.

Inductive step:

Stage 1: The inductive hypothesis asserts that the number of diagonals of a polygon with k vertices is $\frac{1}{2}k(k-3)$.

Stage 2: We want to prove that the number of diagonals of a polygon with $(k+1)$ vertices is $\frac{1}{2}(k+1)[(k+1)-3] = \frac{1}{2}(k+1))(k+2)$.

Stage 3: How can we get to stage 2 from stage 1?

The answer here is to "add another vertex". Let's do this and see if we can count how many additional diagonals can be drawn as a result. Figure 3 will help here.

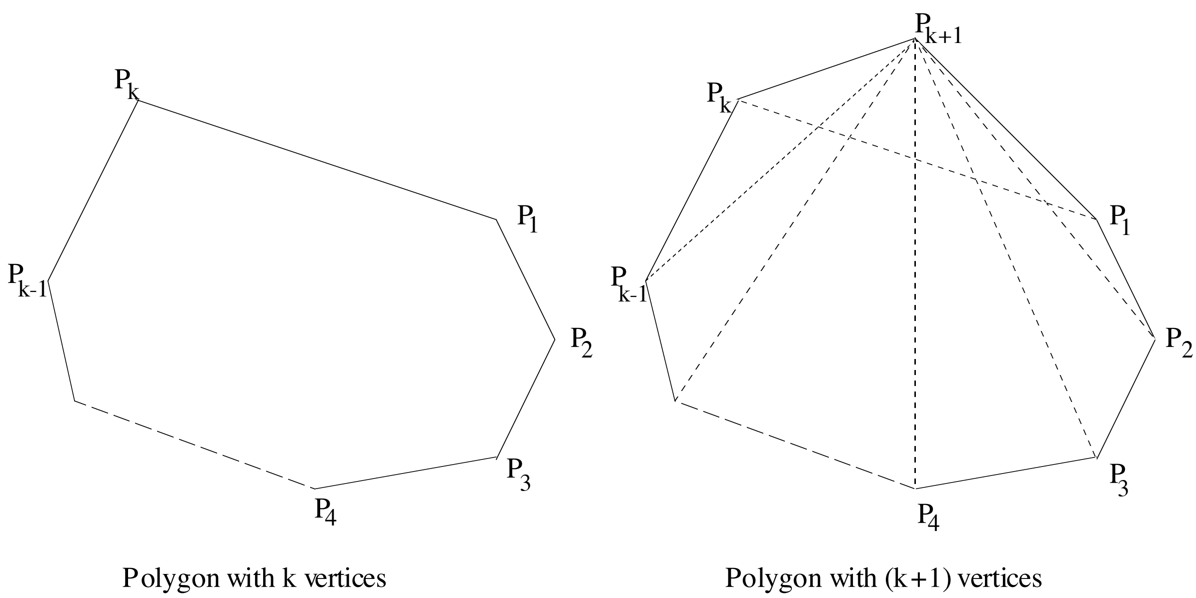

Figure 3: Adding another vertex to a polygon with k vertices.

When we add an extra vertex to a polygon with n vertices, all the lines which were diagonals of the original polygon will still be diagonals of the new one. The inductive hypothesis says that there are $\frac{1}{2}k(k-3)$ existing diagonals. In addition, new diagonals can be drawn from the extra vertex P_{k+1} to all other vertices except the two adjoining it (P_1 and P_k) giving us $k-2$ extra diagonals. Finally, the line joining P_1 and P_k, which used to be a side of the polygon, is now a diagonal.

This gives us a total of

$$\frac{1}{2}k(k-3)+(k-2)+1 \quad \text{diagonals.}$$

But,

$$\begin{aligned}\frac{1}{2}k(k-3)+(k-2)+1 &= \frac{1}{2}[k(k-3)+2k-2]\\ &= \frac{1}{2}(k^2-k-2)\\ &= \frac{1}{2}(k+1)(k+2), \quad \text{as required.}\end{aligned}$$

This completes the inductive step. Hence the result is true for all $n \geq 4$.

At this point, it would be a good idea to go back and read over the explanation of the process of mathematical induction thinking about the general explanation in the light of the two examples we have just completed.

Next, we illustrate this process again, by using mathematical induction to give a proof of an important result, which is frequently used in algebra, calculus, probability and other topics.

