REGULARISATION EFFECTS OF NONLINEAR SEMIGROUPS

THIERRY COULHON AND DANIEL HAUER

ABSTRACT. One introduces natural and simple methods to deduce L^s-L^∞-regularisation estimates for $1 \leq s < \infty$ of nonlinear semigroups holding uniformly for all time with sharp exponents from natural Gagliardo-Nirenberg inequalities. From L^q-L^r Gagliardo-Nirenberg inequalities, $1 \leq q, r \leq \infty$, one deduces L^q-L^r estimates for the semigroup. New nonlinear interpolation techniques of independent interest are introduced in order to extrapolate such estimates to $L^{\tilde{q}}$-L^∞ estimates for some $\tilde{q}$, $1 \leq \tilde{q} < \infty$. Finally one is able to extrapolate to L^s-L^∞ estimates for $1 \leq s < q$. The theory developed in this monograph allows to work with minimal regularity assumptions on solutions of nonlinear parabolic boundary value problems as illustrated in a plethora of examples including nonlocal diffusion processes.

CONTENTS

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1. Introduction

1.1. **The story.** It begins in the linear semigroup theory: let $\{T_t\}_{t\geq 0}$ be a symmetric semigroup with infinitesimal generator $-A$ of linear operators acting on $L^2(\Sigma,\mu)$, where (Σ,μ) is a σ-finite measure space. Assume that $\{T_t\}_{t\geq 0}$ is submarkovian, meaning that $0\leq u\leq 1$ implies $0\leq T_t u\leq 1$ for all $t>0$. If follows that $\{T_t\}_{t\geq 0}$ acts on $L^q(\Sigma,\mu)$ for all $1\leq q\leq\infty$.

In this framework, there has been many works in the last four decades that connect a variety of L^q-L^r, $1\leq q<r\leq\infty$, regularisation properties of $\{T_t\}_{t\geq 0}$ to a variety of abstract Sobolev type inequalities involving A. The first regularisation property of $\{T_t\}_{t\geq 0}$ that attracted much attention was the so-called *hypercontractivity*: for some (all) $1<q<r<\infty$, there exists $t_0=t_0(q,r)>0$ such that T_{t_0} maps L^q to L^r and

$$\|T_{t_0}\|_{q\to r}\leq 1 \tag{1.1}$$

where $\|T\|_{q\to r}:=\sup_{\|u\|_q\leq 1}\|Tu\|_r$ denotes the operator norm of a linear bounded operator $T:L^q\to L^r$, $1\leq q,r\leq\infty$. The theory of hypercontractive semigroups was introduced by Nelson in [73], who also provided the most basic example: for the harmonic oscillator $A=-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2-\frac{1}{2}$ on L^2 equipped with the Gaussian measure $d\mu=(2\pi)^{-1/2}\exp(-\frac{1}{2}x^2)\,dx$ on $\mathbb{R}$, T_t is a linear contraction from L^2 to L^4 if $e^{-t}\leq 1/\sqrt{3}$ (cf. [73, 74]). One reason for the popularity of hypercontractivity was its deep connection to constructive quantum field theory. The ideas in [73] were followed up rapidly and further developed. For instance, Simon and Høegh-Krohn [83] combined the property that the considered semigroup $\|T_t\|_{q\to r}$ is contractive on L^q for all $1\leq q\leq\infty$ with Riesz-Thorin's and Stein's interpolation techniques to *extrapolate* the L^q-L^r-regularisation estimate (1.1) to an $L^{\tilde q}$-$L^{\tilde r}$-regularisation estimate $\|T_{t_0}\|_{\tilde q\to\tilde r}\leq\tilde C$ for some $\tilde q,\tilde r$ such that $1\leq\tilde q<q$ and $r<\tilde r<\infty$. Hypercontractivity is a natural property of some infinite-dimensional semigroups such as Ornstein-Uhlenbeck. Note that Nelson [74] proved that the Ornstein-Uhlenbeck semigroup does not admit an L^q-L^∞-regularisation effect for $1\leq q<\infty$.

In the mid 70's, Gross [52] characterised hypercontractivity in terms of a *single logarithmic Sobolev inequality in* L^2. Then $\{T_t\}_{t\geq 0}$ is hypercontractive if and only if there is some $C>0$ such that

$$\int_\Sigma |u|^2\,\log|u|\,\mathrm{d}\mu\leq C\,\langle Au,u\rangle+\|u\|_2^2\log\|u\|_2 \tag{1.2}$$

for every $u\in D(A)$. Here, $D(A)$ denotes the domain of A in $L^2(\Sigma,\mu)$.

In the 80's, in the context of heat kernels on Lie groups and manifolds, the focus shifted towards a stronger property, namely *ultracontractivity*: for all $t>0$,

T_t maps L^1 to L^∞. The game is then to estimate $\|T_t\|_{1\to\infty}$ from above by an explicit function of t. Of particular interest is the estimate

$$\|T_t\|_{1\to\infty} \le C\, t^{-d/2}, \quad \text{for every } t > 0, \tag{1.3}$$

where $d > 0$ plays the role of a dimension. Davis and Simon [44] (see also [43]) adapted Gross' approach [52] to the ultracontractivity framework and established the equivalence of estimates (1.3) with the following one-parameter family of logarithmic Sobolev inequalities: for every $\varepsilon > 0$,

$$\int_\Sigma |u|^2 \,\log|u| \,\mathrm{d}\mu \le \varepsilon\, \langle Au, u\rangle + \varepsilon^{-d/4}\|u\|_2^2 + \|u\|_2^2 \log\|u\|_2 \tag{1.4}$$

for every $u \in D(A)$. Estimate (1.3) was also characterised in terms of *(d-dimensional) Sobolev inequalities*: if there exists $C > 0$ such that

$$\|u\|^2_{\frac{2d}{d-2}} \le C\,\langle Au, u\rangle \qquad \text{for every } u \in D(A) \tag{1.5}$$

by Varopoulos [89] (see also [35], [36] for simplifications and further developments) and in terms of (d-dimensional) Nash inequalities by Carlen-Kusuoka-Stroock [28]). Further, an intermediate property called *supercontractivity* was also considered ([82]): for all $1 < q < r < \infty$ and all $t > 0$, T_t maps L^q to L^r with the polynomial estimate

$$\|T_t\|_{q\to r} \le C\, t^{-d\left(\frac{1}{q}-\frac{1}{r}\right)} \quad \text{for all } t > 0. \tag{1.6}$$

Note that if the semigroup is uniformly bounded on L^1 and L^∞, then (1.6) implies (1.3) (see [35]).

The above outlined development of characterising L^q-L^r-regularisation estimates of the semigroup $\{T_t\}_{t\ge 0}$ with abstract (logarithmic) Sobolev inequalities is exclusively concerned with *linear semigroups*. Thus, it is interestingly enough that prior to Varopoulos' theorem [89], the fact that an abstract Sobolev type inequality associated with an operator A implies an L^1-L^∞regularisation effect for the semigroup $\{T_t\}_{t\ge 0}$ generated by $-A$ had been discovered in the late 70's by Bénilan ([11], see also [90, p. 25]) in the context of *nonlinear semigroups*: let $\{T_t\}_{t\ge 0}$ be a semigroup of mappings T_t acting on L^q for all $1 \le q \le \infty$, of a σ-finite measure space (Σ, μ), with infinitesimal generator $-A$.

In the paper [11], Bénilan established first L^q-L^r-regularisation estimates ($1 \le q < r \le \infty$) of nonlinear semigroups $\{T_t\}_{t\ge 0}$ generated either by operators of similar type as the *Dirichlet p-Laplace operator* $\Delta_p^D u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, ($1 < p < \infty$), or by operators similar to the *Dirichlet porous media operator* $\Delta^D(u^m) = \mathrm{div}(\nabla u^m)$, ($m > 0$). Here, the name *Dirichlet* and the superscript D refer to the fact that the differential operators Δ_p^D and $\Delta^D(\cdot^m)$ are equipped with homogeneous Dirichlet boundary conditions on a bounded domain Σ of $\mathbb{R}^d$, and u^m is the shorthand of $|u|^{m-1}u$. Bénilan's method employs a truncation technique on the sublevel sets of the resolvent combined with the regularisation effect of the resolvent given by the d-dimensional Sobolev inequality.

Only one year later, Véron [92] simplified Bénilan's method essentially and adapted it to the general nonlinear semigroup framework acting on L^q for $1 \le q \le \infty$. Véron introduces an abstract Sobolev type inequality (similar to (1.5)) satisfied by the generator A, from which one can conclude an L^q-L^r regularity estimate ($1 \le q < r \le \infty$) of the corresponding semigroup $\{T_t\}_{t\ge 0}$. In particular, L^q-L^∞ estimates ($1 \le q < \infty$) of $\{T_t\}_{t\ge0}$ are obtained by using a *one-parameter family of Sobolev type inequalities* satisfied by A combined with an iteration method in the time-variable of $\{T_t\}_{t\ge0}$. To be more precise, one easily sees that, for instance, for $1 < p < d$, the Dirichlet p-Laplace operator $A = -\Delta_p^D$ on L^2 satisfies the following one-parameter family (in $q \ge p$) of Sobolev type inequalities

$$\|u\|^q_{\frac{dq}{d-p}} \le C\,\frac{1}{q-p+1}\,\left(\frac{q}{p}\right)^p\,\langle -\Delta_p^D u, u_{q-p+2}\rangle \tag{1.7}$$

for every $u \in D(A) \cap L^\infty$ and $q \ge p$. To the best of our knowledge, it goes back to Véron [92] who established that the semigroup $\{T_t\}_{t\ge0}$ generated by Δ_p^D satisfies the L^q-L^r-regularisation estimate

$$\|T_t u - T_t\hat{u}\|_r \le C\,t^{-\delta}\,\|u - \hat{u}\|_q^\gamma \tag{1.8}$$

for every $t > 0$, u, $\hat{u} \in L^q$ with exponents δ, $\gamma > 0$ depending on d, p, r and q for every $q \le r \le \infty$ ($1 \le q \le \infty$, $2 \le p < \infty$), and that the semigroup $\{T_t\}_{t\ge0}$ generated by $\Delta^D(\cdot^m)$ satisfies the L^q-L^r-regularisation estimate

$$\|T_t u\|_r \le C\,t^{-\delta}\,\|u\|_q^\gamma \tag{1.9}$$

for every $t > 0$, $u \in L^q$, with $q = 1$, $r = \infty$ and exponents δ, $\gamma > 0$ depending on r, q, $m > 1$ and d. Véron's approach was quickly adapted to many nonlinear parabolic problems (cf. for instance [1, 71]).

The analogue of estimates (1.3) and (1.6) concerning linear semigroups are in the nonlinear semigroup theory the estimates (1.8) or (1.9). To emphasise the fact that these estimates involving nonlinear semigroups appear with an exponent γ at the initial datum $u \in L^q$, which is, in general, different of one, we avoid calling the estimates (1.8) and (1.9) supercontractive or ultracontractive estimates, but rather speak from an L^q-L^r *regularisation estimate* of the nonlinear semigroup $\{T_t\}_{t\ge0}$ if $1 \le q < r \le \infty$ (see also Remark 3.4 in Section 3).

In 2001, Cipriani and Grillo [33] adapted the approach by Davis and Simon [44] to establish L^q-L^∞-regularisation estimates of solutions of parabolic diffusion equations involving quasilinear operators of p-Laplace type equipped with homogeneous Dirichlet boundary conditions on bounded domains. The approach in [33] is essentially based on the following two steps (cf. [33]): firstly, one employs the *classical* Sobolev inequality

$$\|u\|_{\frac{pd}{d-p}} \le C\,\||\nabla u|\|_p \tag{1.10}$$

with respect to the Lebesgue measure, in order to derive a *one-parameter family of logarithmic Sobolev inequalities* in L^p (similar to (1.4) with 2 replaced by p) associated with the energy functional of the Dirichlet-p-Laplace operator Δ_p^D. Then one uses this family of inequalities to show that for a solution u of the parabolic equation under consideration the function

$$y(t) := \log\|u(t)\|_{r(t)} \qquad \text{for } t \ge 0,$$

satisfies a differential inequality from which one can deduce an L^q-L^r-regularisation estimate for $1 \le q < r \le \infty$.

Comparing this method with the one by Véron, the approach in [92] seems to be more direct in order to achieve L^q-L^r-regularisation estimates for $1 \le q < r \le \infty$ with optimal exponents.

Many authors followed the approach in [33]; they derive from the classical Sobolev inequality (1.10) new families of *energy entropy inequalities* (generalising the logarithmic Sobolev inequality) and then apply these inequalities to nonlinear parabolic problems (see, for instance, [47, 46, 84, 19, 20, 21, 69, 94]).

Another approach worth mentioning in this context is [81] by Porzio. In this paper, Porzio employs directly the classical Sobolev inequality to establish L^q-L^r-regularisation estimates ($1 \le q < r \le \infty$) for solutions of nonlinear parabolic equations involving non-autonomous quasilinear differential operators of p-Laplace type.

In order to conclude this section, we want to emphasise that L^q-L^r-regularity estimates of semigroups $\{T_t\}_{t\ge 0}$ have many applications, such as new existence results (see, for instance, [81]), global Hölder continuity (see, for instance, [49, 80, 93]) (and higher regularity (see [49])) of weak energy solutions of the underlying parabolic boundary value problem (see Section 5 and Section 7 of this monograph), finite time of extinction results with respect to the initial data (see, for instance, [8, pp 234] or [91]) or uniqueness of solutions (see [60]), and others.

1.2. **Main results.** In the present monograph, rather than making a detour via a family of Log-Sobolev inequalities, we shall use the tools that are more directly relevant to establish general L^q-L^r-regularity estimates for $1 \le q, r \le \infty$ and L^q-L^∞-regularisation estimates for $1 \le q < \infty$ of nonlinear semigroups $\{T_t\}_{t\ge 0}$, namely, Sobolev type inequalities or, more generally, *Gagliardo-Nirenberg* type inequalities

In the following, (Σ, μ) will be a σ-finite measure space. For $1 \le q \le \infty$, we shall say that A is an *operator on* $L^q(\Sigma, \mu)$ if A is a subset of $L^q(\Sigma, \mu) \times L^q(\Sigma, \mu)$. Further notions and notation used throughout this monograph can be found in Section 2.

Definition 1.1. Let $1 \le q < \infty$ and $1 \le r \le \infty$. We say an operator A on L^q satisfies an L^q-L^r- *Gagliardo-Nirenberg type inequality* for some $\varrho \ge 0$, $\sigma > 0$, $\omega \in \mathbb{R}$ and $(u_0, 0) \in A$ if there is a constant $C > 0$ such that

$$\|u - u_0\|_r^\sigma \le C \left([u - u_0, v]_q + \omega \|u - u_0\|_q^q \right) \|u - u_0\|_q^\varrho \tag{1.11}$$

for every $(u, v) \in A$. Moreover, we say that an operator A on L^q satisfies an L^q-L^r- *Gagliardo-Nirenberg type inequality with differences* for some $\varrho \ge 0$, $\sigma > 0$ and $\omega \in \mathbb{R}$ if there is a constant $C > 0$ such that

$$\|u - \hat{u}\|_r^\sigma \le C \left([u - \hat{u}, v - \hat{v}]_q + \omega \|u - \hat{u}\|_q^q \right) \|u - \hat{u}\|_q^\varrho \tag{1.12}$$

for every (u, v), $(\hat{u}, \hat{v}) \in A$.

For example, the negative Dirichlet p-Laplace operator $-\Delta_p^D$ satisfies the Gagliardo-Nirenberg inequality (1.11) with $u_0 = 0$ if $1 \le p < 2$ and (1.12) if $2 \le p < \infty$ (see Section 6.1) and for $m > 0$, the negative doubly nonlinear operator $-\Delta_p^D(\cdot^m)$ equipped with Dirichlet boundary conditions satisfies the Gagliardo-Nirenberg inequality (1.11) for $u_0 = 0$ (see Section 6.3). Further examples of operators and other type of boundary conditions are discussed in Section 6.

In the paper, we intend to come back to Bénilan's and Véron's viewpoint, and provide a systematic semigroup approach in order to establish L^s-L^∞-regularisation estimates of the form (1.8) and (1.9) for any $1 \le s < \infty$ for (nonlinear) semigroups $\{T_t\}_{t\ge0}$ under the assumption that the corresponding infinitesimal generator $-A$ satisfies an L^q-L^r-Gagliardo-Nirenberg type inequality either without differences (1.11) or with differences (1.12) for some $1 \le q, r \le \infty$.

We simplify Bénilan's and Véron's method by avoiding for a large class of operators the construction of a one-parameter family of Sobolev inequalities (such as the family of inequalities given by (1.7)) to establish L^q-L^∞-regularisation estimates of semigroups $\{T_t\}_{t\ge0}$. We rather tried to make the extrapolation techniques from the *linear* semigroup theory by Simon and Høegh-Krohn [83] available for the *nonlinear* semigroup theory. To achieve this, we have established a new nonlinear interpolation theorem (see Theorem 4.6, Theorem 4.7 and Theorem 4.8 in Section 4.2). Our techniques require the validity of only one L^q-L^r Gagliardo-Nirenberg type inequality satisfied by the generator A for some $1 \le q, r \le \infty$ in order to establish L^s-L^∞-regularisation estimates for $1 \le s < \infty$ of the corresponding semigroup $\{T_t\}_{t\ge0}$. This simplifies essentially the known techniques in the existing literature (cf. [11, 92, 52, 47, 46, 84, 19, 20, 21, 69, 94] and many more), but also allows us to establish L^q-L^r-regularity estimates, $1 \le q$, $r \le \infty$, for solutions of nonlinear parabolic problems involving nonlocal diffusion processes (see Section 6.2 and 6.2.2). Estimates of this type for solutions of nonlinear nonlocal diffusion problems are know to hold only for the fractional porous media equation on the whole space (cf. [45]). Further, we provide a nonlinear version of the methods from [35] and [36] to conclude that if a semigroup $\{T_t\}_{t\ge0}$ satisfies a L^q-L^r-regularisation estimate of the form (1.8) or (1.9) for some $1 < q < r \le \infty$ then the semigroup admits, in particular, a L^1-L^r-regularisation estimate of the form (1.8) or (1.9) (see Theorem 4.1 and Theorem 4.3 in Section 4.1).

Similar to [92], we focus our attention on two important classes of operators generating nonlinear semigroups acting on L^q for all $1 \le q \le \infty$:

- *quasi m-completely accretive operators in L^{q_0} for some $1 \le q_0 < \infty$,*
- *quasi m-T-accretive operators in L^1 with complete resolvent.*

In order to keep this subsection for an overview of the main results of this monograph, we refer for the definition of these two classes of operators to Section 2.2 and Section 2.3 and note briefly that prototypes of the first class of operators are of the form $A + F$, where F is a Lipschitz continuous mapping on L^{q_0} and A is, for instance, the celebrated negative p-Laplace operator $-\Delta_p$ (see Section 6.1) but also the negative nonlocal fractional p-Laplace operator $-(-\Delta_p)^s$ (see [68] and Section 6.2.2) respectively equipped with some boundary conditions and the Dirichlet-to-Neumann operator associated with the p-Laplace operator (see Section 6.2.1). Examples of the second class of operators are also of the form $A + F$, where F is a Lipschitz continuous mapping on L^1 and A is, for instance, the negative porous media operator $-\Delta(\cdot^m)$ and its nonlocal counterpart ([45]) or, more generally, doubly nonlinear operators $\Delta_p(\cdot^m)$ (see Section 6.3), where each of them is equipped with some boundary conditions.

Our first main result is concerned with L^q-L^r-regularity estimates of semigroups $\{T_t\}_{t\ge0}$ generated by $-A$ for an operator A of the first class satisfying L^q-L^r-Gagliardo-Nirenberg type inequality (1.12) with *differences*.

