2 Unit Bridging Course - Day 11

Inverse Functions

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Definition

Consider the function $f(x) = 2x$, whose rule is to simply *double* any input. For instance:

$$\xrightarrow{3} \boxed{\quad f \quad} \xrightarrow{6}$$

The **inverse function** of $f(x)$, denoted $f^{-1}(x)$, 'undoes' f by directing the outputs of f back to their respective inputs.

$$\xrightarrow{6} \boxed{\quad f^{-1} \quad} \xrightarrow{3}$$

Hence $f^{-1}(x) = \frac{1}{2}x$, since the inverse operation of doubling is *halving*.

[**Important:** $f^{-1}(x)$ is **not** the same as $\{f(x)\}^{-1} = \dfrac{1}{f(x)}$.]

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Cancellation Property

But just as how the inverse operation of halving ‘undoes’ or ‘cancels’ out the act of doubling, doubling also undoes or cancels the act of halving.

Hence f and f^{-1} undo or cancel each other and are therefore **mutually inverse functions** of each other.

The cancellation property of inverses can be stated as follows:

Cancelling Property of Inverses

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

That is, f and f^{-1} applied in succession renders the input x unchanged.

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Cancellation Property (cont.)

Example

For $f(x) = 2x$ and $f^{-1}(x) = \frac{1}{2}x$, we have:

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x,$$

i.e. $\xrightarrow{x}$ [doubling] $\xrightarrow{2x}$ [halving] $\xrightarrow{x}$

Also:

$$f(f^{-1}(x)) = f(\frac{1}{2}x) = 2(\frac{1}{2}x) = x,$$

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A Caveat

Caution: Inverse functions don't always exist!

For instance, consider the function

$$y = x^2.$$

Since 3^2 and $(-3)^2$ both equal 9, the output $y = 9$ can be traced back to *two* possible inputs: $x = 3$ and $x = -3$.

But outputs for functions must be *unique*, so would the inverse function of $y = x^2$ direct 9 back to 3 or -3?

This is ambiguous, and hence we say that there does not exist an inverse function for $y = x^2$.

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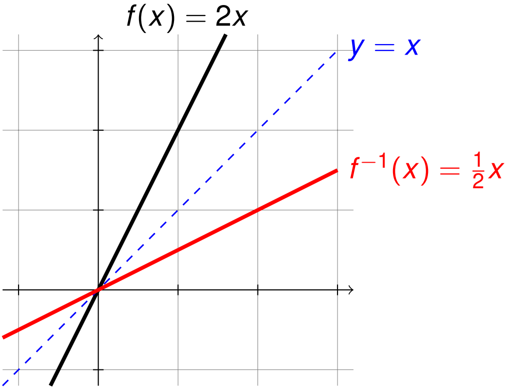
Graphs of Inverse Functions

Important fact: For any function f whose inverse f^{-1} exists, their graphs are symmetric about the diagonal line $y = x$:



Examples

Let's look at a few more examples of inverse functions.

Example

Consider the function

$$f(x) = 3x,$$

where outputs are obtained by multiplying inputs by 3.

Since inputs are recovered through the inverse operation of *division* by 3, the inverse function of f is given by

$$f^{-1}(x) = \frac{1}{3}x.$$

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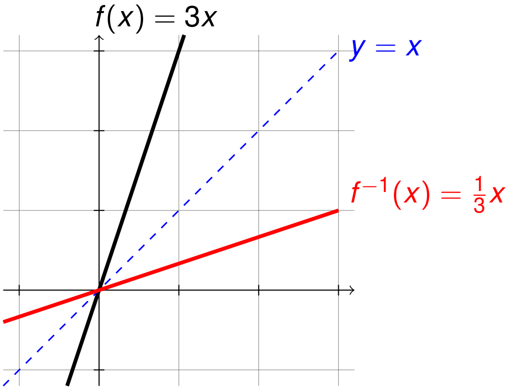
Examples (cont.)

Here are the graphs of $f(x) = 3x$ and $f^{-1}(x) = \frac{1}{3}x$ plotted together. Observe the symmetry of the two graphs about the line $y = x$.



Example

Here's one more example. Consider the function

$$f(x) = x^3,$$

where outputs are obtained by cubing inputs.

Since inputs are recovered through the inverse operation of *cube-rooting*, the inverse function of f is given by

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Examples (cont.)

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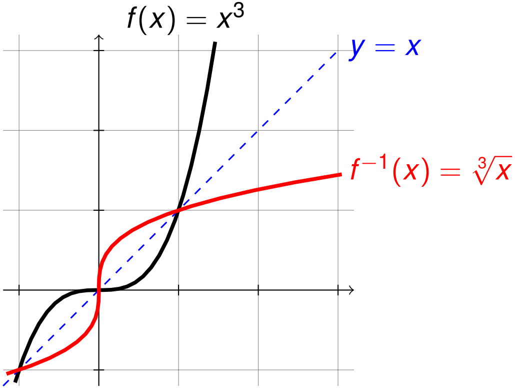
Examples (cont.)

Here are the graphs of $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$. Once again, observe the symmetry between f and f^{-1} about $y = x$.



Obtaining Inverse Functions

For a general function f, how does one obtain its inverse function f^{-1}? There are two main steps:

Step 1: Since f^{-1} recovers inputs from outputs, we first solve the equation of the function for x.

For instance, given

$$y = 3x + 1,$$

we obtain

$$x = \frac{y-1}{3}.$$

Step 2: Finally, we interchange x and y. Hence f^{-1} is given by:

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Obtaining Inverse Functions (cont.)

Practice Questions

Find the inverse functions of the following:

- $f(x) = 6x$
- $f(x) = 4x - 1$
- $f(x) = x^5$.

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Obtaining Inverse Functions (cont.)

Answers

- $f^{-1}(x) = \dfrac{x}{6}$
- $f^{-1}(x) = \dfrac{x+1}{4}$
- $f^{-1}(x) = \sqrt[5]{x}.$

Summary

- Given a function f, its inverse function f^{-1}, if it exists, *undoes* or *cancels* the operation performed by f.
- f and f^{-1} are *mutually inverse functions*.
- The Cancellation Property holds for f and f^{-1}, where $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x.
- The graphs of two mutually inverse functions are symmetric about the diagonal line $y = x$.
- The inverse function for $y = f(x)$ is obtained by solving for x and then interchanging x and y.