1.3 The Binomial Theorem

The Binomial Theorem states that if n is an integer greater than 0,

$$(x+a)^n = x^n+nx^{n-1}a+\frac{n(n-1)}{2!}x^{n-2}a^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}a^3+\cdots+nxa^{n-1}+a^n. \tag{1}$$

For example,

$$(x+a)^3 = x^3+3x^2a+3xa^2+a^3.$$

This useful theorem can be more compactly expressed as

$$(x+a)^n = \sum_{r=0}^{n} nCrx^ra^{n-r}$$

but if you are not familiar with this notation, use equation (1).

Let's prove this theorem by mathematical induction.

Initial step: Let $n=1$. Then the left hand side of (1) is $(x+a)^1$ and the right hand side of (1) is x^1+a^1 which both equal $x+a$.

Inductive step:

Step 1: Assume the theorem is true for $n = k$, ie that

$$(x+a)^k = x^k + kx^{k-1}a + \frac{k(k-1)}{2!}x^{k-2}a^2 + \cdots + kxa^{k-1} + a^k.$$

Stage 2: We want to prove that the theorem is true for $n = k + 1$, ie that

$$(x+a)^{k+1} = x^{k+1} + (k+1)x^k a + \frac{(k+1)(k)}{2!}x^{k-1}a^2 + \cdots + (k+1)xa^k + a^{k+1}. \quad (2)$$

Stage 3: How do we get to stage 2 from stage 1?

Look at the left hand side of (2),

$$(x+a)^{k+1} = (x+a)(x+a)^k.$$

but by the inductive hypothesis,

$$\begin{aligned}
(x+a)(x+a)^k &= (x+a)[x^k + kx^{k-1}a + \frac{k(k-1)}{2!}x^{k-2}a^2 + \cdots + kxa^{k-1} + a^k] \\
&= x^{k+1} + kx^k a + \frac{k(k-1)}{2!}x^{k-1}a^2 + \cdots + kx^2a^{k-1} + xa^k \\
&\qquad + x^k a + \qquad kx^{k-1}a^2 + \qquad \cdots \qquad + kxa^k + a^{k+1} \\
\text{(adding)} \qquad &= x^{k+1} + (k+1)x^k a + \frac{(k+1)(k)}{2!}x^{k-1}a^2 + \cdots + (k+1)xa^k + a^{k+1}.
\end{aligned}$$

This is the right hand side as required. Hence the result is true for all $n \geq 1$.

Now try some of the exercises for yourself.

1.4 Exercises

Prove the following statements by mathematical induction.

Note that many of these statements can also be proved by other methods, and sometimes these other proofs will be neater and simpler. However, the purpose of these exercises is to practice proof by induction, so please try to use this method, even if if you can see an easier way.

1. The n^{th} odd number is $2n - 1$.

2. The sum of the first n odd numbers is n^2.

3. **a.** For every positive integer n, $n(n+1)$ is even.

b. For every positve integer $n \geq 2$, $n^3 - n$ is a multiple of 6.

4. For all $n \geq 1$,

$$\frac{x^{n+1}-1}{x-1} = 1 + x + x^2 + \cdots + x^n \quad \text{where } x \neq 1.$$

5. For all $n \geq 1$, $(1+x)^n \geq 1 + nx$ where x is any real number greater than -1.

6. For all ≥ 1,

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2).$$

ie,

$$1(2) + 2(3) + 3(4) + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

7. For all $n \geq 3$, $n^2 \geq 2n + 1$.

8. For all $n \geq 4$, $2^n \geq n^2$.

(Hint: Use the previous exercise to help you in the inductive step.)

9. For all $n \geq 3$, the sum of the interior angles of a polygon with n vertices is $180(n-2)^\circ$.

10. The number of non-empty subsets of a set of n elements is $2^n - 1$.

1.5 Finding the formula

The problems in the exercises were stated in a form which made it clear that mathematical induction could be used to prove the results: they nearly all began with the words "for all $n \geq \ldots$". Sometimes you will have a more open-ended question to answer, such as the following.

Exercise

What is the maximum numbeer of regions into which a plane can be divided by n straight lines?

In this case, we have to analyse the problem carefully, step by step, looking for a pattern, and then use that pattern as the basis for a proof by induction. Try this yourself, before reading the solution.

Solution

First of all, if two of the lines were parallel, by moving one so that they were no longer parallel, we could increase the number of regions. This is illustrated in Figure 4.