Theorem 1.2. *For some $q \in [1,+\infty)$ and $\omega \geq 0$, let $A+\omega I$ be m-completely accretive in $L^q(\Sigma,\mu)$ with dense domain. If A satisfies the Gagliardo-Nirenberg type inequality (1.12) with parameters q, $1 \leq r \leq \infty$, $\varrho \geq 0$ and $\sigma > 0$, then the semigroup $\{T_t\}_{t\geq 0}$ generated by $-A$ on $L^q(\Sigma,\mu)$ satisfies*

$$\|T_t u - T_t \hat{u}\|_r \leq \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega\,\beta\,t}\, \|u-\hat{u}\|_q^{\gamma} \tag{1.13}$$

for every $t>0$, u, $\hat{u} \in L^q(\Sigma,\mu)$ with exponents $\alpha = \frac{1}{\sigma}$, $\beta = \gamma+1$ and $\gamma = \frac{q+\varrho}{\sigma}$. Moreover, if $1 \leq r < \infty$, $\gamma\, r > q$ and there is $(u_0, 0) \in A$ for some $u_0 \in L^1 \cap L^\infty(\Sigma,\mu)$, then

$$\|T_t u - T_t \hat{u}\|_\infty \lesssim t^{-\alpha_s}\, e^{\omega\,\beta_s\,t}\, \|u-\hat{u}\|_s^{\gamma_s} \tag{1.14}$$

for every $t>0$, u, $\hat{u} \in L^s(\Sigma,\mu)$, $1 \leq s \leq \gamma\, r\, q^{-1}\, m_0$ satisfying $\gamma(1-\frac{sq}{\gamma r m_0}) < 1$ for every $m_0 \geq q\,\gamma^{-1}$ satisfying

$$\left(\tfrac{\gamma\, r}{q} - 1\right) m_0 + q\left(\tfrac{1}{\gamma} - 1\right) > 0, \tag{1.15}$$

with exponents

$$\begin{aligned} \alpha^* &= \frac{\sigma^{-1}\,q\,\gamma^{-1}}{(\frac{\gamma r}{q}-1)\,m_0 + q(\frac{1}{\gamma}-1)}, \quad \beta^* = \frac{\gamma^2 r q^{-1} - 1}{(\frac{\gamma r}{q}-1)\,m_0 + q(\frac{1}{\gamma}-1)} + 1, \quad \gamma^* = \frac{(\gamma r q^{-1}-1)\,m_0}{(\frac{\gamma r}{q}-1)\,m_0 + q(\frac{1}{\gamma}-1)}, \\ \alpha_s &= \frac{\alpha^*}{1-\gamma^*(1-\frac{sq}{\gamma r m_0})}, \quad \beta_s = \frac{\beta^* 2^{-1} + \gamma^* \frac{sq}{\gamma r m_0}}{1-\gamma^*(1-\frac{sq}{\gamma r m_0})}, \quad \gamma_s = \frac{\gamma^* \frac{sq}{\gamma r m_0}}{1-\gamma^*(1-\frac{sq}{\gamma r m_0})}. \end{aligned} \tag{1.16}$$

The proof of Theorem 1.2 follows by combining Theorem 3.3 (Section 3), Theorem 4.10 (Section 4.3) and subsequently by applying Theorem 4.1 (Section 4.1).

Remark 1.3. At first glance, the condition $\gamma\, r > q$ and the choice of $m_0 \geq q\,\gamma^{-1}$ satisfying (1.15) in Theorem 1.2 and in the subsequent two theorems seem rather mysterious. They are sufficient conditions to conclude an L^s-L^∞ regularisation estimate for $s = \gamma\, r\, q^{-1}\, m_0$ from an L^q-L^r regularity estimate for some $1 \leq q$, $r < \infty$ (cf. Remark 4.11 in Chapter 4.3). But on the other hand, the parameters γ, r and q are intimately related with the given operator A. In fact, the condition $\gamma\, r > q$ changes if and only if A changes. This is not the case for the parameter m_0 satisfying $m_0 \geq q\,\gamma^{-1}$ and (1.15) since for sufficiently large m_0 both conditions always hold. In certain cases, but not all, $m_0 = q\,\gamma^{-1}$ satisfies (1.15), in which case this choice of m_0 is optimal. This is well demonstrated by the example of the p-Laplace operator $A = -\Delta_p^{\mathbb{R}^d}$ on $\mathbb{R}^d$ satisfying vanishing conditions at infinity (see Theorem 6.1 in Section 6.1.1).

Our second main result is concerned with L^q-L^r-regularisation estimates of semigroups $\{T_t\}_{t\geq 0}$ generated by $-A$ for an operator A of the first class but satisfying the Gagliardo-Nirenberg type inequality (1.11) without *differences*.

Theorem 1.4. *For some $q \in [1,+\infty)$ and $\omega \geq 0$, let $A+\omega I$ be m-completely accretive in $L^q(\Sigma,\mu)$ with dense domain. If A satisfies the Gagliardo-Nirenberg type inequality (1.11) with parameters q, $1 \leq r \leq \infty$, $\varrho \geq 0$ and $\sigma > 0$ and some $(u_0, 0) \in A$ satisfying $u_0 \in L^1 \cap L^\infty(\Sigma,\mu)$, then the semigroup $\{T_t\}_{t\geq 0}$ generated by $-A$ on $L^q(\Sigma,\mu)$ satisfies*

$$\|T_t u - u_0\|_r \leq \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega\,\beta\,t}\, \|u-u_0\|_q^{\gamma} \tag{1.17}$$

for every $t > 0$, $u \in L^q(\Sigma,\mu)$ *with exponents* $\alpha = \frac{1}{\sigma}$, $\beta = \gamma + 1$ *and* $\gamma = \frac{q+\varrho}{\sigma}$. *Moreover, if* $1 \le r < \infty$ *and* $\gamma\, r > q$, *then*

$$\|T_t u - u_0\|_\infty \lesssim t^{-\alpha_s}\, e^{\omega\,\beta_s\,t}\, \|u - u_0\|_s^{\gamma_s} \tag{1.18}$$

for every $t > 0$, $u \in L^s(\Sigma,\mu)$, $1 \le s \le \gamma\, r\, q^{-1}\, m_0$ *satisfying* $\gamma(1 - \frac{sq}{\gamma r m_0}) < 1$ *for every* $m_0 \ge q\,\gamma^{-1}$ *satisfying* (1.15) *with exponents* (1.16).

The statements of Theorem 1.4 follows from Theorem 3.8 (Section 3) and by Theorem 4.13 (Section 4.3) combined with Theorem 4.3 (Section 4.1).

The last main result of this monograph focuses on the L^q-L^r-regularisation estimates of semigroups $\{T_t\}_{t\ge0}$ generated by $-A$ for an operator A in $L^1(\Sigma,\mu)$ of the second class satisfying the Gagliardo-Nirenberg type inequality (1.11) without *differences*. However, applications show that operators A of the second class, generally, do not satisfy the Gagliardo-Nirenberg type inequality (1.11) for $u \in D(A) \cap L^q(\Sigma,\mu)$ and some $q > 1$ but for $u \in D(A) \cap L^\infty(\Sigma,\mu)$. Thus, it is very useful to introduce the *trace*

$$A_{1\cap\infty} := A \cap ((L^1 \cap L^\infty(\Sigma,\mu)) \times (L^1 \cap L^\infty(\Sigma,\mu)))$$

of A on $L^1 \cap L^\infty(\Sigma,\mu)$. Furthermore, note that the notion of *c-complete resolvent* is defined in Section 2.3.

The nonlinear interpolation theorems used in the proofs of Theorems 1.2 and 1.4 (see Theorem 4.6, Theorem 4.7 and Theorem 4.8 in Section 4.2) cannot be applied to semigroups $\{T_t\}_{t\ge0}$ generated by $-A$ in L^1 for operators of the second class. One essential reason for this is that in the latter case each mapping $T_t : L^1 \cap L^\infty \to L^1 \cap L^\infty$ is, in general, not Lipschitz continuous with respect to the L^∞-norm. Another important observation is that operators satisfying an L^q-L^r Gagliardo-Nirenberg type inequality (1.12) with differences are necessarily quasi-accretive in L^q. But there are operators of the second class, as for instance, the negative porous media operator $-\Delta(\cdot^m)$ (cf. [91]) that are not accretive in L^q for $q > 1$. Hence we have to provide an alternative approach which applies to the second class of operators.

Theorem 1.5. *Let* $A + \omega I$ *be an m-T-accretive operator in* $L^1(\Sigma,\mu)$ *for some* $\omega \ge 0$ *with complete resolvent (respectively, c-complete resolvent and* $\omega = 0$*). Suppose the trace* $A_{1\cap\infty}$ *of* A *on* $L^1 \cap L^\infty(\Sigma,\mu)$ *satisfies the range condition*

$$L^1 \cap L^\infty(\Sigma,\mu) \subseteq Rg(I + (A_{1\cap\infty} + \omega I)). \tag{1.19}$$

and the Gagliardo-Nirenberg type inequality (1.11) *for parameters* $1 \le q, r \le \infty$, $(q < \infty)$, $\sigma > 0$, $\varrho \ge 0$ *and some* $(u_0, 0) \in A_{1\cap\infty}$ *satisfying* $u_0 \in L^1 \cap L^\infty(\Sigma,\mu)$. *Then the semigroup* $\{T_t\}_{t\ge0}$ *generated by* $-A$ *on* $\overline{D(A)}^{L^1}$ *satisfies*

$$\|T_t u - u_0\|_r \le \left(\frac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega\,\beta\,t}\, \|u - u_0\|_q^{\gamma} \tag{1.17}$$

for every $t > 0$, $u \in \overline{D(A)}^{L^1} \cap L^\infty(\Sigma,\mu)$ *with exponents* $\alpha = \frac{1}{\sigma}$, $\beta = \gamma + 1$ *and* $\gamma = \frac{q+\varrho}{\sigma}$. *Moreover, if for parameters* $\kappa > 1$, $m > 0$ *and* $q_0 \ge p \ge 1$ *satisfying* $\kappa m q_0 \ge 1$ *and*

$$(\kappa - 1)q_0 + p - 1 - \tfrac{1}{m} > 0, \tag{1.20}$$

the trace $A_{1\cap\infty}$ satisfies the one-parameter family of Sobolev type inequalities

$$\|u-u_0\|_{\kappa m q}^{mq} \le \frac{C\,(q/p)^p}{q-p+1}\,\Big[[u-u_0,v]_{(q-p+1)m+1}+\omega\|u-u_0\|_{(q-p+1)m+1}^{(q-p+1)m+1}\Big] \tag{1.21}$$

for every $(u,v)\in A_{1\cap\infty}$ and every $q\ge q_0$, then there is a $\beta^*\ge 0$ such that the semigroup $\{T_t\}_{t\ge0}$ satisfies

$$\|T_t u-u_0\|_\infty \lesssim t^{-\alpha_s}\,e^{\omega\,\beta_s\,t}\,\|u-u_0\|_s^{\gamma_s} \tag{1.18}$$

for every $u\in\overline{D(A)}^{L^1}\cap L^\infty(\Sigma,\mu)$, $1\le s\le\kappa m q_0$ satisfying $\gamma^*\left(1-\frac{q}{\kappa m q_0}\right)<1$ with exponents

$$\begin{aligned}\alpha^* &= \frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}, \qquad \gamma^*=\frac{(\kappa-1)q_0}{(\kappa-1)q_0+p-1-\frac{1}{m}},\\ \alpha_s &= \frac{\alpha^*}{1-\gamma^*\left(1-\frac{s}{\kappa m q_0}\right)},\ \beta_s=\frac{\beta^*2^{-1}+\gamma^* s\kappa^{-1}m^{-1}q_0^{-1}}{1-\gamma^*\left(1-\frac{s}{\kappa m q_0}\right)},\ \gamma_s=\frac{\gamma^* s}{\kappa m q_0(1-\gamma^*(1-\frac{s}{\kappa m q_0}))}.\end{aligned} \tag{1.22}$$

Suppose the domain $D(A)$ of the operator A considered in Theorem 1.5 is dense in $L^1(\Sigma,\mu)$. If the measure space (Σ,μ) is finite then a standard density result yields the inequalities (1.17) and (1.18) in Theorem 1.5 hold for all $u\in L^q(\Sigma,\mu)$, respectively, $u\in L^s(\Sigma,\mu)$. If Σ has infinite measure μ and if the inequalities (1.17) and (1.18) hold for $q=1$ and $s=1$ then (1.17) and (1.18) hold also for all $u\in L^1(\Sigma,\mu)$.

By comparing Theorem 1.5 with Theorem 1.4, we note that quasi-m-completely accretive operators A in L^{q_0} for some $q_0\ge1$ (that is, operators of the first class considered in Theorem 1.4), it is not difficult to see that the trace $A_{1\cap\infty}$ of A in $L^1\cap L^\infty$ satisfies the *range condition* (1.19). This is not immediately clear for quasi-m-accretive operators A in L^1 with complete resolvent (that is, the second class of operators considered in Theorem 1.5). Thus, we provide in Proposition 2.18 sufficient conditions yielding that operators $A\phi$ composed of an operator A of the first class and a monotone graph ϕ satisfies range condition (1.19). On the other hand, there are operators of the first class which do not satisfy the one-parameter family of Sobolev type inequalities 1.21 in Theorem 1.5 but they do satisfy all assumptions in Theorem 1.4. Examples of such operators are the nonlinear Dirichlet-to-Neumann operator associated with p-Laplace type operators (see Section 6.2.1) or the fractional p-Laplace operator equipped with some boundary conditions (see Section 6.2.2). Here, it is interesting to note that both operators are of *nonlocal* character (cf. [55, 68]).

The proof of Theorem 1.5 follows from Corollary 3.12 (respectively, Corollary 3.13) and Theorem 4.15. Comparing Theorem 1.5 with the existing literature (cf., for instance, [21, 22, 69, 84, 91]), we note that so far, in order to establish an L^1-L^∞-regularisation estimate (1.9) for semigroups $\{T_t\}_{t\ge0}$ generated by doubly nonlinear operators $\Delta_p(\cdot^m)$ on $L^1(\mathbb{R}^d)$, one had either to assume more regularity on the solution of the associated parabolic problem (cf., for instance, [21, 69, 84]) or to approximate the semigroup $\{T_t\}_{t\ge0}$ on $L^1(\mathbb{R}^d)$ by the semigroup $\{T_t^n\}_{t\ge0}$ generated by the Dirichlet-doubly nonlinear operator $\Delta_p^D(\cdot^m)$ on $L^1(\Sigma_n)$ for a sequence $(\Sigma_n)_n$ of open sets $\Sigma_n\subseteq\mathbb{R}^d$ with finite measure, smooth boundary, and satisfying $\Sigma_n\subseteq\Sigma_{n+1}$ and $\bigcup_{n\ge1}\Sigma_n=\mathbb{R}^d$ (cf., for instance, [91]). Our approach presented in Theorem 1.5 is different to this one and to the one given by Véron [92]. Thus, the statement of Theorem 1.5 unfolds its complete strength in the case (Σ, μ) is an infinite measure space. It is based on the result stated in Proposition 2.18. We emphasise that we generalise in Proposition 2.18 an idea by Crandall and Pierre [41] for the composition $A\phi$ of a *nonlinear* completely accretive operator A and a non-decreasing function $\phi : \mathbb{R} \to \mathbb{R}$.

As an application of the theory developed in this monograph, we provide in Section 5 an abstract approach showing that a global L^1-L^∞-regularisation estimate satisfied by a semigroup $\{T_t\}_{t\ge0}$ implies that the trajectories $u(t) := T_t u_0$ in L^1, $t \ge 0$, for initial values $u_0 \in L^1$, are, in fact, *weak energy solutions* of the corresponding abstract initial value problem (see Definition 5.2 in Section 5). Here, the semigroup $\{T_t\}_{t\ge0}$ is generated by a quasi-m-T-accretive operator in L^1 of the form $A_{1\cap\infty}\phi + F$, where A is the realisation in L^2 of the Gâteaux-derivative $\Psi' : V \to V'$ of a convex real-valued functional Ψ defined on a Banach space $V \hookrightarrow L^2$, and A has the property to be an m-completely accretive operator in L^2. Further, ϕ is assumed to be a continuous, strictly increasing function on $\mathbb{R}$, and F a Lipschitz mapping on L^1. In the case $F \equiv 0$ and $A\phi$ is the celebrated negative porous media operator $-\Delta\phi$, this result is well-known (see [9, 51, 91]). Our results (Theorem 5.6, Theorem 5.7 and Theorem 5.9) in Section 5 extend the known literature by providing a uniform approach which can be applied to general quasi-m-T-accretive operators $A_{1\cap\infty}\phi + F$ in L^1 where the operator $A_{1\cap\infty}$ needs *not to be linear*, but has the important property to be completely accretive in L^2. Concerning the regularity of trajectories $u(t) := T_t u_0$ in L^1 generated by operators $-A\phi$ with similar characteristics as $\Delta_p\phi$, the notion of *entropy solutions* was developed (see [13, 5, 6]). Theorem 5.6 in Section 5 improves the regularity of these trajectories $u(t) := T_t u_0$ in L^1 essentially.

We demonstrate the efficiency of the methods and techniques developed in the Sections 3, 4.1, 4.3 and 4.4 with a plethora of examples gathered in Section 6. Section 6.1 is concerned with establishing L^q-L^r-regularisation estimates of the semigroup generated by a negative Leray-Lions type operator equipped with either homogeneous Dirichlet, Neumann or Robin boundary conditions. Our results in this section improve some known results in the literature. For instance, we prove that the exponents in the L^q-L^r-regularisation estimates remain unchanged by adding a monotone (multi-valued) or a Lipschitz perturbation. Note that our methods yield sharp exponents as one can see in Section 6.1. Section 6.2 is dedicated to establishing the L^q-L^r-regularisation estimates of two nonlocal parabolic problems where known methods fail. In this section, we establish the L^q-L^r-regularisation estimates of the semigroup generated by the negative Dirichlet-to-Neumann operator associated with a Leray-Lions type operator and of the semigroup generated by the fractional p-Laplace operator equipped with either Dirichlet or Neumann boundary conditions. In Section 6.3, we establish the L^q-L^r-regularisation estimates of *mild* solutions of parabolic problems involving the negative doubly nonlinear operator $-\Delta_p(\cdot^m)$ for $m > 0$ equipped with either homogeneous Dirichlet, Neumann or Robin boundary conditions. In this section, we use the properties of the p-Laplace operator established in Section 6.1, to construct the operator $-\Delta_p\phi$.

In Section 7, we employ the L^1-L^∞-regularisation estimate (1.18) satisfied by the semigroup $\{T_t\}_{t\ge0}$ of the doubly nonlinear operator $\Delta_p(\cdot^m)$ equipped with either Dirichlet, Neumann or Robin boundary conditions to show that for every given initial value $u_0 \in L^1$, the mild solution $u(t) := T_t u_0$, $t \ge 0$, in L^1 is a *strong energy solution* (see Theorem 7.3). We give the exact definition of strong solutions in the next section, where we set the general framework in which we are working and briefly review some classical definitions and important results in nonlinear semigroup theory.

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2. Framework

Throughout this monograph, (Σ, μ) denotes a σ-finite measure space and $M(\Sigma, \mu)$ the space (of all classes) of measurable real-valued functions on Σ. We denote by $L^q(\Sigma, \mu)$, $1 \le q \le \infty$, the corresponding standard Lebesgue space with norm $\|\cdot\|_q$. For $1 \le q < \infty$, we identify the dual space $(L^q(\Sigma, \mu))'$ with $L^{q'}(\Sigma, \mu)$ and use the notation $\langle u', u \rangle$ to denote the natural pairing of $u' \in L^{q'}(\Sigma, \mu)$ and $u \in L^q(\Sigma, \mu)$, where q' is the conjugate exponent of q given by $1 = \frac{1}{q} + \frac{1}{q'}$. More generally, for every topological vector space $X \subseteq M(\Sigma, \mu)$, we denote by $\langle \psi, u \rangle$ the value of $\psi \in X'$ at $u \in X$. In the case $1 < q < \infty$, we shall write u_q to denote $|u|^{q-2}u$ for every $u \in L^q(\Sigma, \mu)$.

2.1. **Nonlinear semigroup theory: old and new.** Most of this section can be skipped by readers familiar with classical nonlinear semigroup theory. Most of the results of this theory can be found in the books [8] by Barbu, [70] by Miyadera or in the famous draft [15] of the *unpublished* book by Bénilan, Crandall and Pazy.