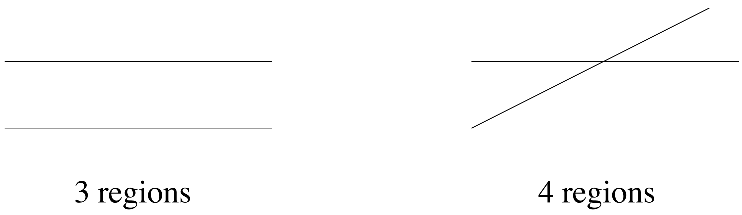

Figure 4: Dividing the plane with two lines.

Secondly, if 3 of the lines were concurrent, we could increase the number of regions by moving one so that they were no longer concurrent. This is illustrated in Figure 5.

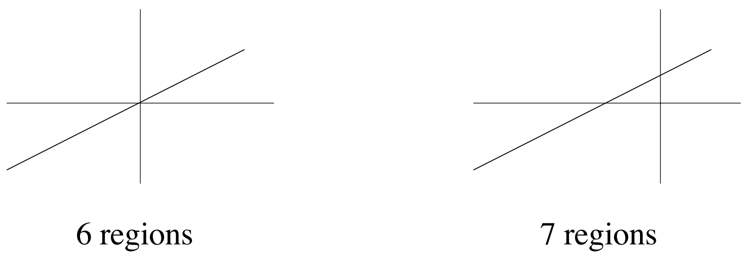

Figure 5: Dividing the plane with three lines.

Hence, we conclude that, to get the *maximum* number of regions, none of the lines can be parallel, and no 3 can be concurrent. We can now work out the first few cases which are illustrated in Figure 6.

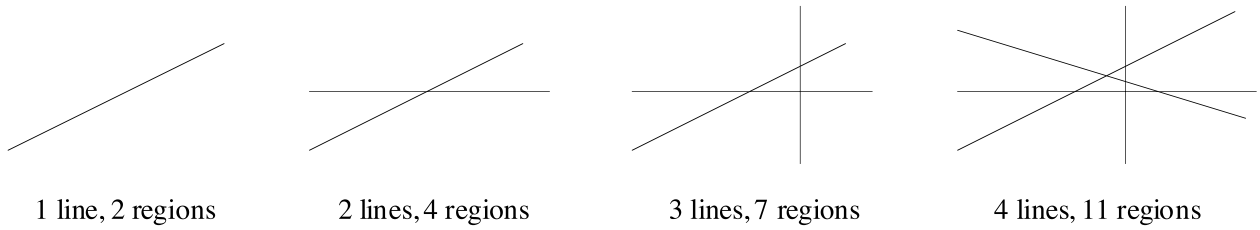

Figure 6: Dividing the plane with one to four lines.

There is no obvious pattern in this, so we need to analyse more carefully what is going on. What happens when we add an extra line?

When we added the third line, it cut each of the other lines at a distinct point. The new line thus passed through three of the existing regions, dividing each in two and creating 3 new regions.

When we added the fourth line, it cut each of the others at a distinct point, and passed through 4 of the existing regions, creating 4 new regions.

And so on ....

When we add an n^{th} line, it will cut through each of the existing $(n-1)$ lines at a distinct point, and will pass through n of the existing regions creating n additional regions.

We can put this information in a table.

Number of lines	1	2	3	4	$\cdots$	n
Number of regions	2	$2+2$	$2+2+3$	$2+2+3+4$		$2+2+3+4+\cdots+n$

From this, we can write down an answer.

$$\begin{aligned}\text{Maximum no. of regions formed by } n \text{ lines} &= 2+2+3+4+\cdots+n \\ &= 1+(1+2+3+4+\cdots+n) \\ &= 1+\frac{1}{2}n(n+1) \qquad \text{by Proposition 1} \\ &= \frac{1}{2}[2+n(n+1)] \\ &= \frac{1}{2}(n^2+n+2).\end{aligned}$$

Not only have we obtained an answer to the question, we also have worked out how to progress from one step to the next, and so we have the basis for a formal proof by induction.

1.6 Further exercises

1. Write out a formal proof by mathematical induction that the maximum number of regions the plane is divided into by n lines is $\frac{1}{2}(n^2+n+2)$.

2. Each of n famous scientists who meet at a conference (where $n \geq 2$) wants to shake hands with all the others. Work out how many handshakes there will be and prove your result by mathematical induction.

2 Solutions to Exercises

Solutions to exercises

Remember there may be may different ways of presenting a solution. The working given is just one way.

1. The n^{th} odd number is $2n-1$.

Initial step: The first odd number is 1 which is $2(1)-1$ so the assertion is true for $n=1$.

Inductive step:

Stage 1: The inductive hypothesis asserts that the n^{th} odd number is $2n-1$.