We call a mapping A from $M(\Sigma, \mu)$ into the set of all subsets of $M(\Sigma, \mu)$, denoted by $2^{M(\Sigma,\mu)}$, an *operator* on $M(\Sigma, \mu)$. As usual, we identify an operator A on $M(\Sigma, \mu)$ with its *graph*, that is, with the set

$$\Big\{ (u, v) \in M(\Sigma, \mu) \times M(\Sigma, \mu) \,\Big|\, v \in Au \Big\},$$

and thus, we shall say that $(u, v) \in A$ if $v \in Au$. The *effective domain* $D(A)$ of A denotes the set of all $u \in M(\Sigma, \mu)$ satisfying $Au \neq \emptyset$ and the *range* $Rg(A)$ of A the set $\bigcup_{u \in D(A)} Au \subseteq M(\Sigma, \mu)$. The *inverse operator* A^{-1} of A is given by the set of all pairs $(u, v) \in M(\Sigma, \mu) \times M(\Sigma, \mu)$ satisfying $u \in Av$, hence $D(A^{-1}) = Rg(A)$ and $Rg(A^{-1}) = D(A)$. Given two operators A and B on $M(\Sigma, \mu)$ and a scalar $\alpha \in \mathbb{R}$, the operator $A + \alpha B$ is given by $(A + \alpha B)u = Au + \alpha(Bu)$ for every $u \in D(A) \cap D(B)$. Further, the *composition* $AB := A \circ B$ of two operators A and B on $M(\Sigma, \mu)$ is defined by

$$AB = \Big\{ (u, v) \in M(\Sigma, \mu) \times M(\Sigma, \mu) \,\Big|\, \text{ there is } z \in Bu \text{ such that } v \in Az \Big\}.$$

Let $X \subseteq M(\Sigma, \mu)$ be a Banach space with norm $\|.\|_X$. Then, an operator A on X, meaning that $A \subseteq X \times X$, is said to be *densely defined* or with *dense domain* if its effective domain $D(A)$ is dense in X. We denote by $\overline{A}$ the closure of the graph of A in X and call $\overline{A}$ the *closure of* A in X. We call A closed if $A = \overline{A}$. Obviously, the domain $D(\overline{A})$ of the closure $\overline{A}$ of A in X is closed in X. For any sequence

$(A_n)_{n\ge 0}$ of operators A_n on X, the *limit inferior of* (A_n) denoted by $\liminf_{n\to\infty} A_n$ is defined by

$$\Big\{(u,v)\in X\times X\,\Big|\,\text{there are } (u_n,v_n)\in A_n \text{ s. t. } \lim_{n\to\infty}(u_n,v_n)=(u,v) \text{ in } X\times X\Big\}.$$

Now, an operator A on X is called *accretive (in X)* if for every (u,v), $(\hat{u},\hat{v})\in A$ and every $\lambda\ge 0$, one has

$$\|u-\hat{u}\|_X\le \|u-\hat{u}+\lambda(v-\hat{v})\|_X. \tag{2.1}$$

In other words, A is accretive in X if and only if for every $\lambda>0$, the *resolvent operator* $J_\lambda=(I+\lambda A)^{-1}$ of A is a single-valued mapping from $Rg(I+\lambda A)$ to $D(A)$, which is *contractive* (also called *nonexpansive*) with respect to the norm of X. Recall a mapping $S:D(S)\to X$ with domain $D(S)\subseteq X$ is call *contractive* with respect to the norm of X or a *contraction* in X if

$$\|Su-S\hat{u}\|_X\le\|u-\hat{u}\|_X,$$

for all $u,\hat{u}\in D(S)$. In order to better grasp the definition of accretive operators, one might first consider accretive operators β on $\mathbb{R}$. If $X=\mathbb{R}$ is equipped with the absolute value $|\cdot|$ then for every (u,v) and $(\hat{u},\hat{v})\in\beta$, inequality (2.1) is equivalent to the inequality

$$(v-\hat{v})(u-\hat{u})\ge 0. \tag{2.2}$$

This shows that a single-valued operator β on $\mathbb{R}$ is accretive if and only if β is non-increasing. From now on we refer to accretive operators on $\mathbb{R}$ as *monotone graphs* (cf. [15, Example (2.3)]). For a given monotone graph β on $\mathbb{R}$ and for every $1\le q\le\infty$, we denote by β_q the *associated accretive operator with β* in $L^q(\Sigma,dx)$ given by

$$\beta_q=\Big\{(u,v)\in L^q\times L^q(\Sigma,\mu)\,\Big|\,v(x)\in\beta(u(x)) \text{ for a.e. } x\in\Sigma\Big\}. \tag{2.3}$$

There are important characterisations of accretivity, which we use from time to time throughout this paper. Here is the first one: an operator A is accretive in X if and only if

$$\begin{cases} \text{for every } (u,v),\,(\hat{u},\hat{v})\in A, \text{ there exists } \psi\in J(u-\hat{u})\\ \text{satisfying } \langle\psi,v-\hat{v}\rangle\ge 0,\end{cases} \tag{2.4}$$

where $J:X\to 2^{X'}$ denotes the *duality mapping* of X, which is given by

$$J(u)=\Big\{\psi\in X'\,\Big|\,\langle\psi,u\rangle=\|u\|_X \text{ and } \|\psi\|_{X'}\le 1\Big\}$$

for every $u\in X$ (cf. [15, Theorem (2.15)] or [8, Proposition 3.1]).

Now, it is not difficult to verify (cf. [15, Example (2.11)]) that for $q=1$, the duality mapping J on $L^1(\Sigma,\mu)$ is given by

$$J(u)=\Big\{\psi\in L^\infty(\Sigma,\mu)\,\big|\,\psi(x)\in\operatorname{sign}(u(x)) \text{ for a.e. } x\in\Sigma\Big\}$$

for every $u\in L^1(\Sigma,\mu)$, where the multi-valued *signum* function is defined by

$$\operatorname{sign}(s):=\begin{cases}1 & \text{if } s>0,\\ [-1,1] & \text{if } s=0,\\ -1 & \text{if } s<0\end{cases}$$

for every $s \in \mathbb{R}$, and for $1 < q < \infty$, the duality mapping J on $L^q(\Sigma,\mu)$ is a well-defined mapping $J : L^q(\Sigma,\mu) \to L^{q'}(\Sigma,\mu)$ given by

$$J(u) = u_q \, \|u\|_q^{1-q} \tag{2.5}$$

for every $u \in L^q(\Sigma,\mu)$. In the case $q = 1$, $J(u)$ is multi-valued exactly when the set $\{u = 0\}$ has strictly positive μ-measure. However, the multi-valued signum function $\operatorname{sign}(\cdot)$ can be approximated by the sequence $(\gamma_\varepsilon)_{\varepsilon>0}$ of piecewise smooth functions $\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\gamma_\varepsilon(r) = \begin{cases} 1 & \text{if } r > \varepsilon, \\ \frac{r}{\varepsilon} & \text{if } -\varepsilon \le r \le \varepsilon, \\ -1 & \text{if } r < -\varepsilon. \end{cases} \tag{2.6}$$

For our purposes, it is convenient to introduce the notion of *q-brackets*. For $1 \le q < \infty$, the *q-bracket* $[\cdot,\cdot]_q : L^q(\Sigma,\mu) \times L^q(\Sigma,\mu) \to \mathbb{R}$ is defined by

$$[u,v]_q = \lim_{\lambda\to 0+} \frac{\frac{1}{q}\|u+\lambda v\|_q^q - \frac{1}{q}\|u\|_q^q}{\lambda}$$

for every $u, v \in L^q(\Sigma,\mu)$. For given $u, v \in L^q(\Sigma,\mu)$, the number $[u,v]_q$ is the right-hand directional derivative of the function $u \mapsto \frac{1}{q}\|u\|_q^q$. Since the function $\lambda \mapsto \frac{1}{q}\|u+\lambda v\|_q^q$ is convex on $\mathbb{R}$, we can define $[\cdot,\cdot]_q$, equivalently, by

$$[u,v]_q = \inf_{\lambda>0} \frac{\frac{1}{q}\|u+\lambda v\|_q^q - \frac{1}{q}\|u\|_q^q}{\lambda} \tag{2.7}$$

for every $u, v \in L^q(\Sigma,\mu)$. The q-bracket $[\cdot,\cdot]_q : L^q(\Sigma,\mu) \times L^q(\Sigma,\mu) \to \mathbb{R}$ is upper semicontinuous (respectively, continuous if $1 < q < \infty$) and

$$[u,v]_q = \langle u_q, v\rangle \qquad \text{for every } u, v \in L^q(\Sigma,\mu) \text{ if } 1 < q < \infty, \tag{2.8}$$

while for $q = 1$, $[\cdot,\cdot]_1$ reduces to the classical *brackets* $[\cdot,\cdot]$ on $L^1(\Sigma,\mu)$ given by

$$[u,v]_1 = \int_{\{u\neq 0\}} \operatorname{sign}_0(u)\, v \,\mathrm{d}\mu + \int_{\{u=0\}} |v| \,\mathrm{d}\mu \tag{2.9}$$

for every $u, v \in L^1(\Sigma,\mu)$, where the *restricted signum* sign_0 is defined by

$$\operatorname{sign}_0(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0 \end{cases}$$

for every $s \in \mathbb{R}$ (cf. [15, Section 2.2 & Example (2.8)] or [8, pp 102]). By characterisations (2.4), (2.5) and (2.8) if $1 < q < \infty$, respectively, by [15, Theorem 2.14] (or, alternatively, [8, p 103 formula (3.15)]) if $q = 1$, we see that an operator A on $L^q(\Sigma,\mu)$ is accretive if and only if

$$[u-\hat{u}, v-\hat{v}]_q \ge 0 \qquad \text{for every } (u,v), (\hat{u},\hat{v}) \in A.$$

In Section 3, we shall need the following two properties of q-brackets:

$$[u,v]_q \le \frac{1}{q}\|u+v\|_q^q - \frac{1}{q}\|u\|_q^q \qquad \text{for every } u, v \in L^q(\Sigma,\mu), \tag{2.10}$$

and

$$[u, \alpha\, v + \omega u]_q = \alpha\,[u,v]_q + \omega\,\|u\|_q^q \tag{2.11}$$

for every u, $v \in L^q(\Sigma,\mu)$, ω, $\alpha \in \mathbb{R}$. Here, note that inequality (2.10) is an immediate consequence of (2.7). Property (2.11) is shown for $q = 1$ in [15, Proposition (2.5)] (or, alternatively, [8, Proposition 3.7]) and if $1 < q < \infty$ then (2.11) can be easily deduced from (2.8).

If $X \subseteq M(\Sigma,\mu)$ is a Banach lattice, then we shall denote the usual lattice operations $u \vee \hat{u}$ and $u \wedge \hat{u}$ to be the almost everywhere pointwise supremum and infimum of u and $\hat{u} \in X$. In addition, $u^+ = u \vee 0$ is the positive part, $u^- = (-u) \vee 0$ the negative part, and $|u| = u^+ + u^-$ the absolute value of an element $u \in X$. For every $u, \hat{u} \in X$, one denotes by $u \leq \hat{u}$ the usual order relation on X. In this framework, a mapping $S : D(S) \to X$ with domain $D(S) \subseteq X$ is called *order preserving* if $Su \leq S\hat{u}$ for every $u \leq \hat{u}$, *positive* if $Su \geq 0$ for every $u \geq 0$, and a T*-contraction* if

$$\|[Su - S\hat{u}]^+\|_X \leq \|[u - \hat{u}]^+\|_X$$

for every $u, \hat{u} \in D(S)$. Note that if S is T-contractive then it is order-preserving and that the converse holds if S is contractive and satisfies $u \vee \hat{u}$ and $u \wedge \hat{u} \in D(S)$ for every u, $\hat{u} \in D(S)$ (see [15, Lemma (19.11)]). We shall say that an operator A on X is T*-accretive* if for every $\lambda > 0$, the resolvent J_λ of A defines a T-contraction with domain $D(J_\lambda) = Rg(I + \lambda A)$.

Note, without further assumptions (cf. [15, Proposition (19.13)]), T-contractive does not imply contractive, nor does T-accretive imply accretive, and vice-versa. However, if the norm $\|\cdot\|_X$ on X satisfies the implication

$$\|u^+\|_X \leq \|\hat{u}^+\|_X, \quad \|u^-\|_X \leq \|\hat{u}^-\|_X \quad \text{implies} \quad \|u\|_X \leq \|\hat{u}\|_X \tag{2.12}$$

for every u, $\hat{u}$, then every T-contraction is also a contraction (cf. [15, p 267]). One easily verifies that this implication holds, for instance, for the space $X = L^q(\Sigma,\mu)$ for every $1 \leq q \leq \infty$. Thus, for the rest of this monograph, if we speak about T-contractive or T-accretive operators on X, then we automatically assume that the underlying space X is a Banach lattice satisfying (2.12).

In the space $X = L^1(\Sigma,\mu)$, the property that an operator A in $L^1(\Sigma,\mu)$ is T-accretive can be characterised as follows: for every (u,v), $(\hat{u},\hat{v}) \in A$, there is a $w \in L^\infty(\Sigma,\mu)$ satisfying $w(x) \in \mathrm{sign}^+(u(x) - \hat{u}(x))$ for a.e. $x \in \Sigma$ and

$$\int_\Sigma w\,(v - \hat{v})\,\mathrm{d}\mu \geq 0,$$

where for every $s \in \mathbb{R}$,

$$\mathrm{sign}^+(s) := \begin{cases} 1 & \text{if } s > 0,\\ [0,1] & \text{if } s = 0,\\ 0 & \text{if } s < 0, \end{cases}$$

or equivalently (cf. [10]) for every (u,v), $(\hat{u},\hat{v}) \in A$, one has

$$[u - \hat{u}, v - \hat{v}]_+ := \int_{\{u=\hat{u}\}} [v - \hat{v}]^+\,\mathrm{d}\mu + \int_{\{u>\hat{u}\}} (v - \hat{v})\,\mathrm{d}\mu \geq 0.$$

In order to conclude that the sum $A + B$ of two operators A and B in X is accretive, the assumption that A and B are both accretive is not sufficient (cf. [15, Exercise E2.3]). For this to be true, we need that at least one of the two operators

A and B admits the following stronger property. We call an operator A *s-accretive* in X if for every (u,v), $(\hat{u},\hat{v}) \in A$ and for every $\psi \in J(u-\hat{u})$,

$$\langle \psi, v - \hat{v}\rangle \geq 0.$$

Now, if A is an operator on X then the sum $A+B$ is accretive in X for every accretive operator B on X if and only if A is s-accretive (cf. [15, Proposition (2.20)]). Obviously, for $1<q<\infty$, every accretive operator A in $L^q(\Sigma,\mu)$ is s-accretive in $L^q(\Sigma,\mu)$. Unfortunately, this is not true for accretive operators A in $L^1(\Sigma,\mu)$. A counter example is, for instance, given by the accretive operator β_1 in $L^1((-2,1),\mathrm{d}x)$ for $\beta(s) := \mathrm{sign}(s)$ (cf. [15, Exercise E2.25]). On the other hand, a prototype of s-accretive operators in $L^1(\Sigma,\mu)$ is provided by the accretive operator β_1 in $L^1(\Sigma,\mu)$ associated with a *non-decreasing function* $\beta : \mathbb{R} \to \mathbb{R}$. To see that β_1 is s-accretive in $L^1(\Sigma,\mu)$, we need to check that for every (u,v), $(\hat{u},\hat{v}) \in \beta_1$ every $\psi \in L^\infty(\Sigma,\mu)$ satisfying $\psi(x) \in \mathrm{sign}(u(x)-\hat{u}(x))$ for a.e. $x \in \Sigma$, one has

$$\int_\Sigma \psi\,(v-\hat{v})\,\mathrm{d}\mu \geq 0. \tag{2.13}$$

Since β is assumed to be real-valued function, we have that $v = \beta(u)$ and $\hat{v} = \beta(\hat{u})$. Thus, and by the monotonicity of β, for a.e. $x \in \Sigma$, the condition $v(x) > \hat{v}(x)$ implies $u(x) > \hat{u}(x)$ and so, $\psi(x) = 1$ hence $\psi(x)\,(v(x)-\hat{v}(x)) \geq 0$. Analogously, for a.e. $x \in \Sigma$, the condition $v(x) < \hat{v}(x)$ implies $u(x) < \hat{u}(x)$ and so $\psi(x) = -1$ hence $\psi(x)\,(v(x)-\hat{v}(x)) \geq 0$. Therefore, (2.13) holds.

Next, an operator A on X is called *m-(T)-accretive in* X if A is (T)-accretive in X and satisfies the *range condition*

$$Rg(I+\lambda A) = X \qquad \text{for some (or equivalently all) } \lambda > 0. \tag{2.14}$$

Coming back to the example of a monotone graph β in $\mathbb{R}$, we set $\beta(r+) = \inf \beta(]r,\infty[)$ and $\beta(r-) = \sup \beta(]-\infty,r[)$ for every $r \in \mathbb{R}$, where, as usual, $\inf\emptyset := +\infty$ and $\sup\emptyset := -\infty$. Then, a monotone graph β in $\mathbb{R}$ is m-accretive if and only if for every $r \in \mathbb{R}$, one has

$$\beta(r) = [\beta(r-),\beta(r+)] \cap \mathbb{R}.$$

Therefore, a monotone graph β in $\mathbb{R}$ is m-accretive if and only if the graph of β is the maximal monotone set in $\mathbb{R}\times\mathbb{R}$ containing β itself. If β is m-accretive in $\mathbb{R}$ and either $(0,0) \in \beta$ or (Σ,μ) is finite, then for every $1 \leq q \leq \infty$, the associated operator β_q on $L^q(\Sigma,\mu)$ is m-T-accretive (cf. [15, Examples (8.4) & (8.5)] or [8, Section 3.2]). Moreover, the resolvent operator J_λ of β_q is given by $(J_\lambda u)(x) = (1+\lambda\beta)^{-1}u(x)$ and the so-called *Yosida operator* $\beta_\lambda(\cdot) := \lambda^{-1}(I-J_\lambda)$ is given by $\beta_\lambda(u)(x) = \lambda^{-1}(1-(1+\lambda\beta)^{-1})u(x)$ for a.e. $x \in \Sigma$. If $1<q<\infty$ then for a given accretive operator A on $L^q(\Sigma,\mu)$, the sum $A+\beta_q$ is accretive in $L^q(\Sigma,\mu)$. This is an immediate consequence of the fact that the duality mapping J of $L^q(\Sigma,\mu)$ is single-valued on $L^q(\Sigma,\mu)$. However, in order to conclude that for an m-accretive operator A in $L^q(\Sigma,\mu)$, $(1<q<\infty)$, satisfying $D(A)\cap D(\beta_q) \neq \emptyset$, the sum $A+\beta_q$ is m-accretive in $L^q(\Sigma,\mu)$, one needs an additional condition. A possible one is the following (cf. [8, Proposition 3.8]):

$$[\beta_\lambda(u),v]_q \geq 0 \qquad \text{for every } \lambda > 0,\ (u,v) \in A. \tag{2.15}$$

Another important example of m-accretive operators in $L^2(\Sigma,\mu)$ is given by the *subgradient*

$$\partial_{L^2}\Psi := \Big\{(u,v)\in L^2\times L^2(\Sigma,\mu)\,\Big|\,\langle v,\xi-u\rangle \le \Psi(\xi)-\Psi(u) \text{ for all } \xi\in L^2(\Sigma,\mu)\Big\}$$

in $L^2(\Sigma,\mu)$ of a functional $\Psi : L^2(\Sigma,\mu)\to\mathbb{R}\cup\{+\infty\}$ which is convex, lower semicontinuous and proper (see [24] and also [31]).

More generally, an operator A on X is called *quasi (T)-accretive* if there is an $\omega\in\mathbb{R}$ such that $A+\omega I$ is (T)-accretive in X. Obviously, if $A+\omega I$ is (T)-accretive for some $\omega\in\mathbb{R}$ then $A+\tilde{\omega}I$ is (T)-accretive for every $\tilde{\omega}\ge\omega$. Thus, there is no loss of generality in assuming that $A+\omega I$ is (T)-accretive for some $\omega\ge 0$. Finally, we call A *quasi m-(T)-accretive* if $A+\omega I$ is m-(T)-accretive for some $\omega\in\mathbb{R}$. It is easy to check that $A+\omega I$ is (T)-accretive for some $\omega\in\mathbb{R}$ if and only if for every $\lambda>0$ satisfying $\lambda\omega<1$, the resolvent J_λ of A satisfies

$$\|J_\lambda u - J_\lambda\hat{u}\|_X \le (1-\lambda\omega)^{-1}\,\|u-\hat{u}\|_X$$

for every u, $\hat{u}\in Rg(I+\lambda A)$ (respectively, one has

$$\|[J_\lambda u - J_\lambda\hat{u}]^+\|_X \le (1-\lambda\omega)^{-1}\,\|[u-\hat{u}]^+\|_X$$

for every u, $\hat{u}\in Rg(I+\lambda A)$). It is important to know that if $A+\omega I$ is m-accretive for some $\omega\in\mathbb{R}$, then for every $\lambda>0$ such that $\omega\lambda<1$ and $u\in\overline{D(A)}^X$, the closure of $D(A)$ in X, one has $J_\lambda u\in D(A)$ and

$$\lim_{\lambda\to 0} J_\lambda u = u \qquad \text{in } X \tag{2.16}$$

(cf. [15, Proposition (4.4)] or [8, Proposition 3.2]). Prototype examples of quasi (T)-accretive operators A on $X=L^q(\Sigma,\mu)$, $1\le q\le\infty$, are of the form $A=B+F$, where B denotes a (T)-accretive operator on $L^q(\Sigma,\mu)$ and $F : L^q(\Sigma,\mu)\to L^q(\Sigma,\mu)$ is defined by $F(u)(x) := f(x,u(x))$ for every $u\in L^q(\Sigma,\mu)$ of a given $f : \Sigma\times\mathbb{R}\to\mathbb{R}$ with the properties that $f(\cdot,u) : \Sigma\to\mathbb{R}$ is measurable on Σ for every $u\in\mathbb{R}$, $f(x,0)=0$ for a.e. $x\in\Sigma$, and there is a constant $\omega\ge 0$ such that

$$|f(x,u)-f(x,\hat{u})| \le L\,|u-\hat{u}| \qquad \text{for all } u,\,\hat{u}\in\mathbb{R} \text{ and a.e. } x\in\Sigma. \tag{2.17}$$

A real-valued function f satisfying such properties (or slightly weaker ones) is also called a *Carathéodory function*, and the mapping F given by $F(u)(x) := f(x,u(x))$ for every $u\in L^q(\Sigma,\mu)$ the *Nemytski operator* on $L^q(\Sigma,\mu)$ associated with f.

If $q=1$ then an approximation argument with the sequence $(\gamma_\varepsilon)_{\varepsilon>0}$ given by (2.6) and if $1<q<\infty$, using that the duality map J on $L^q(\Sigma,\mu)$ is single-valued, one sees that the operator $B+F+\omega I$ is (T)-accretive in $L^q(\Sigma,\mu)$. On the other hand, for the same ω, the operator $F+\omega I$ on $L^q(\Sigma,\mu)$ is accretive and Lipschitz continuous. Hence a standard fixed point argument shows that $F+\omega I$ is m-(T)-accretive in $L^q(\Sigma,\mu)$. Therefore, if B is m-(T)-accretive in $L^q(\Sigma,\mu)$ for some $1\le q<\infty$, then $B+F$ is quasi m-(T)-accretive in $L^q(\Sigma,\mu)$ (cf. [8, Theorem 3.1]).

For an accretive operator $A+\omega I$ on X, $\omega\in\mathbb{R}$, one easily verifies that the following properties hold (cf. [15, Proposition 2.18]):

(2.18) *The closure $\overline{A+\omega I}$ of $A+\omega I$ in X coincides with $\overline{A}+\omega I$ and is accretive.*

(2.19) *If A is closed, then $Rg(I+\lambda(A+\omega I))$ is closed for every $\lambda>0$.*

(2.20) *If there is a* $\lambda > 0$ *such that* $Rg(I + \lambda(A + \omega I))$ *is closed, then* A *is closed.*

(2.21) *If* $A \subseteq B$, $Rg(I + (B + \omega I)) \subseteq Rg(I + (A + \omega I))$ *and* $B + \omega I$ *is accretive, then* $A = B$.

By the celebrated Crandall-Liggett theorem [40, Theorem I], the condition A *is quasi m-accretive in* X ensures that for all $u_0 \in \overline{D(A)}^X$, the abstract initial value problem

$$\frac{du}{dt} + Au \ni 0, \quad u(0) = u_0 \tag{2.22}$$

is well-posed in the sense of mild solutions. In particular, if A is quasi m-T-accretive in X, then for every u_0, $\hat{u}_0 \in \overline{D(A)}^X$ satisfying $u_0 \le \hat{u}_0$, the corresponding mild solutions u and $\hat{u}$ of (2.22) satisfy $u(t) \le \hat{u}(t)$ for all $t \ge 0$ (cf. [15, Proposition (19.12)]). To be more precise, we first recall that for given $u_0 \in X$, a function $u \in W^{1,1}_{loc}((0,\infty); X) \cap C([0,\infty); X)$ satisfying $u(0) = u_0$, $u(t) \in D(A)$ and $-\frac{du}{dt}(t) \in Au(t)$ a *strong solution of* (2.22). Now, a *mild solution* u *of Cauchy problem* (2.22) is a function $u \in C([0,\infty); X)$ with the following property: for every T, $\varepsilon > 0$, for every partition $0 = t_0 < \cdots < t_N = T$ of the interval $[0, T]$ such that $t_i - t_{i-1} < \varepsilon$ for every $i = 1, \dots, N$, there exists a piecewise constant function $u_{\varepsilon,N} : [0, T] \to X$ given by

$$u_{\varepsilon,N}(t) = u_0\, \mathbb{1}_{\{t_0=0\}}(t) + \sum_{i=1}^{N} u_{\varepsilon,i}\, \mathbb{1}_{(t_{i-1},t_i]}(t)$$

where the values u_i on $(t_{i-1}, t_i]$ solve recursively the finite difference equation

$$u_i + (t_i - t_{i-1})Au_i \ni u_{i-1} \qquad \text{for every } i = 1, \dots, N$$

and

$$\sup_{t\in[0,T]} \|u(t) - u_{\varepsilon,N}(t)\|_X \le \varepsilon.$$

If $A + \omega I$ is m-accretive in X for some $\omega \in \mathbb{R}$, then for every element u_0 of $\overline{D(A)}^X$, there is a unique mild solution u of (2.22) which can be given by *exponential formula*

$$u(t) = \lim_{n\to\infty} \left(I + \tfrac{t}{n}A\right)^{-n} u_0 \tag{2.23}$$

uniformly in t on compact intervals. For every $u_0 \in \overline{D(A)}^X$, setting $T_t u_0 = u(t)$, $t \ge 0$, defines a (non-linear) *strongly continuous semigroup* $\{T_t\}_{t\ge0}$ of *Lipschitz continuous mappings* $T_t : \overline{D(A)}^X \to \overline{D(A)}^X$ with constant $e^{\omega t}$. More precisely, there is a family $\{T_t\}_{t\ge0}$ of mappings T_t on $\overline{D(A)}^X$ obeying the following three properties:

- (*semigroup property*)

$$T_{t+s} = T_t \circ T_s \qquad \text{for every } t, s \ge 0, \tag{2.24}$$

- (*strong continuity*)

$$\lim_{t\to0+} \|T_t u - u\|_X = 0 \qquad \text{for every } u \in \overline{D(A)}^X,$$

- (*exponential growth property in* X)

$$\|T_t u - T_t v\|_X \le e^{\omega t}\|u - v\|_X \qquad \text{for all } u, v \in \overline{D(A)}^X, t \ge 0.$$

In addition, if A is quasi m-T-accretive in X, then for every $t \ge 0$, T_t satisfies

- (*exponential T-growth property in X*)
$$\|[T_tu - T_tv]^+\|_X \le e^{\omega t}\|[u-v]^+\|_X \qquad \text{for all } u, v \in \overline{D(A)}^X,\, t \ge 0,$$

in particular, the semigroup $\{T_t\}_{t\ge 0}$ is order-preserving, that is, every T_t is order-preserving.

To express that the semigroup $\{T_t\}_{t\ge0}$ has been obtained by the above construction we say that $\{T_t\}_{t\ge0}$ has been *generated by* $-A$ *on* $\overline{D(A)}^X$ and we denote $\{T_t\}_{t\ge0} \sim -A$. If A is m-accretive in X, then each mapping T_t of the semigroup $\{T_t\}_{t\ge0} \sim -A$ becomes *contractive* in X.

With these preliminaries in mind, we turn now to one of the main topics of this article. It is not difficult to see ([8, p. 130]) that every strong solution of Cauchy problem (2.22) is a mild solution. But, it is still not well understood under which conditions on the operator A and the Banach space X, for each $u_0 \in \overline{D(A)}^X$, the mild solution $u(t) = T_tu_0$, $t \ge 0$, of (2.22) is a strong one. Obviously, this problem involves a *regularisation effect* since the solution u gains a posteriori in regularity, namely, the property to be differentiable at a.e. $t > 0$ with values in X. The current state of knowledge in the literature concerning this problem is the following one (cf. [8, Theorem 4.6] or [15, Corollary (7.11)]): if A is quasi-m-accretive in X and if the Banach space X and its dual X' are uniformly convex, then for every initial value $u_0 \in D(A)$ the mild solution $u(t) := T_tu_0$, $t \ge 0$, of (2.22) belongs to the space $W^{1,\infty}_{loc}([0,\infty);X)$, is almost everywhere differentiable on $(0,\infty)$, differentiable from the right at every $t \ge 0$, the right-hand side derivative $\frac{du}{dt}_+(t)$ is right continuous on $[0,\infty)$ and for every $t \ge 0$,

$$u(t) \in D(A) \qquad \text{and} \qquad \tfrac{d}{dt}_+u(t) + A^\circ u(t) = 0. \tag{2.25}$$

Here, A° denotes the *principal section* of A which assigns to every $u \in D(A)$ the element $A^\circ u$ of Au with minimal norm among all elements of Au. Thus, under these assumptions, the mild solution $u(t) = T_tu_0$, $t \ge 0$, of (2.22) for every $u_0 \in D(A)$ is a strong solution of (2.22). Recall that the space $X = L^1(\Sigma,\mu)$ is not uniformly convex. Further, it is natural to ask whether this statement holds true if $u_0 \in \overline{D(A)}^X$ for general quasi m-accretive operators A. Thanks to the pioneering result [23] by Brezis, the answer of this question is affirmative provided A is the subgradient $\partial_{L^2}\Psi$ in $L^2(\Sigma,\mu)$ of a convex, proper, lower semicontinuous functional $\Psi : L^2(\Sigma,\mu) \to \mathbb{R}\cup\{+\infty\}$. Semigroups $\{T_t\}_{t\ge0}$ generated by *positive homogeneous operators* A *of order* $\alpha > 0$ *with* $\alpha \neq 1$ on $L^q(\Sigma,\mu)$ for $1 < q < \infty$ admit the same regularisation effect (cf. [14]). As a by-product of our other results (Theorem 1.5) we can show that the semigroup $\{T_t\}_{t\ge0}$ in $L^1(\Sigma,\mu)$ has also this regularisation effect provided its infinitesimal generator A is the closure $\overline{(\partial_{L^2}\Psi)_{1\cap\infty}\phi}$ of $(\partial_{L^2}\Psi)_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$, where $\Psi : L^2(\Sigma,\mu) \to \mathbb{R}\cup\{+\infty\}$ is a convex, proper, lower semicontinuous functional and $\phi : \mathbb{R} \to \mathbb{R}$ a strictly increasing function such that ϕ and ϕ^{-1} are locally Lipschitz continuous (see Theorem 5.7 for the exact statement).

As mentioned in the introduction, throughout this monograph, we deal with the following two classes of accretive operators A on $M(\Sigma,\mu)$ generating nonlinear semigroups acting on all L^q spaces for $1 \le q \le \infty$:

- *quasi m-completely accretive operators in* L^{q_0} *for some* $1 \le q_0 < \infty$
- *quasi m-T-accretive operators in* L^1 *with complete resolvent*.

Besides the already mentioned prototypes, typical examples of the first class of operators are, for instance, quasi-linear operators of second order of p-Laplace type (so-called *Leray-Lions operators* [63]), nonlocal diffusion operators of p-Laplace type (see, e.g., [7]), but also the total variational flow operator of local and nonlocal type (cf., for instance, [6] and [56]). Typical examples of the second class of operators are easily obtained by considering the *composition operator* $A\phi$ in L^1 of an m-completely accretive operator A in L^1 and a strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi(0) = 0$. Operators of both classes are equipped with some boundary conditions when required and may be perturbed by a monotone (multi-valued) or Lipschitz continuous lower order term.

In the following two subsections, we introduce these two classes of operators in more details and list some of their properties relevant for this monograph. We establish L^q-L^r-regularisation estimates for several examples of both classes in Section 6.

2.2. **Completely accretive operators.** The class of *completely accretive* operators is the nonlinear analogue of the class of linear operators generating a submarkovian semigroup in the sense that the semigroup they generate extrapolates to L^p, $1 < p < \infty$ (see Proposition 2.8 below) and is order preserving. This class of nonlinear operators was introduced by Benilan and Crandall [9].

The notion of complete accretivity we use is the same as in [9] and will be introduced now. We denote by $\mathcal{J}_0$ the set of all convex, lower semicontinuous functions $j : \mathbb{R} \to [0, \infty]$ satisfying $j(0) = 0$.

Definition 2.1. A mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a *complete contraction* if

$$\int_\Sigma j(Su - S\hat{u})\,\mathrm{d}\mu \le \int_\Sigma j(u - \hat{u})\,\mathrm{d}\mu$$

for all $j \in \mathcal{J}_0$ and every $u, \hat{u} \in D(S)$.

Remark 2.2. Choosing $j(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$ if $1 \le q < \infty$ and $j(\cdot) = [[\cdot]^+ - k]^+ \in \mathcal{J}_0$ for $k \ge 0$ large enough if $q = \infty$ shows that each complete contraction S is T-contractive in $L^q(\Sigma, \mu)$ for every $1 \le q \le \infty$.

Now, we can state the definition of completely accretive operators.

Definition 2.3. An operator A on $M(\Sigma, \mu)$ is called *completely accretive* if for every $\lambda > 0$, the resolvent operator J_λ of A is a complete contraction. If X is a linear subspace of $M(\Sigma, \mu)$ and A an operator on X, then A is *m-completely accretive on X* if A is completely accretive and satisfies the range condition (2.14). Further, we call an operator A on $M(\Sigma, \mu)$ *quasi completely accretive* if there is an $\omega \in \mathbb{R}$ such that $A + \omega I$ is completely accretive. Finally, an operator A on a linear subspace X is called *quasi m-completely accretive* if $A + \omega I$ is m-completely accretive on X for some $\omega \in \mathbb{R}$.

As a matter of fact, in most applications the following characterisation is used to verify whether a given operator A on $X = L^q(\Sigma, \mu)$ is completely accretive (see also [7, Corollary A.43]). Here, we state [9, Proposition 2.2] only in a special case since it is more convenient for us.

Proposition 2.4 ([9, Proposition 2.2])**.** *Let P_0 denote the set of all functions $T \in C^\infty(\mathbb{R})$ satisfying $0 \le T' \le 1$, T' is compactly supported, and $x = 0$ is not contained* *in the support* $\mathrm{supp}(T)$ *of* T. *Then for* $u,\, v \in L^1(\Sigma,\mu) + L^\infty(\Sigma,\mu)$ *with* $\mu(\{|u| > k\}) < \infty$ *for all* $k > 0$, *one has*

$$\int_\Sigma j(u)\, d\mu \le \int_\Sigma j(u+\lambda v)\, d\mu$$

for every $j \in \mathcal{J}_0$ *and* $\lambda > 0$ *if and only if*

$$\int_\Sigma T(u)\, v \,\mathrm{d}\mu \ge 0 \tag{2.26}$$

for every $T \in P_0$. *As a consequence, an operator* A *on* $L^q(\Sigma,\mu)$ *for* $1 \le q < \infty$ *is completely accretive if and only if*

$$\int_\Sigma T(u-\hat{u})(v-\hat{v})\,\mathrm{d}\mu \ge 0 \tag{2.27}$$

for every $T \in P_0$ *and every* (u,v), $(\hat{u},\hat{v}) \in A$.

For any given monotone graph β on $\mathbb{R}$ and $1 \le q < \infty$, the associated accretive operator β_q on $L^q(\Sigma,\mu)$ is, in fact, completely accretive. To see this, note first that every $T \in P_0$ is continuous and non-decreasing. Thus for all (u,v), $(\hat{u},\hat{v}) \in \beta_q$ and every $T \in P_0$, one has

$$T(u-\hat{u})(v-\hat{v}) \ge 0 \qquad \text{a.e. on } \Sigma.$$

Integrating this inequality over Σ yields inequality (2.27) in Proposition 2.4.

Further, the property *completely accretive* is preserved under perturbation of a Lipschitz continuous mapping. This result seems to be known, but we could not find a reference in the literature. It provides an important example of completely accretive operators (cf. [9, Corollary 2.4]).

Proposition 2.5. *Let* $1 \le q < \infty$, B *be a completely accretive operator on* $L^q(\Sigma,\mu)$ *and* $F : L^q(\Sigma,\mu) \to L^q(\Sigma,\mu)$ *the Nemytski operator of a Carathéodory function* $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ *satisfying* (2.17) *for some constant* $\omega \ge 0$ *and* $F(0) \in L^q(\Sigma,\mu)$. *Then the following statements hold:*

(1) *The operator* $A := B + F + \omega I$ *is completely accretive.*

(2) *Let* $1 < q < \infty$ *and* β *be an m-accretive graph on* $\mathbb{R}$ *such that either* $(0,0) \in \beta$ *or* (Σ,μ) *is finite. If* B *satisfies the range condition* (2.14) *in* $L^q(\Sigma,\mu)$ *with* $D(B) \cap D(\beta_q) \neq \emptyset$ *and if the Yosida operator* $\beta_\lambda(\cdot)$ *of* β_q *satisfies*

$$[\beta_\lambda(u), v]_q \ge 0 \qquad \textit{for all } (u,v) \in A \textit{ and } \lambda > 0, \tag{2.28}$$

then $A := B + \beta_q + F$ *is quasi m-completely accretive in* $L^q(\Sigma,\mu)$.

Proof. Let $T \in P_0$ and (u,v), $(\hat{u},\hat{v}) \in A$. Then, in order to apply Proposition 2.4, we need to show that inequality (2.27) holds. By assumption, there are $w \in Bu$ and $\hat{w} \in B\hat{u}$ such that $v = w + f(x,u) + L\,u$ and $\hat{v} = \hat{w} + f(x,\hat{u})$. Also, T is non-decreasing and $T(0) = 0$. Hence $T(u-\hat{u}) \ge 0$ if $u \ge \hat{u}$ and $T(u-\hat{u}) \le 0$ if $u < \hat{u}$. Using this together with inequality (2.17) and since, by assumption, B is completely accretive, we see that

$$\begin{aligned}
&\int_\Sigma T(u-\hat{u})(v + \omega u - (\hat{v} + \omega \hat{u}))\,\mathrm{d}\mu \\
&\quad = \int_\Sigma T(u-\hat{u})\,(w-\hat{w})\,\mathrm{d}\mu + \int_\Sigma T(u-\hat{u})\,(f(x,u) - f(x,\hat{u})\,\mathrm{d}\mu \\
&\qquad\qquad + L \int_\Sigma T(u-\hat{u})\,(u-\hat{u})\,\mathrm{d}\mu
\end{aligned}$$

$$
\begin{aligned}
&\geq -L\int_{\{u\geq\hat{u}\}} T(u-\hat{u})\,|u-\hat{u}|\,\mathrm{d}\mu + L\int_{\{u<\hat{u}\}} T(u-\hat{u})\,|u-\hat{u}|\,\mathrm{d}\mu\\
&\qquad\qquad\qquad + L\int_{\Sigma} T(u-\hat{u})\,(u-\hat{u})\,\mathrm{d}\mu\\
&= -L\int_{\Sigma} T(u-\hat{u})\,(u-\hat{u})\,\mathrm{d}\mu + L\int_{\Sigma} T(u-\hat{u})\,(u-\hat{u})\,\mathrm{d}\mu = 0.
\end{aligned}
$$

Thus claim (1) holds. For any given monotone graph β on $\mathbb{R}$, one sees by using (2.2) together with the fact that $T\in P_0$ is monotonically increasing and by proceeding as before that A is completely accretive. If B is m-accretive in $L^q(\Sigma,\mu)$ and $1<q<\infty$, then we know that $B+F+\omega I$ is m-accretive in $L^q(\Sigma,\mu)$. Now, under the assumptions of claim (2) and by using the definition of q-brackets $[\cdot,\cdot]_q$, it follows by [8, Proposition 3.8], $B+F+\omega I+\beta_q$ is m-accretive in $L^q(\Sigma,\mu)$. This completes the proof. □

The last proposition of this subsection states that a semigroup $\{T_t\}_{t\geq 0}$ generated by a quasi m-completely accretive operator $-A$ on $L^{q_0}(\Sigma,\mu)$ for some $1\leq q_0<\infty$ has, in fact, *exponential growth* in all L^q spaces.

Proposition 2.6. *Let $1\leq q_0<\infty$ and $\omega\in\mathbb{R}$ such that $A+\omega I$ is completely accretive in $L^{q_0}(\Sigma,\mu)$. Then for every $\lambda>0$ satisfying $\lambda\,\omega<1$, the resolvent operator J_λ of A satisfies*

$$\|J_\lambda u - J_\lambda \hat{u}\|_{\tilde{q}} \leq (1-\lambda\omega)^{-1}\|u-\hat{u}\|_{\tilde{q}} \tag{2.29}$$

for every u, $\hat{u}\in Rg(I+\lambda A)$ and every $1\leq\tilde{q}\leq\infty$. If $A+\omega I$ is m-completely accretive in $L^{q_0}(\Sigma,\mu)$, then for all $1\leq\tilde{q}\leq\infty$, the semigroup $\{T_t\}_{t\geq0}\sim -A$ on $\overline{D(A)}^{L^{q_0}}$ satisfies the "exponential growth property"

$$\|T_t u - T_t\hat{u}\|_{\tilde{q}} \leq e^{\omega t}\|u-\hat{u}\|_{\tilde{q}} \tag{2.30}$$

for every u, $\hat{u}\in\overline{D(A)}^{L^{q_0}}\cap L^{\tilde{q}}(\Sigma,\mu)$ and $t>0$. Moreover, if there are $u_0\in L^{q_0}\cap L^{\tilde{q}}(\Sigma,\mu)$ and $t\geq 0$ such that $T_tu_0\in L^{\tilde{q}}(\Sigma,\mu)$ then T_t can be uniquely extended to a Lipschitz continuous mapping on $\overline{\overline{D(A)}^{L^{q_0}}\cap L^{\tilde{q}}(\Sigma,\mu)}^{L^{\tilde{q}}}$ with constant $e^{\omega t}$.

Remark 2.7. Throughout this monograph, we often assume that there is

$$(u_0,0)\in A \qquad \text{for some } u_0\in L^{q_0}\cap L^{\tilde{q}}(\Sigma,\mu) \tag{2.31}$$

and some $1\leq q_0,\tilde{q}\leq\infty$ (see, for instance, Definition 1.1, Theorem 1.4 or Theorem 1.5). Condition (2.31) is equivalent to $u_0\in L^{q_0}\cap L^{\tilde{q}}(\Sigma,\mu)$ is a *fixed point* for the resolvent J_λ of A, that is, $J_\lambda u_0=u_0$ for all $\lambda>0$. Thus and by using the exponential formula (2.23), one sees that condition (2.31) is equivalently to the fact that $u_0\in L^{q_0}\cap L^{\tilde{q}}(\Sigma,\mu)$ is a *fixed point* for the semigroup $\{T_t\}\sim -A$, that is, $T_tu_0=u_0$ for all $t\geq 0$. Moreover, it is worth noting that under condition (2.31), the exponential growth property (2.30) of the semigroup $\{T_t\}$ reduces to

$$\|T_t u - u_0\|_{\tilde{q}} \leq e^{\omega t}\|u-u_0\|_{\tilde{q}} \tag{2.32}$$

for every $u\in\overline{D(A)}^{L^{q_0}}\cap L^{\tilde{q}}(\Sigma,\mu)$ and $t>0$.