Stage 2: We want to prove that the $(n+1)^{th}$ odd number is $2(n+1)-1$, ie $2n+1$.

Stage 3: How can we get to Stage 2 from Stage 1?

By the inductive hypothesis, the n^{th} odd number is $2n-1$. The next odd number is 2 more than this, ie $2n-1+2=2n+1$ as required.

Thus the result is true for all $n \geq 1$.

2. The sum of the first n odd numbers is n^2.

Initial step: For $n=1$, the sum is 1 which is 1^2.

Inductive step:

Stage 1: The inductive hypothesis asserts that

$$1+3+5+\cdots+(2n-1)=n^2.$$

(Note, we are using the result of question 1 here.)

Stage 2: We want to prove that the sum of the first $(n+1)$ odd numbers is $(n+1)^2$.

Stage 3: Getting to stage 2 from stage 1.

The sum of the first $(n+1)$ odd numbers is equal to the sum of the first n odd numbers plus the next odd number, namely, $1+3+5+\cdots+(2n-1)+(2n+1)$, as each odd number is two more than the previous one. So,

$$\begin{aligned} 1+3+5+\cdots+(2n-1)+(2n+1) &= n^2+(2n+1) && \text{the inductive hypothesis} \\ &= n^2+2n+1 \\ &= (n+1)^2 && \text{as required.} \end{aligned}$$

The result is therefore true for all $n \geq 1$.

3. a. For every positive integer n, $n(n+1)$ is even (ie is divisible by 2).

Initial step: Let $n = 1$. Then $n(n+1) = 2$, which is even, so the proposition is true for $n = 1$.

Inductive Step:
Assume that for $n = k$, $k(k+1)$ is even. If a number is even, it is a multiple of 2. So, we can assume that $k(k+1) = k^2 + k = 2m$ for some integer m.

We want to prove that the theorem is true for $n = k+1$, ie that $(k+1)[(k+1)+1]$ is even.

Now

$$\begin{aligned} (k+1)[(k+1)+1] &= (k+1)(k+2) \\ &= k^2 + k + 2k + 2 \\ &= 2m + 2k + 2 && \text{using the inductive hypothesis} \\ &= 2(m+k+1) && \text{which is even as required.} \end{aligned}$$

Hence the result is true for all $n \geq 1$.

b. For every positive integer $n \geq 2$, $n^3 - n$ is a multiple of 6.

Initial step: For $n = 2$, $n^3 - n = 2^3 - 2 = 6$ which is a multiple of 6 (6×1).

Inductive step:
The inductive hypothesis asserts that $n^3 - n$ is a multiple of 6, so let $n^3 - n = 6m$ for some integer m. We want to prove that $(n+1)^3 - (n+1)$ is a multiple of 6.

$$\begin{aligned} (n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= (n^3 - n) + (3n^2 + 3n) && \text{rearranging the previous line} \\ &= 6m + 3n(n+1) && \text{using the inductive hypothesis} \\ &= 6m + 3 \times 2l && \text{we proved } n(n+1) \text{ even in 3(a)} \\ &= 6(m + l) && \text{which is divisible by 6.} \end{aligned}$$

The result is therefore true for all $n \geq 2$.

4. For all $n \geq 1$,

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \cdots + x^n \quad \text{where } x \neq 1. \tag{1}$$

(The reason why x must not equal 1 is to avoid dividing by zero and has nothing to do with the inductive process.)

Initial step: For $n = 1$, the left hand side of equation (1) is

$$\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x - 1} = x + 1, \qquad x \neq 1.$$

The right hand side of (1) is $1 + x^1$, so the proposition is true for $n = 1$.

Inductive step:

We assume the proposition is true for $n = k$, ie

$$\frac{x^{k+1}-1}{x-1} = 1 + x + x^2 + \cdots + x^k \qquad x \neq 1.$$

We want to show that the proposition is true for $n = k + 1$. That is,

$$\frac{x^{k+2}-1}{x-1} = 1 + x + x^2 + \cdots + x^k + x^{k+1} \qquad x \neq 1.$$

Let us look at the right hand side of the equation first.

$$\begin{aligned} 1 + x + x^2 + \cdots + x^k + x^{k+1} &= \frac{x^{k+1}-1}{x-1} + x^{k+1} && \text{using inductive hypothesis} \\ &= \frac{x^{k+1}-1+x^{k+1}(x-1)}{x-1} \\ &= \frac{x^{k+1}-1+x^{k+2}-x^{k+1}}{x-1} \\ &= \frac{x^{k+2}-1}{x-1} && \text{the left hand side as required.} \end{aligned}$$

The proposition is thus true for all $n \geq 1$.