Proof of Proposition 2.6. By assumption, the resolvent operator of $A+\omega I$ is a complete contraction. Let $1\leq\tilde{q}\leq\infty$ and $\lambda>0$ such that $\lambda\omega<1$. Then by Remark 2.2,

$$\|u-\hat{u}\|_{\tilde{q}} \leq \|u-\hat{u}+\lambda(\omega(u-\hat{u})+(v-\hat{v}))\|_{\tilde{q}}$$

for every (u,v), $(\hat{u},\hat{v}) \in A$. Thus,

$$\begin{aligned}\|u-\hat{u}+\lambda(v-\hat{v})\|_{\tilde{q}} &= (1-\lambda\omega)\|u-\hat{u}+\tfrac{\lambda}{1-\lambda\omega}(\omega(u-\hat{u})+(v-\hat{v}))\|_{\tilde{q}}\\ &\geq (1-\lambda\omega)\,\|u-\hat{u}\|_{\tilde{q}}\end{aligned}$$

for every (u,v), $(\hat{u},\hat{v}) \in A$ and every $\lambda > 0$ such that $\lambda\,\omega < 1$, proving that the resolvent operator J_λ of A satisfies (2.29). In order to see that the semigroup $\{T_t\}_{t\geq 0} \sim -A$ on $\overline{D(A)}^{L^{q_0}}$ satisfies (2.30), for given $t > 0$ and $n \in \mathbb{N}$ large enough, one takes $\lambda = t/n$ such that $\frac{t}{n}\omega < 1$. Then, replacing u and $\hat{u}$ by $J_{t/n}^{n-1}u$ and $J_{t/n}^{n-1}\hat{u}$ in (2.29) yields

$$\begin{aligned}\|J_{t/n}^{n}u - J_{t/n}^{n}\hat{u}\|_{\tilde{q}} &\leq (1-\tfrac{t\omega}{n})^{-1}\|J_{t/n}^{n-1}u - J_{t/n}^{n-1}\hat{u}\|_{\tilde{q}}\\ &\;\;\vdots\\ &\leq (1-\tfrac{t\omega}{n})^{-n}\|u-\hat{u}\|_{\tilde{q}}.\end{aligned}$$

By the exponential formula (2.23), one has

$$\lim_{n\to\infty} J_{t/n}^{n}u - J_{t/n}^{n}\hat{u} = T_t u - T_t\hat{u} \qquad \text{in } L^{q_0}(\Sigma,\mu).$$

Since the $L^{\tilde{q}}$ norm on $L^{q_0}(\Sigma,\mu)$ is lower semicontinuous, sending $t \to \infty$ in the previous estimates shows that inequality (2.30) holds. In order to see that the last statement of this proposition holds, note that the existence of $u_0 \in L^{q_0} \cap L^{\tilde{q}}(\Sigma,\mu)$ and $t \geq 0$ such that $T_t u_0 \in L^{\tilde{q}}(\Sigma,\mu)$ together with (2.30) imply that T_t maps $\overline{D(A)}^{L^{q_0}} \cap L^{\tilde{q}}(\Sigma,\mu)$ into $L^{q_0} \cap L^{\tilde{q}}(\Sigma,\mu)$. Thus (2.30) and a standard density argument yield that T_t has a unique Lipschitz continuous extension from $\overline{\overline{D(A)}^{L^{q_0}} \cap L^{\tilde{q}}(\Sigma,\mu)}^{L^{\tilde{q}}}$ to $\overline{\overline{D(A)}^{L^{q_0}} \cap L^{\tilde{q}}(\Sigma,\mu)}^{L^{\tilde{q}}}$ with constant $e^{\omega t}$. This completes the proof. $\square$

The fundamental property of completely accretive operators A is given by the *extrapolation property* stated in the next proposition, which is especially meaningful when (Σ,μ) is not a finite measure space. The first main steps towards the statement of this proposition have been established by Bénilan and Crandall [9]. The main idea of the proof in [9] relies on the following fundamental property of sequences of functions in $L^q(\Sigma,\mu)$ for $1 \leq q < \infty$:

$$\tag{2.33}\begin{aligned}&\text{for any sequence } (u_n)_{n\geq 1} \subseteq L^q(\Sigma,\mu) \text{ satisfying}\\ &\quad (i)\quad \int_\Sigma |u_n|^q\,\mathrm{d}\mu \leq \int_\Sigma |u|^q\,\mathrm{d}\mu \quad \text{for all } n \geq 1,\\ &\quad (ii)\quad \lim_{n\to\infty} u_n(x) = u(x) \quad \text{for a.e. } x \in \Sigma,\\ &\text{one has}\quad \lim_{n\to\infty} u_n = u \text{ in } L^q(\Sigma,\mu).\end{aligned}$$

Statement (2.33) follows from Fatou's lemma in combination with either the uniform convexity of $L^q(\Sigma,\mu)$ if $q > 1$ or with Young's theorem (cf., for instance, [17, Theorem 2.8.8]) if $q = 1$. Furthermore (2.33) yields the statements of our next proposition. We leave its easy proof to the interested reader as an exercise.

Proposition 2.8. *Let $1 \leq q_0 < \infty$ and A be a m-completely accretive in $L^{q_0}(\Sigma,\mu)$ with dense domain and satisfying $(0,0) \in A$. Let $A_{1\cap\infty}$ be the trace of A on $L^1 \cap L^\infty(\Sigma,\mu)$ and for $1 \leq q < \infty$, let $\overline{A_{1\cap\infty}}^{L^q}$ be the closure of $A_{1\cap\infty}$ in $L^q(\Sigma,\mu)$. Then the following statements hold true.*

(1) *For every* $1 \le q < \infty$, $\overline{A_{1\cap\infty}}^{L^q}$ *is m-completely accretive in* $L^q(\Sigma,\mu)$ *with dense domain and* $(0,0) \in \overline{A_{1\cap\infty}}^{L^q}$.
(2) *For every* $1 \le q < \infty$ *and for all* $\lambda > 0$, *the resolvent* J_λ *of* A *admits a unique extension on* $L^q(\Sigma,\mu)$ *and this extension coincides with the resolvent operator* J_λ^q *of* $\overline{A_{1\cap\infty}}^{L^q}$.
(3) *For every* $1 \le q < \infty$ *and for every* $t > 0$, *the mapping* T_t *of the semigroup* $\{T_t\}_{t\ge0} \sim -A$ *on* $L^{q_0}(\Sigma,\mu)$ *admits a unique extension on* $L^q(\Sigma,\mu)$ *and this extension coincides with the mapping* T_t^q *on* $L^q(\Sigma,\mu)$ *of the semigroup* $\{T_t^q\}_{t\ge0} \sim -\overline{A_{1\cap\infty}}^{L^q}$ *on* $L^q(\Sigma,\mu)$.

If $F : L^{q_0}(\Sigma,\mu) \to L^{q_0}(\Sigma,\mu)$ *is a Lipschitz mapping satisfying* $F(0) = 0$, *then the same statements hold for the quasi m-completely accretive operator* $A + F$ *on* $L^{q_0}(\Sigma,\mu)$.

2.3. **T-accretive operators in L^1 with complete resolvent.** The class of *T-accretive operators in L^1 with complete resolvent* was developed in connection with the porous media equation by Bénilan [10]. To the best of our knowledge, the first important result in direction to this class of operators has been the interpolation theorem in [26] due to Brezis and Strauss in order to treat semilinear elliptic equations in L^1. This interpolation theorem has been extended by Bénilan and Crandall in [9] to introduce the class of completely accretive operators and further investigated by many others (for instance, see also [10, Partie II], [42, 41] and [91]).

For the sake of brevity, we merely state the results in this section and refer for their proofs to Appendix A.

We begin by introducing the notion of *complete maps* on $M(\Sigma,\mu)$ in accordance with [9, Definition 1.7] (see also [10, Définition 2.1]).

Definition 2.9. Let $D(S)$ be a subset of $M(\Sigma,\mu)$. A mapping $S : D(S) \to M(\Sigma,\mu)$ is called *complete* if

$$\int_\Sigma j(Su)\,\mathrm{d}\mu \le \int_\Sigma j(u)\,\mathrm{d}\mu \tag{2.34}$$

for every $j \in \mathcal{J}_0$ and $u \in D(S)$. Further, let $\mathcal{J}$ be the set of all convex, lower semicontinuous functions $j : \mathbb{R} \to [0,\infty]$. We call a mapping $S : D(S) \to M(\Sigma,\mu)$ *c-complete* if S satisfies inequality (2.34) for all $u \in D(S)$ and $j \in \mathcal{J}$.

Since the set $\mathcal{J}_0$ is contained in $\mathcal{J}$, every c-complete mapping is a complete mapping. In particular, we have the following characterisation.

Proposition 2.10. *Let* P *denote the set of all functions* $T \in C^\infty(\mathbb{R})$ *satisfying* $0 \le T' \le 1$ *and* T' *is compactly supported. Suppose* (Σ,μ) *is a finite measure space. Then for* $u, v \in L^1(\Sigma,\mu)$, *one has*

$$\int_\Sigma j(u)\,d\mu \le \int_\Sigma j(u+\lambda v)\,d\mu \tag{2.35}$$

for every $j \in \mathcal{J}$ *and* $\lambda > 0$ *if and only if*

$$\int_\Sigma T(u)\,v\,\mathrm{d}\mu \ge 0 \tag{2.36}$$

for every $T \in P$. *As a consequence, an operator* A *on* $L^1(\Sigma,\mu)$ *has a c-complete resolvent if and only if inequality* (2.36) *holds for every* $(u,v) \in A$.

For a proof of this characterisation we refer to Appendix A.

Remark 2.11. By taking $j(\cdot) = |\cdot|^q \in \mathcal{J}_0$ if $1 \leq q < \infty$ and $j(\cdot) = [|\cdot| - k]^+ \in \mathcal{J}_0$ for $k \geq 0$ large enough if $q = \infty$, we see that every complete mapping S has *non-increasing* L^q norm for all $1 \leq q \leq \infty$. More precisely,

$$\|Su\|_q \leq \|u\|_q$$

for all $u \in D(S)$ and every $1 \leq q \leq \infty$. More generally, since for every constant $c \in \mathbb{R}$ and every $1 \leq q < \infty$, the function $j(\cdot) := |\cdot - c|^q \in \mathcal{J}$ and since $j(\cdot) := [|\cdot - c| - k]^+ \in \mathcal{J}$ for every $k \geq 0$, one has that if S is c-complete, then

$$\|Su - c\|_q \leq \|u - c\|_q$$

for all $u \in D(S)$, $c \in \mathbb{R}$ and $1 \leq q \leq \infty$.

Now, we can define the class of accretive operators in L^1 with (c-)complete resolvent.

Definition 2.12. An operator A on $L^1(\Sigma, \mu)$ is called *(m-) accretive in L^1 with complete resolvent* if A is (m-) accretive in $L^1(\Sigma, \mu)$ and for every $\lambda > 0$, the resolvent operator $J_\lambda : Rg(I + \lambda A) \to D(A)$ of A is a complete mapping. Further, we call an operator A on $M(\Sigma, \mu)$ *quasi (m-) accretive in L^1 with complete resolvent* if there exists $\omega \in \mathbb{R}$ such that $A + \omega I$ is (m-) accretive in $L^1(\Sigma, \mu)$ and for every $\lambda > 0$, the resolvent J_λ of $A + \omega I$ is a complete mapping. Similarly, we call an operator A on $L^1(\Sigma, \mu)$ *(m-) accretive in L^1 with c-complete resolvent* if A is (m-) accretive in $L^1(\Sigma, \mu)$ and for every $\lambda > 0$, the resolvent $J_\lambda : Rg(I + \lambda A) \to D(A)$ of A is a c-complete mapping.

Remark 2.13. Note that, in contrast to completely accretive operators in L^1, an accretive operator A in L^1 with (c)-complete resolvent does not admit, in general, an order-preserving resolvent J_λ on L^1. For this, one needs the additional assumption A is *T-accretive* in L^1.

Note that it does not make much sense to introduce the notion of *quasi (m-) accretive operators in L^1 with c-complete resolvent*. This becomes more clear by the following result due to Bénilan [10, Corollaire 2.3]. To be more precise, consider the following situation. Let $B = -\Delta^N$ be the Neumann Laplace operator on L^1 on a bounded Lipschitz domain $\Sigma \subseteq \mathbb{R}^d$ and f a real-valued Carathéodory function on $\Sigma \times \mathbb{R}$ satisfying $f(x, 0) = 0$ for a.e. $x \in \Sigma$ and Lipschitz condition (2.17) for some $\omega \geq 0$. Then B is accretive in L^1 has a c-complete resolvent and satisfies (2.37). Let F denote the Nemytski operator on L^1 associated with f. Then for $\omega = L$, $A + \omega I$ is accretive in L^1 and has a complete resolvent. If one assumes that the resolvent of A is c-complete, then our next proposition implies that $f \equiv -\omega I_{\mathbb{R}}$.

Proposition 2.14 ([10])**.** *Suppose that (Σ, μ) is a finite measure space and A is an accretive operator in $L^1(\Sigma, \mu)$ satisfying $L^\infty(\Sigma, \mu) \subseteq Rg(I + A)$. Then A has a c-complete resolvent if and only if*

$$(c, 0) \in A \text{ for all } c \in \mathbb{R}. \tag{2.37}$$

Due to Proposition 2.14, a typical example of accretive operator in L^1 with c-complete resolvent on a finite measure space (Σ, μ) is given by any second order (nonlinear) diffusion operator equipped with homogeneous Neumann boundary conditions on a bounded Lipschitz domain.

In the next proposition, we state some important properties of accretive operators in L^1 with (c-)complete resolvent for later reference.

Proposition 2.15. *If $A+\omega I$ is accretive in L^1 with complete resolvent for some $\omega \in \mathbb{R}$, then the closure $\overline{A}+\omega I$ is accretive in L^1 with complete resolvent. Further, if (Σ,μ) is a finite measure space and A is accretive in L^1 with c-complete resolvent, then the closure $\overline{A}$ is accretive in L^1 with c-complete resolvent.*

Proof. The operator $\overline{A}+\omega I$ is accretive in $L^1(\Sigma,\mu)$ by (2.18) and since $\overline{A+\omega I} = \overline{A}+\omega I$. Now, suppose that the resolvent J_λ of $A+\omega I$ is a complete mapping $J_\lambda : Rg(I+\lambda(\omega I + A)) \to D(A)$ for every $\lambda > 0$. Let $(u,v) \in \overline{A}$. Then there are sequences (u_n) and (v_n) such that $(u_n,v_n) \in A$ and u_n converges to u and v_n converges to v in $L^1(\Sigma,\mu)$. By assumption, for every $\lambda > 0$, the resolvent operator J_λ of $A+\omega I$ is a complete mapping, that is, by Proposition 2.4, for every $(u_n,v_n) \in A$, one has

$$\int_\Sigma T(u_n)\,(\omega u_n + v_n)\,\mathrm{d}\mu \geq 0 \tag{2.38}$$

for every $T \in P_0$. Since every $T \in P_0$ is Lipschitz continuous and bounded,

$$T(u_n)(u_n + \lambda(\omega u_n + v_n)) \to T(u_n)(u_n + \lambda(\omega u_n + v_n))$$

in $L^1(\Sigma,\mu)$ as $n \to \infty$. Thus, sending $n \to \infty$ in (2.38) yields

$$\int_\Sigma T(u)\,(\omega u + v)\,\mathrm{d}\mu \geq 0,$$

showing that the first statement of this proposition holds. In the case that the measure space (Σ,μ) is finite and A with c-complete resolvent, the same arguments show that $\overline{A}$ has a c-complete resolvent. □

A semigroup $\{T_t\}_{t\geq 0} \sim -A$ on $\overline{D(A)}^{L^1}$ of a quasi m-accretive operator in L^1 with complete resolvent has *exponential growth* in all $L^{\tilde{q}}$-norms

$$\|T_t u\|_{\tilde{q}} \leq e^{\omega t}\|u\|_{\tilde{q}} \quad \text{for all } t > 0,\, u \in \overline{D(A)}^{L^1} \cap L^{\tilde{q}}(\Sigma,\mu), \tag{2.39}$$

and $1 \leq \tilde{q} \leq \infty$. Similarly, a semigroup $\{T_t\}_{t\geq 0} \sim -A$ on $\overline{D(A)}^{L^1}$ of a m-accretive operator in L^1 with c-complete resolvent has *modulo a constant "c" non-increasing* $L^{\tilde{q}}$-norm

$$\|T_t u - c\|_{\tilde{q}} \leq \|u - c\|_{\tilde{q}} \quad \text{for all } t > 0,\, u \in \overline{D(A)}^{L^1} \cap L^{\tilde{q}}(\Sigma,\mu), \tag{2.40}$$

$c \in \mathbb{R}$ and $1 \leq \tilde{q} \leq \infty$. We omit the proof of these statements since they are shown similarly as the ones of Proposition 2.6.

Proposition 2.16. *Let $A+\omega I$ be an (m-) accretive operator in L^1 with complete resolvent for some $\omega \in \mathbb{R}$. Then for every $\lambda > 0$ such that $\lambda\,\omega < 1$ and every $1 \leq \tilde{q} \leq \infty$, the resolvent operator J_λ of A satisfies*

$$\|J_\lambda u\|_{\tilde{q}} \leq (1-\lambda\omega)^{-1}\|u\|_{\tilde{q}} \tag{2.41}$$

for every $u \in Rg(I+\lambda A) \cap L^{\tilde{q}}(\Sigma,\mu)$ and the semigroup $\{T_t\}_{t\geq 0} \sim -A$ on $\overline{D(A)}^{L^1}$ satisfies (2.39). If A is (m-) accretive operator in L^1 with c-complete resolvent, then for every $\lambda > 0$ and $1 \leq \tilde{q} \leq \infty$, the resolvent operator J_λ of A satisfies

$$\|J_\lambda u - c\|_{\tilde{q}} \leq \|u - c\|_{\tilde{q}} \tag{2.42}$$

for every $c \in \mathbb{R}$, $u \in Rg(I+\lambda A) \cap L^{\tilde{q}}(\Sigma,\mu)$ *and the semigroup* $\{T_t\}_{t\ge 0} \sim -A$ *on* $\overline{D(A)}^{L^1}$ *satisfies* (2.40).

The next proposition outlines the construction of an operator of the second class by taking the composition $A\phi$ of an operator A of the first class and a continuous non-decreasing function ϕ on $\mathbb{R}$. In the case $\varepsilon = 0$, the statements of this proposition are well-known (cf. [79, Proposition 11] or [10, Proposition 2.5]).

Proposition 2.17. *Let A be an accretive operator in $L^1(\Sigma,\mu)$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Suppose that one of the following hypotheses hold:*

(i) A is s-accretive in $L^1(\Sigma,\mu)$ and single-valued.
(ii) ϕ is injective.

Then, the following statements hold.

(1) For every $\varepsilon \ge 0$, the operator $\varepsilon\phi_1 + A\phi$ is accretive in $L^1(\Sigma,\mu)$.
(2) If, in addition, A has a complete resolvent and ϕ is continuous satisfying $\phi(0) = 0$ (respectively, A has a c-complete resolvent and (Σ,μ) is a finite measure space), then for every $\varepsilon \ge 0$, $\varepsilon\phi_1 + A\phi$ is accretive in L^1 with complete resolvent (respectively, with c-complete resolvent).
(3) If, in addition, A is T-accretive in $L^1(\Sigma,\mu)$ and ϕ is injective, then for every $\varepsilon \ge 0$, $\varepsilon\phi_1 + A\phi$ is T-accretive in $L^1(\Sigma,\mu)$.

Our next result provides sufficient conditions to ensure that the composition operator $A\phi$ of an operator A of the first class and a non-decreasing function ϕ on $\mathbb{R}$ satisfies the range condition (2.14) and so, $-A\phi$ generates a strongly continuous semigroup on $L^1(\Sigma,\mu)$. This result generalises [41, Proposition 2] to operators $A\phi$ for (possibly nonlinear) *m-completely accretive operators* A in L^1. For the proof of this result, we refer the interested reader to the Appendix A of this monograph.

Proposition 2.18. *Suppose A is an m-completely accretive operator in $L^q(\Sigma,\mu)$ for some $1 < q < \infty$ with $(0,0) \in A$ and $A_{1\cap\infty}$ be the trace of A on $L^1 \cap L^\infty(\Sigma,\mu)$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous, non decreasing function and for every $\lambda > 0$, β_λ be the Yosida operator of $\beta = \phi^{-1}$. Suppose that*

$$\phi(0) = 0,\ A \text{ and } \beta_\lambda \text{ satisfy (2.15) in } L^q,\ A_{1\cap\infty} \text{ and } \beta_\lambda \text{ satisfy (2.15) in } L^1, \tag{2.43}$$

and that one of the following hypotheses holds:

(i) ϕ is injective.
(ii) A is s-accretive in $L^1(\Sigma,\mu)$ and single-valued, and there are $r_0 > 0$, $K > 0$ such that

$$|\phi(s)| \le K\,|s| \qquad \text{for every } |s| \le r_0.$$

(iii) A is s-accretive in $L^1(\Sigma,\mu)$ and single-valued, and the measure space (Σ,μ) is finite.