5. For all $n \geq 1$, $(1 + x)^n \geq 1 + nx$, where x is any real number greater than -1. Note this means that $(1 + x) > 0$.

Initial step: For $n = 1$, $(1+x)^1 = 1+x$ and $1+nx = 1+x$, so $(1+x)^n = 1+nx \geq 1+nx$ is true for $n = 1$.

Inductive step:

We assume that the proposition is true for $n = k$. That is, $(1 + x)^k \geq 1 + kx$, when $x \geq -1$.

We want to prove that $(1 + x)^{k+1} \geq 1 + (k + 1)x$. (1)

Let us look at the inequality and try to relate it to the inductive hypothesis.

Now the left hand side of (1) is

$$\begin{aligned} (1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) && \text{using the inductive hypothesis and } (1+x) > 0 \\ &= 1 + kx + x + kx^2 \\ &\geq 1 + kx + x && \text{since } kx^2 \geq 0 \\ &= 1 + (k+1)x. \end{aligned}$$

Hence $(1 + x)^{k+1} \geq 1 + (k + 1)x$, as required.

So the result is true for all $n \geq 1$.

6. For all $n \geq 1$,

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2).$$

That is,

$$1(2) + 2(3) + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

Initial step: For $n = 1$, the left hand side is $1(2) = 2$ and the right hand side is $\frac{1}{3}1(2)(3) = 2$, so the assertion is true for $n = 1$.

Inductive step:

Assume that

$$1(2) + 2(3) + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2).$$

We want to prove that

$$1(2) + 2(3) + \cdots + k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3).$$

Starting with the left hand side,

$$\begin{aligned} 1(2) + 2(3) + \cdots + k(k+1) + (k+1)(k+2) &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= (k+1)(k+2)[\frac{1}{3}k+1] \\ &= \frac{1}{3}(k+1)(k+2)(k+3) \qquad \text{as required.} \end{aligned}$$

Thus the proposition is true for all $n \geq 1$.

7. For all $n \geq 3$, $n^2 \geq 2n+1$.

Initial step: For $n = 3$, $n^2 = 9$, $2n+1 = 7$, so $n^2 \geq 2n+1$ for $n = 3$.

Inductive step:

Assume $k^2 \geq 2k+1$.

We want to prove that $(k+1)^2 \geq 2(k+1)+1$.

Now, starting with the left hand side

$$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 \\ &\geq (2k+1) + 2k + 1 \qquad \text{using inductive hypothesis} \\ &= 2(k+1) + 2k \\ &\geq 2(k+1) + 1 \qquad \text{since } 2k \geq 1 \text{ for } k \geq 1. \end{aligned}$$

This is the right hand side.

Thus the result is true for all $n \geq 3$.

8. For all $n \geq 4$, $2^n \geq n^2$.

Initial step: Let $n = 4$. Then $2^n = 16$ and $n^2 = 16$ so the proposition is true for $n = 4$.

Inductive step:

Assume that $2^k \geq k^2$.

We want to prove that $2^{k+1} \geq (k+1)^2$.

Now,

$$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &\geq 2 \times k^2 && \text{using inductive hypothesis} \\ &= k^2 + k^2 \\ &\geq k^2 + 2k + 1 && k^2 \geq 2k+1 \text{ for all } k \geq 3 \text{ from Question 7} \\ &= (k+1)^2 \end{aligned}$$

Thus the proposition is true for all $n \geq 4$.

9. For $n \geq 3$, the sum of the interior angles of a polygon with n vertices is $180(n-2)^\circ$.

Initial step: For $n = 3$, the polygon is a triangle and the sum of the angles of a triangle is 180° which is $180(3-2)^\circ$, as required.

Inductive step:

Assume that the sum of the interior angles of a polygon with k vertices is $180(k-2)^\circ$.

We want to prove that if a polygon has $k+1$ vertices, the sum of its interior angles is $180(k-1)^\circ$.