Then, the closure $\overline{A_{1\cap\infty}\phi}$ of $A_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$ is m-accretive in $L^1(\Sigma,\mu)$ with complete resolvent. Moreover, under the hypotheses (ii) and (iii), one has

$$\begin{aligned} &\text{for every } \lambda > 0,\ f \in L^1 \cap L^\infty(\Sigma,\mu), \text{ there is } u \in L^1 \cap L^\infty(\Sigma,\mu)\\ &\text{such that } \phi(u) \in D(A_{1\cap\infty}) \text{ with } u + \lambda A_{1\cap\infty}\phi(u) \ni f. \end{aligned} \tag{2.44}$$

The class of accretive operators in L^1 with complete resolvent is invariant under perturbation by a Lipschitz continuous mapping. This is shown similarly as in the proof of Proposition 2.5. Thus, we omit the proof of the first statement of the following proposition.

Proposition 2.19. *Let A be an accretive operator in L^1 with complete resolvent and $(0,0) \in A$. Further, suppose $F : L^1(\Sigma,\mu) \to L^1(\Sigma,\mu)$ is the Nemytski operator of a Carathéodory function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ satisfying $f(x,0) = 0$ for a.e. $x \in \Sigma$ and Lipschitz condition (2.17) for some constant $\omega \geq 0$. Then, the following statements hold:*

1. *The operator $A + F + \omega I$ is accretive in L^1 with complete resolvent.*
2. *Suppose A and $\phi : \mathbb{R} \to \mathbb{R}$ satisfy the hypotheses of Proposition 2.18 and $\overline{A_{1\cap\infty}\phi}$ be the closure of $A_{1\cap\infty}\phi$. Then, $\overline{A_{1\cap\infty}\phi} + F + \omega I$ is m-accretive in $L^1(\Sigma,\mu)$ and for every $\lambda > 0$ satisfying $\lambda\omega < 1$, one has that*

$$L^1 \cap L^\infty(\Sigma,\mu) \subseteq Rg(I + \lambda(A_{1\cap\infty}\phi + F)). \tag{2.45}$$

Proof. By employing the same notation as in Proposition 2.18, $\overline{A_{1\cap\infty}\phi}$ is m-accretive in $L^1(\Sigma,\mu)$ with complete resolvent. Since $F + \omega I$ is accretive and Lipschitz continuous in $L^1(\Sigma,\mu)$, a standard fixed point argument shows that $F + \omega I$ is m-accretive in $L^1(\Sigma,\mu)$. By the continuity of $F + \omega I$ and since $\overline{A_{1\cap\infty}\phi}$ is m-accretive in $L^1(\Sigma,\mu)$, [8, Theorem 3.1] implies that $\overline{A_{1\cap\infty}\phi} + F + \omega I$ is m-accretive in $L^1(\Sigma,\mu)$.

Now, let $\lambda > 0$ such that $\lambda\omega < 1$. Then, Proposition 2.16 yields that the resolvent operator J_λ of $\overline{A_{1\cap\infty}\phi} + F$ satisfies (2.41) with respect to the L^∞-norm. Thus, for every $v \in L^1 \cap L^\infty(\Sigma,\mu)$, there is a $u \in L^\infty(\Sigma,\mu) \cap D(\overline{A_{1\cap\infty}\phi})$ such that $u + \lambda(\overline{A_{1\cap\infty}\phi}(u) + F(u)) = v$ and so, if $J_\lambda^{A_1\phi}$ denotes the resolvent of $\overline{A_{1\cap\infty}\phi}$, $J_\lambda^{\overline{A_{1\cap\infty}\phi}}[v - \lambda F(u)] = u$. On the other hand, since $v - \lambda F(u) \in L^1 \cap L^\infty(\Sigma,\mu)$ and since $A_{1\cap\infty}\phi$ satisfies the range condition (2.44), there is a $\tilde{u} \in L^1 \cap L^\infty(\Sigma,\mu)$ such that $\phi(\tilde{u}) \in D(A_{1\cap\infty})$ and $J_\lambda^{A_1\phi}[v - \lambda F(u)] = \tilde{u}$. Since $A_{1\cap\infty}\phi \subseteq \overline{A_{1\cap\infty}\phi}$, the resolvents $J_\lambda^{\overline{A_{1\cap\infty}\phi}}$ and $J_\lambda^{A_{1\cap\infty}\phi}$ coincide on $Rg(I + \lambda A_{1\cap\infty}\phi)$ and since $J_\lambda^{\overline{A_1\phi}}$ is contractive on $L^1(\Sigma,\mu)$, we obtain that $\tilde{u} = u$, implying that u satisfies $u + \lambda(A_{1\cap\infty}\phi(u) + F(u)) = v$. This shows that also the second statement of this proposition holds. □

3. Gagliardo-Nirenberg type inequalities & L^q-L^r-regularity

This section is concerned with establishing L^q-L^r-regularisation estimates for $1 \leq q, r \leq \infty$ of semigroups $\{T_t\}_{t\geq 0}$ provided their infinitesimal generator $-A$ satisfies a *Gagliardo-Nirenberg type inequality* of the form (1.11) or (1.12).

Remark 3.1. We note that for $\omega = 0$, the Gagliardo-Nirenberg type inequality (1.11) reduces to

$$\|u - u_0\|_r^\sigma \leq C\,[u - u_0, v]_q\,\|u - u_0\|_q^\varrho$$

for all $(u,v) \in A$, and the Gagliardo-Nirenberg type inequality with differences (1.12) becomes

$$\|u - \hat{u}\|_r^\sigma \leq C\,[u - \hat{u}, v - \hat{v}]_q\,\|u - \hat{u}\|_q^\varrho \tag{3.1}$$

for all (u,v), $(\hat{u},\hat{v}) \in A$, which are similar to the classical one (cf. [75]).

Further, similar to the *classical case*, for $\varrho = 0$, Gagliardo-Nirenberg type inequalities (1.11) and (1.12) reduce to the following so-called Sobolev type inequalities.

Definition 3.2. We say an operator A on $L^q(\Sigma,\mu)$ for some $1 \le q < \infty$ satisfies a *Sobolev type inequality for some* $(u_0, 0) \in A$ (respectively, with *differences*) if there exist $1 \le r \le \infty$, $\sigma > 0$, and $C > 0$ such that $(u_0, 0) \in A$ and

$$\|u - u_0\|_r^\sigma \le C\left([u - u_0, v]_q + \omega\|u - u_0\|_q^q\right)$$

for every $(u, v) \in A$ (respectively,

$$\|u - \hat{u}\|_r^\sigma \le C\left([u - \hat{u}, v - \hat{v}]_q + \omega\|u - \hat{u}\|_q^q\right)$$

for every (u, v), $(\hat{u}, \hat{v}) \in A$).

Our first main theorem of this section applies to the class of operators considered in Section 2.2.

Theorem 3.3. *Let $A + \omega I$ be an m-accretive operator on $L^q(\Sigma,\mu)$ for some $1 \le q < \infty$ and $\omega \ge 0$. Suppose A satisfies the Gagliardo-Nirenberg type inequality (1.12) for some $1 \le r \le \infty$, $\varrho \ge 0$ and $\sigma > 0$, and the semigroup $\{T_t\}_{t\ge0} \sim -A$ on $\overline{D(A)}^{L^q}$ has exponential growth (2.30) for $\tilde{q} = r$. Then $\{T_t\}_{t\ge0}$ satisfies*

$$\|T_t u - T_t \hat{u}\|_r \le \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega\beta t}\, \|u - \hat{u}\|_q^\gamma \tag{1.13}$$

for every $t > 0$ and u, $\hat{u} \in \overline{D(A)}^{L^q}$ with exponents $\alpha = \frac{1}{\sigma}$, $\beta = \gamma + 1$ and $\gamma = \frac{q+\varrho}{\sigma}$.

Remark 3.4. If $1 \le q < r \le \infty$ and if there is an element $u_0 \in \overline{D(A)}^{L^q}$ such that $T_t u_0 \in L^r(\Sigma,\mu)$ for some (all) $t > 0$, then inequality (1.13) implies that $\{T_t\}_{t\ge0}$ enjoys an *L^q-L^r-regularisation effect* in the sense that for some (all) $t > 0$, T_t maps $\overline{D(A)}^{L^q}$ into $L^r(\Sigma,\mu)$. Thus we call inequality (1.13) an *L^q-L^r-regularisation estimate* if $r > q$. If $q \ge r$ then we call (1.13) an *L^q-L^r-regularity estimate*. For example, the semigroup $\{T_t\}_{t\ge0}$ associated with the total variational flow (see [56]) satisfies inequality (1.13) for some $r < q$ and some $u_0 \in \overline{D(A)}^{L^q} \cap L^\infty(\Sigma,\mu)$ satisfying $T_t u_0 = u_0$ for all $t \ge 0$.

Remark 3.5. We want to emphasise that Theorem 3.3 implies that the parameters $1 \le r \le \infty$, $1 \le q < \infty$ and exponents $\sigma > 0$ and $\varrho \ge 0$ in L^q-L^r-regularisation estimate (1.13) are *stable* under a monotone or Lipschitz continuous perturbation. To be more specific, suppose B is an accretive operator on $L^q(\Sigma,\mu)$ satisfying the Gagliardo-Nirenberg type inequality (3.1), F be the Nemytski operator on $L^q(\Sigma,\mu)$ of a Carathéodory function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ satisfying $f(x, 0) = 0$ for a.e. $x \in \Sigma$ and Lipschitz condition (2.17) for some constant $\omega \ge 0$ and β_q the accretive operator on $L^q(\Sigma,\mu)$ associated with a monotone graph β on $\mathbb{R}$ (if $q = 1$ suppose, in addition, that $B + \beta_1$ is accretive). We set $A := B + \beta_q + F$. Then, by property (2.11) of the q-bracket $[\cdot,\cdot]_q$ and since β_q and $F + \omega I$ are accretive in $L^q(\Sigma,\mu)$, we see that

$$\begin{aligned}
&[u - \hat{u}, (v_1 + v_2 + F(u)) - (\hat{v}_1 + \hat{v}_2 + F(\hat{u}))]_q + \omega\|u - \hat{u}\|_q^q \\
&\quad = [u - \hat{u}, v_1 - \hat{v}_1]_q + [u - \hat{u}, v_2 - \hat{v}_2]_q \\
&\qquad\qquad + [u - \hat{u}, (F(u) + \omega u) - (F(\hat{u}) + \omega\hat{u})]_q
\end{aligned}$$

$$\geq [u-\hat{u}, v_1-\hat{v}_1]_q$$

for every u, $\hat{u} \in D(A)\cap D(\beta_q)$, $v_1 \in Bu$, $\hat{v}_1 \in B\hat{u}$, $v_2 \in \beta_q(u)$, $\hat{v}_2 \in \beta_q(\hat{u})$. Thus, the L^q-L^r-regularisation effect (1.13) for $1 \leq q < r \leq \infty$ of a semigroup $\{T_t\}_{t\geq 0} \sim -A$ for $A = B+F$ is only determined by B.

Remark 3.6. The statement of Theorem 3.3 remains unchanged if one replaces the constant $e^{\omega t}$ in condition (2.30) for $\tilde{q} = s$ by $M\,e^{\omega t}$ for some constant $M > 0$. Then the constant C in (1.13) has to be changed accordingly.

A common situation in applications is the one where A *is quasi m-completely accretive on* $L^2(\Sigma,\mu)$. Also, we shall see in Section 6.1 how to derive a Gagliardo-Nirenberg type inequality (1.12) for $q=2$. Therefore in practice we shall often use the following special case of Theorem 3.3.

Corollary 3.7. *Let $A+\omega I$ be an m-completely accretive operator on $L^2(\Sigma,\mu)$ for some $\omega \geq 0$. Suppose there are $1\leq r\leq\infty$, $\varrho \geq 0$, $\sigma > 0$ and $C > 0$ such that*

$$\|u-\hat{u}\|_r^{\sigma} \leq C\left[[u-\hat{u}, v-\hat{v}]_2 + \omega\|u-\hat{u}\|_2^2\right]\,\|u-\hat{u}\|_2^{\varrho} \tag{3.2}$$

for every (u,v), $(\hat{u},\hat{v}) \in A$. Then the semigroup $\{T_t\}_{t\geq 0} \sim -A$ on $\overline{D(A)}^{L^2}$ satisfies

$$\|T_tu - T_t\hat{u}\|_r \leq \left(\tfrac{C}{2}\right)^{1/\sigma}\; t^{-\alpha}\, e^{\omega\,\beta\,t}\,\|u-\hat{u}\|_2^{\gamma}$$

for every $t>0$ and u, $\hat{u} \in \overline{D(A)}^{L^2}$ with $\alpha = \frac{1}{\sigma}$, $\beta = \gamma+1$ and $\gamma = \frac{2+\varrho}{\sigma}$.

Now, we turn to the proof of Theorem 3.3. For this, we first consider the case $q > 1$. Then by (2.8), the q-brackets $[u-\hat{u}, v-\hat{v}]_q$ can be replaced by $\langle (u-\hat{u})_q, v-\hat{v}\rangle$ in inequality (1.12). Moreover, the Lebesgue space $L^q(\Sigma,\mu)$ and its dual space are uniformly convex Banach spaces and so for every $u \in D(A)$, the mild solution $t\mapsto T_tu$ is almost everywhere differentiable, everywhere differentiable from the right on $[0,\infty)$ with values in $L^q(\Sigma,\mu)$, and satisfies (2.25). Using this leads to the following short proof of Theorem 3.3 in this situation (cf. [36] in the case of linear semigroups for $\omega = 0$ and $\varrho = 1$).

First proof of Theorem 3.3 for $q>1$. First, let u, $\hat{u} \in D(A)$. By hypothesis, one has

$$\|T_tu - T_t\hat{u}\|_{\tilde{q}} \leq e^{\omega(t-s)}\|T_su - T_s\hat{u}\|_{\tilde{q}} \tag{3.3}$$

for every $t \geq s > 0$ and for every $\tilde{q} \in \{q,r\}$. Combining this with inequality (1.12) and the fact that $\frac{d}{dt_+}T_tu = -A^0T_tu$ for every $t\geq 0$ (cf. (2.25)), we see that

$$\begin{aligned}
\|u-\hat{u}\|_q^{q+\varrho} &\geq \left[\|u-\hat{u}\|_q^q - e^{-\omega\,q\,t}\|T_tu-T_t\hat{u}\|_q^q\right]\,\|u-\hat{u}\|_q^{\varrho}\\
&= \left[-\int_0^t \tfrac{d}{ds}\left(e^{-\omega\,q\,s}\|T_su-T_s\hat{u}\|_q^q\right)\,\mathrm{d}s\right]\,\|u-\hat{u}\|_q^{\varrho}\\
&= \Big[q\int_0^t e^{-\omega\,q\,s}\big(\langle (T_su-T_s\hat{u})_q, A^\circ T_su - A^\circ T_s\hat{u}\rangle\\
&\qquad\qquad + \omega\,\|T_su-T_s\hat{u}\|_q^q\big)\mathrm{d}s\Big]\,\|u-\hat{u}\|_q^{\varrho}\\
&\geq q\int_0^t e^{-\omega\,(q+\varrho)\,s}\Big[\langle (T_su-T_s\hat{u})_q, A^\circ T_su - A^\circ T_s\hat{u}\rangle\\
&\qquad\qquad + \omega\,\|T_su-T_s\hat{u}\|_q^q\Big]\,\|T_su-T_s\hat{u}\|_q^{\varrho}\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{q}{C} \int_0^t e^{-\omega\,(q+\varrho)\,s} \|T_s u - T_s \hat{u}\|_r^\sigma \,\mathrm{d}s\\
&\geq \frac{q}{C} \left(\int_0^t e^{-\omega\,(q+\varrho)\,s}\, e^{-\omega(t-s)\sigma}\,\mathrm{d}s\right) \|T_t u - T_t \hat{u}\|_r^\sigma.\\
&\geq \frac{q}{C} \left(\int_0^t e^{-\omega\,(q+\varrho)\,s}\, e^{-\omega t\sigma}\,\mathrm{d}s\right) \|T_t u - T_t \hat{u}\|_r^\sigma.\\
&\geq \frac{q}{C'}\, t\, e^{-\omega(q+\varrho+\sigma)\,t} \,\|T_t u - T_t \hat{u}\|_r^\sigma,
\end{aligned}$$

showing that inequality (1.13) holds for $u, \hat{u} \in D(A)$.

Now, let $u, \hat{u} \in \overline{D(A)}^{L^q}$. Then, there are sequences (u_n) and $(\hat{u}_n)$ in $D(A)$ such that u_n converges to u and $\hat{u}_n$ converges to $\hat{u}$ in $L^q(\Sigma,\mu)$. Since the semigroup $\{T_t\}_{t\geq 0}$ has exponential growth (2.30) for $\tilde{q} = q$, for every $t \geq 0$, the sequence $S_n(t) := T_t u_n - T_t \hat{u}_n$ converges to $S(t) := T_t u - T_t \hat{u}$ in $L^q(\Sigma,\mu)$. Moreover, by the first step of this proof, inequality (1.13) implies that

$$\|S_n(t)\|_r \leq \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega \frac{q+\varrho+\sigma}{\sigma}\,t}\, \|u_n - \hat{u}_n\|_q^{\frac{q+\varrho}{\sigma}}$$

for every n. Since the L^r-norm is lower semicontinuous on $L^q(\Sigma,\mu)$, sending $n \to \infty$ in the previous inequality yields $S(t) \in L^r(\Sigma,\mu)$ and

$$\|S(t)\|_r \leq \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega \frac{q+\varrho+\sigma}{\sigma}\,t}\, \|u - \hat{u}\|_q^{\frac{q+\varrho}{\sigma}}.$$

Therefore inequality (1.13) holds for every $u, \hat{u} \in \overline{D(A)}^{L^q}$, completing the proof of Theorem 3.3 for $q > 1$. □

Our second proof of Theorem 3.3 is rather technical and uses the definition of mild solutions (cf. [92] in the case $\omega = \varrho = 1$).

Second proof of Theorem 3.3. Let $u, \hat{u} \in \overline{D(A)}^{L^q}$. For given $t > 0$, we choose $N \geq 1$ large enough such that $\frac{\omega\, q\, t}{N} < \frac{1}{2}$ and set $t_n = n\,\frac{t}{N}$ for every $n = 0, \dots, N$, $u_0 = u$ and $\hat{u}_0 = \hat{u}$. By hypothesis, $Rg(I + \frac{\lambda}{1-\lambda\omega}A) = L^q(\Sigma,\mu)$ for every $0 < \lambda < \frac{1}{\omega}$. Thus, there are u_1, $\hat{u}_1 \in D(A)$ solving $u_1 + \frac{t}{N}Au_1 \ni u_0$ and $\hat{u}_1 + \frac{t}{N}A\hat{u}_1 \ni \hat{u}_0$. Iteratively, for every $n = 1, \dots, N$, there are solutions u_n and $\hat{u}_n \in D(A)$ of

$$u_n + \tfrac{t}{N}Au_n \ni u_{n-1} \quad \text{and} \quad \hat{u}_n + \tfrac{t}{N}A\hat{u}_n \ni \hat{u}_{n-1}, \tag{3.4}$$

respectively. We set

$$U_N(s) = u_0 \mathbb{1}_{\{t_0=0\}}(s) + \sum_{n=1}^N u_n\, \mathbb{1}_{(t_{n-1},t_n]}(s)$$

and

$$\hat{U}_N(s) = \hat{u}_0 \mathbb{1}_{\{t_0=0\}}(s) + \sum_{n=1}^N \hat{u}_n\, \mathbb{1}_{(t_{n-1},t_n]}(s)$$

for every $s \in [0,t]$. Further, for $v_n = (u_{n-1} - u_n)\frac{N}{t}$ and $\hat{v}_n = (\hat{u}_{n-1} - \hat{u}_n)\frac{N}{t}$, both inclusions in (3.4) can be rewritten as $v_n \in Au_n$ and $\hat{v}_n \in A\hat{u}_n$, or as $J_{t/N}u_{n-1} = u_n$ and $J_{t/N}\hat{u}_{n-1} = \hat{u}_n$ for every $n = 1, \dots, N$. Hence by Gagliardo-Nirenberg type inequalities (1.12), (2.11) and (2.10), we see that

$$\|u_n - \hat{u}_n\|_r^\sigma$$

$$
\begin{aligned}
&\leq C\Big([u_n - \hat{u}_n, v_n - \hat{v}_n]_q + \omega \|u_n - \hat{u}_n\|_q^q \Big) \, \|u_n - \hat{u}_n\|_q^{\varrho} \\
&= C\,\tfrac{N}{t}\Big([u_n - \hat{u}_n, (u_{n-1} - u_n) - (\hat{u}_{n-1} - \hat{u}_n)]_q + \tfrac{\omega\, t}{N} \|u_n - \hat{u}_n\|_q^q \Big) \, \|u_n - \hat{u}_n\|_q^{\varrho} \\
&= C\,\tfrac{N}{t}\Big([u_n - \hat{u}_n, (u_{n-1} - \hat{u}_{n-1}) - (u_n - \hat{u}_n)]_q + \tfrac{\omega\, t}{N} \|u_n - \hat{u}_n\|_q^q \Big) \, \|u_n - \hat{u}_n\|_q^{\varrho} \\
&\leq C\,\tfrac{N}{t}\Big(\tfrac{1}{q}\|u_{n-1} - \hat{u}_{n-1}\|_q^q - (1 - \tfrac{\omega\, q\, t}{N})\tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q \Big) \, \|u_n - \hat{u}_n\|_q^{\varrho}
\end{aligned}
$$

for every $n = 1, \dots, N$. By assumption, $J_{t/N}$ satisfies inequality (2.29) for $\tilde{q} = q$. Hence

$$
\begin{aligned}
\|u_n - \hat{u}_n\|_q &= \|J_{t/N} u_{n-1} - J_{t/N} \hat{u}_{n-1}\|_q \\
&\leq (1 - \tfrac{t\omega}{N})^{-1} \, \|u_{n-1} - \hat{u}_{n-1}\|_q \\
&\;\;\vdots \\
&\leq (1 - \tfrac{t\omega}{N})^{-n} \|u_0 - \hat{u}_0\|_q \\
&\leq (1 - \tfrac{t\omega}{N})^{-N} \|u_0 - \hat{u}_0\|_q
\end{aligned}
$$