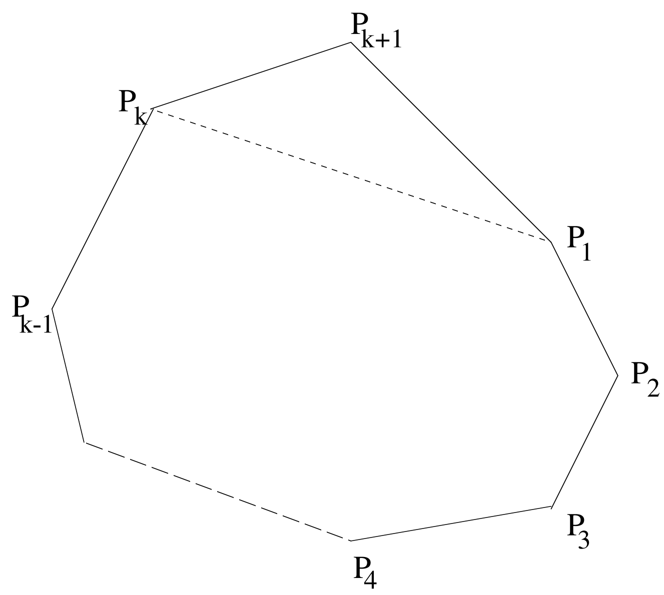

Figure 7: A polygon with $k+1$ vertices.

Let us join the first and k^{th} vertex in the polygon with $k+1$ vertices shown in Figure 7. This divides the polygon into a triangle ($\triangle P_1 P_k P_{k+1}$) and a polygon with k vertices.

The sum of the interior angles of the polygon so formed is $180(k-2)^\circ$, by our inductive hypothesis, and the sum of the angles of the triangle is 180°.

Thus the sum of the interior angles of a polygon with $k+1$ vertices is

$180(k-2)^\circ + 180^\circ = 180(k-1)^\circ$ as required.

Thus the proposition is true for all $n \geq 3$.

10. The number of non-empty subsets of a set of n elements is $2^n - 1$, for any positive integer n.

Initial step: If a set has one element in it then the only non-empty subset is the set itself. Thus there is one non-empty subset and $2^1 - 1 = 1$, so the assertion is true for $n = 1$.

Inductive step:

Assume that if a set has k elements, then it has $2^k - 1$ non-empty subsets.

We want to prove that if a set has $k+1$ elements, it has $2^{k+1} - 1$ non-empty subsets.

Let us suppose the set has k elements in it plus a new element we will denote by p as shown in Figure 8.

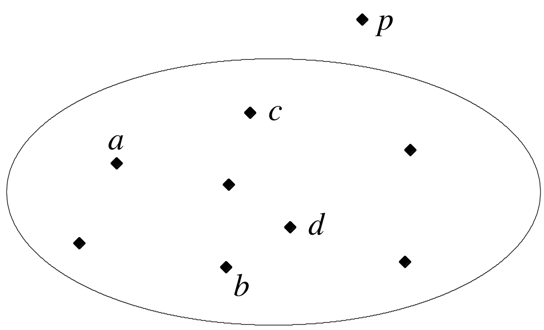

Figure 8: A set with k elements and a new element p.

Now the number of non-empty subsets that can be formed that do not contain p is $2^k - 1$ by the inductive hypothesis.

We can adjoin the element p to each of these subsets to form $2^k - 1$ new subsets. Finally there is one subset whose only element is p.

Hence the total number of subsets is

$$\begin{aligned}(2^k - 1) + (2^k - 1) + 1 &= 2(2^k) - 1 \\ &= 2^{k+1} - 1 \qquad \text{as required.}\end{aligned}$$

Thus the proposition is true for all $n \geq 1$.

Short solutions or hints to further exercises

1. We want to prove that the maximum number of regions a plane can be subdivided into by n lines is $\frac{1}{2}(n^2 + n + 2)$.

Initial step: If $n = 1$, the plane is divided into 2 regions and $\frac{1}{2}(1^2 + 1 + 2) = 2$, so the assertion is true for $n = 1$.

Inductive step:

Now suppose we have n lines and $\frac{1}{2}(n^2 + n + 2)$ regions. (This is the inductive hypothesis.)

Add one more line, which meets the existing lines at n distinct points. These n points subdivide the new line into $n + 1$ segments. Each segment divides one of the existing regions into two, so the number of regions is increased by $n + 1$.

Now check that

$$\frac{1}{2}(n^2 + n + 2) + (n + 1) = \frac{1}{2}[(n + 1)^2 + (n + 1) + 2].$$

This completes the proof.

2. (Hint) If there are only 2 scientists, there will be only one handshake. If one more arrives, he or she will shake hands with both of the others, making 3 handshakes in all.

Now imagine there are n people in the room, and one more arrives and shakes hands with all the others. There will be n additional handshakes.