Using this in order to estimate the term $\|u_n - \hat{u}_n\|_q^{\varrho}$ in the previous inequality and multiplying the resulting inequality by $\frac{t}{N}\,(1 - \frac{\omega\, q\, t}{N})^{-1}$ yields

$$
\begin{aligned}
&\tfrac{t}{N}(1 - \tfrac{\omega\, q\, t}{N})^{-1} \|u_n - \hat{u}_n\|_r^{\sigma} \\
&\quad\leq C\Big((1 - \tfrac{\omega\, q\, t}{N})^{-1} \tfrac{1}{q}\|u_{n-1} - \hat{u}_{n-1}\|_q^q - \tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (1 - \tfrac{t\omega}{N})^{-N\varrho} \|u_0 - \hat{u}_0\|_q^{\varrho}.
\end{aligned}
$$

Rearranging the last inequality gives

$$
\tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q \leq (1 - \tfrac{\omega\, q\, t}{N})^{-1} \, \tfrac{1}{q}\|u_{n-1} - \hat{u}_{n-1}\|_q^q + b_n
$$

for every $n = 1, \dots, N$, where we set

$$
b_n := -\tfrac{t}{N}(1 - \tfrac{\omega\, q\, t}{N})^{-1} \, \|u_n - \hat{u}_n\|_r^{\sigma} \, C^{-1} \, (1 - \tfrac{t\omega}{N})^{N\varrho} \, \|u_0 - \hat{u}_0\|_q^{-\varrho}. \tag{3.5}
$$

It is easy to see that

$$
\left\{
\begin{array}{l}
\text{for sequences } (\lambda_n) \subseteq [0, \infty) \text{ and } (a_n), (b_n) \subseteq \mathbb{R} \text{ satisfying} \\
a_n \leq \lambda_n a_{n-1} + b_n \text{ for all } n = 1, \dots, N, \text{ one has that} \\
\qquad a_N \leq a_0 \left(\displaystyle\prod_{n=1}^{N} \lambda_n \right) + \displaystyle\sum_{n=1}^{N} b_n \left(\displaystyle\prod_{k=n+1}^{N} \lambda_n \right)
\end{array}
\right. \tag{3.6}
$$

(cf. [15, Exercise E3.8]). Applying this to $\lambda_n = (1 - \frac{\omega\, q\, t}{N})^{-1}$, $a_n = \frac{1}{q}\|u_n - \hat{u}_n\|_q^q$ and b_n given by (3.5), we obtain

$$
\tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q \leq (1 - \tfrac{\omega\, q\, t}{N})^{-N} \, \tfrac{1}{q}\|u_0 - \hat{u}_0\|_q^q + \sum_{n=1}^{N} (1 - \tfrac{\omega\, q\, t}{N})^{-(N-(n+1))} b_n.
$$

Using that $(1 - \frac{\omega\, q\, t}{N})^n \leq (1 - \frac{\omega\, q\, t}{N})^N$ and rearranging this inequality yields

$$
(1 - \tfrac{\omega\, q\, t}{N})^{N} \tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q
$$

$$+ C^{-1}\,(1-\tfrac{t\omega}{N})^{N\varrho}\,\|u_0-\hat{u}_0\|_q^{-\varrho}\,(1-\tfrac{\omega\,q\,t}{N})^N\sum_{n=1}^{N}\tfrac{t}{N}\|u_n-\hat{u}_n\|_r^{\sigma}$$
$$\le \tfrac{1}{q}\|u_0-\hat{u}_0\|_q^q$$

so that

$$(1-\tfrac{\omega\,q\,t}{N})^N\tfrac{1}{q}\|U_N(t)-\hat{U}_N(t)\|_q^q$$
$$+ C^{-1}\,(1-\tfrac{t\omega}{N})^{N\varrho}\,\|u-\hat{u}\|_q^{-\varrho}\,(1-\tfrac{\omega\,q\,t}{N})^N\int_0^t\|U_N(s)-\hat{U}_N(s)\|_r^{\sigma}\,\mathrm{d}s$$
$$\le \tfrac{1}{q}\|u-\hat{u}\|_q^q$$

By the Crandall-Liggett theorem,

$$\lim_{N\to\infty} U_N = T_t u \quad \text{in } L^q(\Sigma,\mu) \quad \text{and} \quad \lim_{N\to\infty}\hat{U}_N = T_t\hat{u} \quad \text{in } L^q(\Sigma,\mu)$$

respectively uniformly on $[0,t]$. Thus, sending $N\to\infty$ in the previous estimate and using the lower semicontinuity of the L^r-norm on $L^q(\Sigma,\mu)$ yields

$$e^{-\omega\,q\,t}\,\tfrac{1}{q}\|T_tu-T_t\hat{u}\|_q^q + C^{-1}\,e^{-\omega\,\varrho\,t}\,\|u-\hat{u}\|_q^{-\varrho}\,e^{-\omega\,q\,t}\int_0^t\|T_su-T_s\hat{u}\|_r^{\sigma}\,\mathrm{d}s$$
$$\le \tfrac{1}{q}\|u-\hat{u}\|_q^q$$

and so

$$C^{-1}\,e^{-\omega\,\varrho\,t}\,\|u-\hat{u}\|_q^{-\varrho}\,e^{-\omega\,q\,t}\int_0^t\|T_su-T_s\hat{u}\|_r^{\sigma}\,\mathrm{d}s\le\tfrac{1}{q}\|u-\hat{u}\|_q^q$$

By assumption, $\{T_t\}_{t\ge0}$ satisfies (3.3) for $\tilde{q}=r$ from where we can deduce that (1.13) holds. □

Even for the class of quasi-m-accretive operators A on L^q, there are situations in which the operator A merely satisfies the Gagliardo-Nirenberg type inequality (1.11) for some $(u_0,0)\in A$. In this situation, we can state the following result.

Theorem 3.8. *Let $A+\omega I$ be an m-accretive operator on $L^q(\Sigma,\mu)$ for some $1\le q<\infty$ and $\omega\ge 0$. Suppose A satisfies the Gagliardo-Nirenberg type inequality (1.11) for parameters $1\le r\le\infty$, $\varrho\ge0$, $\sigma>0$ and some $(u_0,0)\in A$ satisfying $u_0\in L^q\cap L^r(\Sigma,\mu)$, and the semigroup $\{T_t\}_{t\ge0}\sim -A$ on $\overline{D(A)}^{L^q}$ has exponential growth (2.30) for $\tilde{q}=r$. Then the semigroup $\{T_t\}_{t\ge0}$ satisfies*

$$(1.17)\qquad \|T_tu-u_0\|_r\le\left(\tfrac{C}{q}\right)^{1/\sigma}\;t^{-\alpha}\,e^{\omega\beta t}\,\|u-u_0\|_q^{\gamma}$$

for every $t>0$, $u\in L^q(\Sigma,\mu)$ with exponents $\alpha=\frac{1}{\sigma}$, $\beta=\gamma+1$ and $\gamma=\frac{q+\varrho}{\sigma}$.

We omit the proof of Theorem 3.8 since it proceeds along the lines of the second proof of Theorem 3.3.

Analogously, as above, the important case $q=2$ and A is quasi m-completely accretive operator on $L^2(\Sigma,\mu)$ follows immediately from Theorem (3.8).

Corollary 3.9. *Let $A+\omega I$ be m-completely accretive operator on $L^2(\Sigma,\mu)$ for some $\omega\ge0$. Suppose there are $(u_0,0)\in A$, $2<r\le\infty$, $\varrho\ge0$, $\sigma>0$ and $C>0$ such that*

$$(3.7)\qquad \|u-u_0\|_r^{\sigma}\le C\left[[u-u_0,v]_2+\omega\|u-u_0\|_2^2\right]\,\|u-u_0\|_2^{\varrho}$$

for every $(u,v) \in A$. Then the semigroup $\{T_t\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^2}$ satisfies

$$\|T_t u - u_0\|_r \le \left(\tfrac{C}{2}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega\beta t}\, \|u-u_0\|_2^{\gamma}$$

for every $t>0$ and $u \in \overline{D(A)}^{L^2}$ with exponents $\alpha = \frac{1}{\sigma}$, $\beta = \gamma+1$ and $\gamma = \frac{2+\varrho}{\sigma}$.

Our third main theorem of this section considers the second class of operators introduced in Section 2.3. As a matter of fact, many examples show that the Gagliardo-Nirenberg type inequality (1.11) is not satisfied by a quasi m-accretive operator A in $L^1(\Sigma,\mu)$ with (c-)complete resolvent. But in order to obtain L^q-L^r-regularisation estimates with $1\le q,r\le\infty$ for the semigroup $\{T_t\}_{t\ge0}\sim -A$ on $\overline{D(A)}^{L^1}$, it turns out that it is sufficient that for some $1\le q\le q_0\le\infty$, the *trace*

$$A_{1\cap q_0} := A \cap ((L^1\cap L^{q_0}(\Sigma,\mu)) \times (L^1\cap L^{q_0}(\Sigma,\mu))$$

of A on $L^1\cap L^{q_0}(\Sigma,\mu)$ satisfies (1.11). Note that, for $1\le q\le q_0\le\infty$, $L^1\cap L^{q_0}(\Sigma,\mu)$ injects continuously into $L^q(\Sigma,\mu)$. Hence, then trace $A_{1\cap q_0}$ is contained in the *part* $A_q := A\cap(L^q\times L^q(\Sigma,\mu))$ of A in $L^q(\Sigma,\mu)$.

Theorem 3.10. *Let $A+\omega I$ be m-accretive in $L^1(\Sigma,\mu)$ for some $\omega\ge 0$. Suppose, there are $1\le q\le q_0\le\infty$, ($q<\infty$), such that the trace $A_{1\cap q_0}$ of A on $L^1\cap L^{q_0}(\Sigma,\mu)$ satisfies the range condition*

$$L^1\cap L^{q_0}(\Sigma,\mu) \subseteq Rg(I+(A_{1\cap q_0}+\omega I)), \tag{3.8}$$

and the Gagliardo-Nirenberg type inequality (1.11) for some $1\le r\le\infty$, $\varrho\ge 0$, $\sigma>0$ and $(u_0,0)\in A_{1\cap q_0}$, and for every $\lambda>0$ satisfying $\lambda\omega<1$, the resolvent J_λ of A satisfies

$$\|J_\lambda u - u_0\|_{\tilde q} \le (1-\lambda\omega)^{-1}\, \|u-u_0\|_{\tilde q} \tag{3.9}$$

for $\tilde q = r$, every $u\in Rg(I+\lambda A_{1\cap q_0})$, and for $\tilde q = q$ provided $\varrho>0$. Then the semigroup $\{T_t\}_{t\ge0}\sim -A$ on $\overline{D(A)}^{L^1}$ satisfies inequality (1.17) for every $t>0$ and $u\in\overline{D(A)}^{L^1}\cap L^{q_0}(\Sigma,\mu)$ with exponents $\alpha=\frac{1}{\sigma}$, $\beta=\gamma+1$ and $\gamma=\frac{q+\varrho}{\sigma}$.

Remark 3.11. One easily verifies that a similar statement as given in Remark 3.5 holds for accretive operators in $L^1(\Sigma,\mu)$. More precisely, for an m-accretive operator A on $L^1(\Sigma,\mu)$ satisfying the hypotheses of Theorem 3.10 with $\omega=0$ and a Lipschitz continuous mapping $F: L^1(\Sigma,\mu)\to L^1(\Sigma,\mu)$ with $F(0)=0$ and Lipschitz constant $L>0$, if the trace $A_{1\cap q_0}$ of A on $L^1\cap L^{q_0}(\Sigma,\mu)$ satisfies (3.8) for $\omega=0$ and satisfies the Gagliardo-Nirenberg type inequality (1.11) for $(u_0,0)$ and $\omega=0$, then $A_{1\cap q_0}+F$ satisfies the Gagliardo-Nirenberg type inequality (1.11) for $(u_0,0)$ and $\omega=L$.

From Theorem 3.10, we can immediately conclude the following result concerning quasi m-accretive operators in L^1 with complete resolvent.

Corollary 3.12. *Let $A+\omega I$ be m-accretive operator in $L^1(\Sigma,\mu)$ with complete resolvent for some $\omega\ge0$. Suppose, there are $1\le q\le q_0\le\infty$, ($q<\infty$), such that the trace $A_{1\cap q_0}$ of A in $L^1\cap L^{q_0}(\Sigma,\mu)$ satisfies range condition (3.8) and Gagliardo-Nirenberg type inequality (1.11) for some $1\le r\le\infty$, $\varrho\ge0$, $\sigma>0$ and $(0,0)\in A_{1\cap q_0}$. Then the semigroup $\{T_t\}_{t\ge0}\sim -A$ on $\overline{D(A)}^{L^1}$ satisfies*

$$\|T_t u\|_r \le \left(\frac{C}{q}\right)^{1/\sigma} e^{\omega(\gamma+1)t}\, t^{-\alpha} \|u\|_q^{\gamma} \tag{3.10}$$

for every $t > 0$ *and* $u \in \overline{D(A)}^{L^1} \cap L^{q_0}(\Sigma,\mu)$ *with exponents* $\alpha = \frac{1}{\sigma}$, $\beta = \gamma + 1$ *and* $\gamma = \frac{q+\varrho}{\sigma}$.

Furthermore, by Theorem 3.10, we can deduce the following result concerning m-accretive operators in L^1 with c-complete resolvent.

Corollary 3.13. *Let A be an m-accretive operator in $L^1(\Sigma,\mu)$ with c-complete resolvent. Suppose, there are $1 \le q \le q_0 \le \infty$, $(q < \infty)$, such that the trace $A_{1\cap q_0}$ of A on $L^1 \cap L^{q_0}(\Sigma,\mu)$ satisfies the range condition (3.8) and the Gagliardo-Nirenberg type inequality (1.11) for some $1 \le r \le \infty$, $\varrho \ge 0$, $\sigma > 0$ and $c \in \mathbb{R}$ with $(c, 0) \in A_{1\cap q_0}$. Then the semigroup $\{T_t\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^1}$ satisfies*

$$\|T_t u - c\|_r \le \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha} \|u - c\|_q^{\gamma}$$

for every $t > 0$ *and* $u \in \overline{D(A)}^{L^1} \cap L^{q_0}(\Sigma,\mu)$ *with exponents* $\alpha = \frac{1}{\sigma}$ *and* $\gamma = \frac{q+\varrho}{\sigma}$.

Proof of Theorem 3.10. Let $u \in \overline{D(A)}^{L^1} \cap L^{q_0}(\Sigma,\mu)$ and for given $t > 0$, let $N \ge 1$ be large enough such that $\frac{t\omega q}{N} < \frac{1}{2}$. Then, we set $t_n = n\,\frac{t}{N}$ for every $n = 0, \dots, N$ and $\hat{u}_0 = u$. By range condition (3.8), for every $n = 1, \dots, N$, there is iteratively a $\hat{u}_n \in D(A_{1\cap q_0})$ satisfying

$$\hat{u}_n + \tfrac{t}{N} A_{1\cap q_0} \hat{u}_n \ni \hat{u}_{n-1}. \tag{3.11}$$

We set

$$\hat{U}_N(s) = \hat{u}_0 \mathbb{1}_{\{t_0=0\}}(s) + \sum_{n=1}^{N} \hat{u}_n \, \mathbb{1}_{(t_{n-1},t_n]}(s)$$

for every $s \in [0,t]$ and $\hat{v}_n = (\hat{u}_{n-1} - \hat{u}_n)\frac{N}{t}$. Then, inclusions (3.11) can be rewritten as $\hat{v}_n \in A_{1\cap q_0}\hat{u}_n$ or as $J_{t/N}\hat{u}_{n-1} = \hat{u}_n$ for every $n = 1, \dots, N$. Hence, since $A_{1\cap q_0}$ satisfies Gagliardo-Nirenberg type inequalities (1.11) with $(u_0, 0) \in A_{1\cap q_0}$, we see that by using (2.11) and (2.10) that

$$\begin{aligned}
&\|\hat{u}_n - u_0\|_r^{\sigma} \\
&\le C \left([\hat{u}_n - u_0, \hat{v}_n]_q + \omega \|\hat{u}_n - u_0\|_q^q \right) \|\hat{u}_n - u_0\|_q^{\varrho} \\
&= C\,\tfrac{N}{t} \left([\hat{u}_n - u_0, \hat{u}_{n-1} - \hat{u}_n]_q + \tfrac{\omega t}{N} \|\hat{u}_n - u_0\|_q^q \right) \|\hat{u}_n - u_0\|_q^{\varrho} \\
&= C\,\tfrac{N}{t} \left([\hat{u}_n - u_0, (\hat{u}_{n-1} - u_0) - (\hat{u}_n - u_0)]_q + \tfrac{\omega t}{N} \|\hat{u}_n - u_0\|_q^q \right) \|\hat{u}_n - u_0\|_q^{\varrho} \\
&\le C\,\tfrac{N}{t} \left(\tfrac{1}{q} \|\hat{u}_{n-1} - u_0\|_q^q - (1 - \tfrac{\omega q t}{N}) \tfrac{1}{q} \|\hat{u}_n - u_0\|_q^q \right) \|\hat{u}_n - u_0\|_q^{\varrho}
\end{aligned}$$

for every $n = 1, \dots, N$. By assumption, the resolvent operator $J_{t/N}$ of A satisfies inequality (3.9) for $\tilde{q} = q$ provided $\varrho > 0$. Then,

$$\begin{aligned}
\|\hat{u}_n - u_0\|_q &= \|J_{t/N}\hat{u}_{n-1} - u_0\|_q \\
&\le (1 - \tfrac{t\omega}{N})^{-1} \|\hat{u}_{n-1} - u_0\|_q \\
&\ \ \vdots \\
&\le (1 - \tfrac{t\omega}{N})^{-n} \|\hat{u}_0 - u_0\|_q \\
&\le (1 - \tfrac{t\omega}{N})^{-N} \|\hat{u}_0 - u_0\|_q.
\end{aligned}$$

Applying this to the previous inequality, in order to estimate $\|\hat{u}_n - u_0\|_q^{\varrho}$ and multiplying the resulting inequality by $\frac{t}{N}\,(1-\frac{\omega\, q\, t}{N})^{-1}$ yields

$$\begin{aligned}
&\tfrac{t}{N}(1-\tfrac{\omega\, q\, t}{N})^{-1}\|\hat{u}_n - u_0\|_r^{\sigma}\\
&\quad\le C\left((1-\tfrac{\omega\, q\, t}{N})^{-1}\,\tfrac{1}{q}\|\hat{u}_{n-1} - u_0\|_q^{q} - \tfrac{1}{q}\|\hat{u}_n - u_0\|_q^{q}\right)\\
&\qquad\qquad\qquad\qquad\times(1-\tfrac{t\omega}{N})^{-N\varrho}\|\hat{u}_0 - u_0\|_q^{\varrho}.
\end{aligned}$$

Rearranging this inequality yields

$$\tfrac{1}{q}\|\hat{u}_n - u_0\|_q^{q} \le (1-\tfrac{\omega\, q\, t}{N})^{-1}\,\tfrac{1}{q}\|\hat{u}_{n-1} - u_0\|_q^{q} + b_n$$

with

$$b_n := -\tfrac{t}{N}(1-\tfrac{\omega\, q\, t}{N})^{-1}\,\|\hat{u}_n - u_0\|_r^{\sigma}\,C^{-1}\,(1-\tfrac{t\omega}{N})^{N\varrho}\,\|\hat{u}_0 - u_0\|_q^{-\varrho}.$$

for every $n = 1, \dots, N$. By auxiliary inequality (3.6),

$$\tfrac{1}{q}\|\hat{u}_n - u_0\|_q^{q} \le (1-\tfrac{\omega\, q\, t}{N})^{-N}\,\tfrac{1}{q}\|\hat{u}_0 - u_0\|_q^{q} + \sum_{n=1}^{N}(1-\tfrac{\omega\, q\, t}{N})^{-(N-(n+1))}b_n.$$

Rearranging this inequality and using that $(1-\frac{\omega\, q\, t}{N})^{n} \le (1-\frac{\omega\, q\, t}{N})^{N}$ gives

$$\begin{aligned}
&(1-\tfrac{\omega\, q\, t}{N})^{N}\tfrac{1}{q}\|\hat{u}_n - u_0\|_q^{q}\\
&\qquad + C^{-1}\,(1-\tfrac{t\omega}{N})^{N\varrho}\,\|\hat{u}_0 - u_0\|_q^{-\varrho}\,(1-\tfrac{\omega\, q\, t}{N})^{N}\sum_{n=1}^{N}\tfrac{t}{N}\|\hat{u}_n - u_0\|_r^{\sigma}\\
&\le \tfrac{1}{q}\|\hat{u}_0 - u_0\|_q^{q}
\end{aligned}$$

and so,

$$\begin{aligned}
&(1-\tfrac{\omega\, q\, t}{N})^{N}\tfrac{1}{q}\|\hat{U}_N(t) - u_0\|_q^{q}\\
&\qquad + C^{-1}\,(1-\tfrac{t\omega}{N})^{N\varrho}\,\|u - u_0\|_q^{-\varrho}\,(1-\tfrac{\omega\, q\, t}{N})^{N}\int_0^t\|\hat{U}_N(s) - u_0\|_r^{\sigma}\,\mathrm{d}s\\
&\le \tfrac{1}{q}\|u - u_0\|_q^{q}.
\end{aligned}\tag{3.12}$$

Recall that $A_{1\cap q_0} \subseteq A$ and, by assumption, $A + \omega I$ is m-accretive in $L^1(\Sigma, \mu)$. Thus, the Crandall-Liggett theorem yields

$$\lim_{N\to\infty}\hat{U}_N = T_t u \qquad \text{in } L^1(\Sigma,\mu) \text{ uniformly on } [0,t].$$

Since the L^r- and L^q-norm on $L^1(\Sigma, \mu)$ are lower semicontinuous in $L^1(\Sigma, \mu)$, sending $N \to \infty$ in (3.12) and applying Fatou's Lemma yields

$$\begin{aligned}
&e^{-\omega\, q\, t}\,\tfrac{1}{q}\|T_t u - u_0\|_q^{q} + C^{-1}\,e^{-\omega\,\varrho\, t}\,\|u - u_0\|_q^{-\varrho}\,e^{-\omega\, q\, t}\int_0^t\|T_s u - u_0\|_r^{\sigma}\,\mathrm{d}s\\
&\qquad\le \tfrac{1}{q}\|u - u_0\|_q^{q}
\end{aligned}$$

and so

$$C^{-1}\,e^{-\omega\,\varrho\, t}\,\|u - u_0\|_q^{-\varrho}\,e^{-\omega\, q\, t}\int_0^t\|T_s u - u_0\|_r^{\sigma}\,\mathrm{d}s \le \tfrac{1}{q}\|u - u_0\|_q^{q}.\tag{3.13}$$

Now, fix $\tilde{u} \in \overline{D(A)}^{L^1} \cap L^r(\Sigma,\mu)$. Then, applying (3.9) iteratively for $\tilde{q} = r$, we see that

$$\begin{aligned}
\|J^n_{t/n}\tilde{u} - u_0\|_r &= \|J^{n-1}_{t/n}(J_{t/n}\tilde{u}) - u_0\|_r \\
&\le (1 - \tfrac{t\omega}{n})^{-1}\,\|J^{n-1}_{t/n}\tilde{u} - u_0\|_r \\
&\;\;\vdots \\
&\le (1 - \tfrac{t\omega}{n})^{-n}\|\tilde{u} - u_0\|_r
\end{aligned} \tag{3.14}$$

for every $t > 0$ and integer $n \ge 1$ such that $\frac{t}{n}\omega < 1$. By Euler's formula (2.23) and since $\tilde{u} \in \overline{D(A)}^{L^1}$,

$$\lim_{n\to\infty} J^n_{t/n}\tilde{u} = T_t\tilde{u} \qquad \text{in } L^1(\Sigma,\mu)$$

for every $t > 0$. Since the L^r-norm is lower semicontinuous on $L^1(\Sigma,\mu)$, sending $n \to \infty$ in (3.14) yields

$$\|T_t\tilde{u} - u_0\|_r \le e^{\omega t}\|\tilde{u} - u_0\|_r$$

for every $t > 0$ hence, by using the semigroup property of $\{T_t\}_{t\ge 0}$, it follows that

$$\|T_t\tilde{u} - u_0\|_r \le e^{\omega(t-s)}\|T_s\tilde{u} - u_0\|_r$$

for every $t \ge s > 0$ and $\tilde{u} \in \overline{D(A)}^{L^1} \cap L^r(\Sigma,\mu)$. Applying this inequality to the integrand in (3.13), we see that (1.17) holds. □

4. Nonlinear extrapolation

The aim of this section is to provide simple and sufficient conditions such that an L^q-L^r-regularisation estimate of the type (1.13) or (1.17) for some $1 < q < r \le \infty$ satisfied by a nonlinear semigroup $\{T_t\}_{t\ge 0}$ can be extrapolated to an L^s-L^r-regularisation estimate for every $1 \le s < q$ (this we call below *extrapolation towards* L^1) and such that an L^q-L^r-regularity estimate of the type (1.13) or (1.17) for some $1 < q, r < \infty$ can be extrapolated to an $L^{\tilde{q}}$-L^∞-regularisation estimate for some $1 \le \tilde{q} < \infty$ (*extrapolation towards* L^∞). We note that in order to extrapolate towards L^∞, the relation $r > q$ is not important, but one rather needs that the relation $\gamma r > q$ holds for the exponent $\gamma > 0$ in the estimates (1.13) and (1.17) (cf. Theorems 1.2 and 1.4 or Theorems 4.10 and 4.13). Further, the iteration method (Lemma 4.12 and Lemma 4.14) used to establish $L^{\tilde{q}}$-L^∞-regularisation works if $1 \le \tilde{q} < \infty$ is chosen sufficiently large. Thus, if one starts from an L^q-L^r-regularity estimate for some $1 < q, r < \infty$ then, first, one extrapolates towards L^∞, and then one extrapolates towards L^1.

The extrapolation towards L^∞ being more involved, we shall begin in Section 4.1 by extrapolating towards L^1, or, more precisely, towards L^s for any $1 \le s < q$. Section 4.2 is concerned with a new nonlinear interpolation result which provides the fundamental auxiliary tool to establish our *extrapolation result towards* L^∞ presented in Section 4.3.

4.1. **Extrapolation towards L^1.** This subsection is dedicated to giving a nonlinear version of [35, Lemme 1] (see also [36, Section I]). The first extrapolation result of this subsection is adapted to semigroups generated by completely accretive operators (see Section 2.2) satisfying the L^q-L^r-regularising effect (1.13) for *differences* and $1 < q < r \le \infty$.

Theorem 4.1. *Let $1 \le s < q < r \le \infty$ and $\{T_t\}_{t\ge 0}$ be a semigroup acting on some subset D of $L^q(\Sigma,\mu)$ with exponential growth (2.30) for $\tilde{q} = s$ and some $\omega \ge 0$. Suppose there exist $\alpha > 0$, β, $\gamma > 0$ and $C > 0$ such that*

$$\|T_t u - T_t\hat{u}\|_r \le C\, t^{-\alpha}\, e^{\omega\beta t}\, \|u-\hat{u}\|_q^{\gamma} \tag{1.13}$$

for every $t > 0$ and u, $\hat{u} \in D$. For $\theta_s = \frac{(r-q)s}{q(r-s)} > 0$ if $r < \infty$ and $\theta_s = \frac{s}{q}$ if $r = \infty$, assume that

$$\gamma(1-\theta_s) < 1. \tag{4.1}$$

Then one has

$$\|T_t u - T_t\hat{u}\|_r \le \big(C\, 2^{\frac{\alpha}{1-\gamma(1-\theta_s)}}\big)^{\frac{1}{1-\gamma(1-\theta_s)}}\; t^{-\alpha_s}\; e^{\omega\beta_s t}\|u-\hat{u}\|_s^{\gamma_s} \tag{4.2}$$

for every $t > 0$ and u, $\hat{u} \in D \cap L^s(\Sigma,\mu)$ with exponents

$$\alpha_s = \frac{\alpha}{1-\gamma(1-\theta_s)},\qquad \beta_s = \frac{(\beta/2)+\gamma\,\theta_s}{1-\gamma(1-\theta_s)},\qquad \gamma_s = \gamma\frac{\theta_s}{1-\gamma(1-\theta_s)}. \tag{4.3}$$

Remark 4.2. The statement of Theorem 4.1 remains unchanged if one replaces the constant $e^{\omega t}$ in condition (2.30) for $\tilde{q} = s$ by $M\, e^{\omega t}$ for some constant $M > 0$. Then the constant C in (4.2) has to be changed accordingly.

Proof of Theorem 4.1. We outline the proof only for $r < \infty$ since the case $r = \infty$ is treated similarly. Then, set $\theta_s = \frac{(r-q)s}{q(r-s)}$ and assume that (4.1) holds. For $\theta := 1-\gamma(1-\theta_s)$, u, $\hat{u} \in L^s(\Sigma,\mu)\cap D$ satisfying $u \neq \hat{u}$ and $T > 0$, set

$$C_{u,\hat{u},T} := \sup_{t\in[0,T]} \frac{t^{\alpha/\theta}\|T_t u - T_t\hat{u}\|_r}{e^{\omega\gamma_s t}\,\|u-\hat{u}\|_s^{\gamma_s}}.$$

By (1.13) and since θ_s satisfies $\frac{1}{q} = \frac{(1-\theta_s)}{r} + \frac{\theta_s}{s}$, Hölder's inequality imply

$$\begin{aligned}\|T_t u - T_t\hat{u}\|_r &\le C\, e^{\omega\beta\frac{t}{2}}\, \left(\tfrac{t}{2}\right)^{-\alpha} \|T_{t/2}u - T_{t/2}\hat{u}\|_q^{\gamma}\\ &\le C\, e^{\omega\beta\frac{t}{2}}\, \left(\tfrac{t}{2}\right)^{-\alpha} \|T_{t/2}u - T_{t/2}\hat{u}\|_r^{\gamma(1-\theta_s)}\, \|T_{t/2}u - T_{t/2}\hat{u}\|_s^{\gamma\theta_s}\end{aligned}$$

Since $\{T_t\}_{t\ge 0}$ satisfies (2.30) for $\tilde{q} = s$ and some $\omega \ge 0$,

$$\|T_t u - T_t\hat{u}\|_r \le C\, e^{\omega(\beta+\gamma\theta_s)\frac{t}{2}}\, \left(\tfrac{t}{2}\right)^{-\alpha} \|T_{t/2}u - T_{t/2}\hat{u}\|_r^{\gamma(1-\theta_s)}\, \|u-\hat{u}\|_s^{\gamma\theta_s}$$

and so by definition of $C_{u,\hat{u},T}$,

$$\|T_t u - T_t\hat{u}\|_r \le C\; e^{\omega(\beta+\gamma\theta_s+\gamma_s\gamma(1-\theta_s))\frac{t}{2}}\; \left(\tfrac{t}{2}\right)^{-\alpha-\alpha\frac{\gamma(1-\theta_s)}{\theta}}\; C_{u,\hat{u},T}^{\gamma(1-\theta_s)}\; \|u-\hat{u}\|_s^{\gamma(\theta_s+\gamma_0(1-\theta_s))}$$

for every $t \in [0,2T]$. Since $\gamma\theta_s + \gamma_s\gamma(1-\theta_s) = \gamma_s$ and $1 + \frac{\gamma(1-\theta_s)}{\theta} = \frac{1}{\theta}$, the previous estimate becomes

$$\|T_t u - T_t\hat{u}\|_r \le C\; e^{\omega(\beta+\gamma_s)\frac{t}{2}}\; 2^{\frac{\alpha}{\theta}}\; t^{-\frac{\alpha}{\theta}}\; C_{u,\hat{u},T}^{\gamma(1-\theta_s)}\; \|u-\hat{u}\|_s^{\gamma_s}$$

and so

$$\|T_t u - T_t\hat{u}\|_r \le C\, e^{\omega\beta\frac{t}{2}}\, 2^{\frac{\alpha}{\theta}}\, C_{u,\hat{u},T}^{\gamma(1-\theta_s)}\; e^{\omega\gamma_s t}\; t^{-\frac{\alpha}{\theta}}\; \|u-\hat{u}\|_s^{\gamma_s}$$

for every $t \in [0,T]$. Dividing this inequality by $e^{\omega\gamma_s t}\, t^{-\frac{\alpha}{\theta}}\, \|u-\hat{u}\|_s^{\gamma_s}$ and taking the supremum over $[0,T]$ on the left hand-side of the resulting inequality yields

$$C_{u,\hat{u},T} \le C\, e^{\omega\beta\frac{T}{2}}\, 2^{\frac{\alpha}{\theta}}\, C_{u,\hat{u},T}^{\gamma(1-\theta_s)}.$$

Since $\gamma(1-\theta_s) < 1$, this implies that $C_{u,\hat{u},T}$ is uniformly bounded in u, $\hat{u}$ by constant $(C\,2^{\frac{\alpha}{\theta}})^{\frac{1}{\theta}}\, e^{\omega\frac{\beta}{\theta}\frac{T}{2}} > 0$ with $\theta = 1-\gamma(1-\theta_s)$. In other words,

$$\|T_t u - T_t\hat{u}\|_r \le (C\,2^{\frac{\alpha}{\theta}})^{\frac{1}{\theta}}\, e^{\omega\frac{\beta}{\theta}\frac{T}{2}}\, e^{\omega\gamma_s t}\, t^{-\frac{\alpha}{\theta}}\, \|u-\hat{u}\|_s^{\gamma_s}$$

for every $t \in [0,T]$ and u, $\hat{u} \in D \cap L^s(\Sigma,\mu)$, where $T > 0$ was arbitrary. Taking $t = T$ in this inequality, we can conclude that inequality (4.2) holds for every $t > 0$ and u, $\hat{u} \in D \cap L^s(\Sigma,\mu)$. □

Our second extrapolation result of this subsection is adapted to semigroups enjoying the L^q-L^r-regularising effect (1.17) for $1 < q < r \le \infty$ and some $u_0 \in L^r \cap L^s(\Sigma,\mu)$ generated by either quasi m-completely accretive operators on $L^q(\Sigma,\mu)$ (Section 2.2) or quasi m-accretive operators in L^1 with (c-)complete resolvent (Section 2.3).

Theorem 4.3. *Let $1 \le s < q < r \le \infty$ and $\{T_t\}_{t\ge 0}$ be a semigroup acting on a subset D of $L^q(\Sigma,\mu)$ and satisfies the exponential growth property (2.32) for $\tilde{q} = s$, some $\omega \ge 0$ and $u_0 \in L^s \cap L^r(\Sigma,\mu)$, $\{T_t\}_{t\ge 0}$. Suppose there exist $C > 0$ and exponents $\alpha > 0$, β, $\gamma > 0$ such that*

$$\|T_t u - u_0\|_r \le C\, t^{-\alpha}\, e^{\omega\beta t}\, \|u-u_0\|_q^{\gamma} \tag{1.17}$$

for every $t > 0$ and $u \in D$. For $\theta_s = \frac{(r-q)s}{q(r-s)} > 0$ if $r < \infty$ and $\theta_s = \frac{s}{q}$ if $r = \infty$, assume that $\gamma(1-\theta_s) < 1$. Then one has

$$\|T_t u - u_0\|_r \le \left(C\,2^{\frac{\alpha}{1-\gamma(1-\theta_s)}}\right)^{\frac{1}{1-\gamma(1-\theta_s)}}\, t^{-\alpha_s}\, e^{\omega\beta_s t}\, \|u-u_0\|_s^{\gamma_s} \tag{4.4}$$

for every $t > 0$ and $u \in D \cap L^{q_0}(\Sigma,\mu)$ with exponents (4.3).

Proof of Theorem 4.3. By using the same arguments as outlined in the proof of Theorem 4.1, where one replaces $\hat{u}$ and $T_t\hat{u}$ by u_0 and condition (2.30) by (2.32), one sees that the statement of Theorem 4.3 holds. □

We continue this section by establishing a new nonlinear interpolation theorem of independent interest.

4.2. **A nonlinear interpolation theorem.**

In this subsection, we state our nonlinear interpolation theorem, which generalises both Peetre's ([77, Theorem 3.1]) and Tartar's (cf. [85, Théorème 4]) nonlinear interpolation results. Our nonlinear interpolation theorem complements the existing literature in three ways, namely, by introducing additional parameters p_0, r_0, r_1, by treating the borderline cases $p_0 = \infty$, $p_1 < \infty$ and $p_0 < \infty$, $p_1 = \infty$, and by giving exact constants.

We begin by recalling some basic definitions, notations and results from the classical interpolation theory (cf., for instance, [78] or [27, Chapter 3]). Let X_0 and X_1 be two real or complex Banach spaces such that both are continuously embedded into a Hausdorff topological vector space $\mathcal{X}$. A pair $\{X_0, X_1\}$ of Banach spaces X_0 and X_1 satisfying these conditions is called an *interpolation couple*. We equip the *intersection space* $X_0 \cap X_1$ and the *sum space*

$$X_0 + X_1 := \left\{ x \,\middle|\, \text{there are } x_0 \in X_0,\ x_1 \in X_1 \text{ s.t. } x = x_0 + x_1 \right\}$$

respectively with the norm $\|x\|_{X_0\cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$ and

$$\|x\|_{X_0+x_1} := \inf\left\{ \|x_0\|_{X_0} + \|x_1\|_{X_1} \;\middle|\; x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1 \right\}.$$

Then $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces and

$$X_0 \cap X_1 \hookrightarrow Z \hookrightarrow X_0 + X_1 \tag{4.5}$$

for $Z = X_0$ and $Z = X_1$ each with linear continuous embeddings (cf. [27, Proposition 3.2.1]). A Banach space Z satisfying (4.5) is called an *intermediate space* (of X_0 and X_1).

For any Banach space X equipped with norm $\|\cdot\|_X$ and for every $1 \le q \le \infty$, we denote by $L^q_*(X)$ the Banach space of all (classes of) strongly $\mathrm{d}t/t$-measurable functions $f : (0,\infty) \to X$ having finite norm

$$\|f\|_{L^q_*(X)} := \begin{cases} \left\{ \int_0^\infty \|f(t)\|_X^q \frac{\mathrm{d}t}{t} \right\}^{1/q} & \text{if } 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{t\in(0,\infty)} \|f(t)\|_X & \text{if } q = \infty. \end{cases}$$

We shall make use of the so-called *mean-method*, which was introduced by J.-L. Lions and Peetre ([65, 66]) and further elaborated, for instance, in [76, 61].

We begin by introducing the *mean spaces* (*espaces de moyennes*). Let (X_0, X_1) be an interpolation couple. Then for every $0 < \theta < 1$ and $1 \le p_0, p_1 \le \infty$, the *mean space* $(X_0, X_1)_{\theta,p_0,p_1}$ is defined by the space of all elements $u \in X_0 + X_1$ with the property

$$\begin{cases} \text{for } i = 0,1, \text{ there is a measurable function } v_i : (0,\infty) \to X_i \\ \text{satisfying } u = v_0(t) + v_1(t) \text{ in } X_0 + X_1 \text{ for a.e. } t \in (0,\infty), \\ t^{-\theta} v_0 \in L^{p_0}_*(X_0) \text{ and } \ t^{1-\theta} v_1 \in L^{p_1}_*(X_1). \end{cases} \tag{4.6}$$

We equip the mean space $(X_0, X_1)_{\theta,p_0,p_1}$ with the norm

$$\|u\|_{\theta,p_0,p_1} := \inf_{a=v_0(t)+v_1(t)} \max\left\{ \|t^{-\theta} v_0\|_{L^{p_0}_*(X_0)}, \|t^{1-\theta} v_1\|_{L^{p_1}_*(X_1)} \right\},$$

where the infimum is taken of all representation pairs (v_0, v_1) satisfying (4.6). Then, it is not difficult to see that each mean space $(X_0, X_1)_{\theta,p_0,p_1}$ is an intermediate space (cf. [65, p. 9]). Moreover, the spaces $(X_0, X_1)_{\theta,p_0,p_1}$ admits the so-called *interpolation property* (cf. [88, p. 63]), that is, for every linear mapping $T : X_0 + X_1 \to X_0 + X_1$ such that its restriction to X_i yields a linear and bounded operator from X_i into itself, where $i = 0, 1$, one has that the restriction of T to $(X_0, X_1)_{\theta,p_0,p_1} X_i$ yields a linear and bounded operator from $(X_0, X_1)_{\theta,p_0,p_1}$ into itself ([65, Théorème (3.1)]). In particular, one has

$$\|u\|_{\theta,p_0,p_1} = \inf_{a=v_0(t)+v_1(t)} \|t^{-\theta} v_0\|^{1-\theta}_{L^{p_0}_*(X_0)} \, \|t^{1-\theta} v_1\|^{\theta}_{L^{p_1}_*(X_1)}, \tag{4.7}$$

for every $u \in (X_0, X_1)_{\theta,p_0,p_1}$, where the infimum is taken of all representation pairs (v_0, v_1) satisfying (4.6) (cf. [65, Lemme (3.1)]). In addition, the following continuous embedding is valid.

Lemma 4.4 ([65, Théorème (5.3)]). *Let $0 < \theta < 1$ and $1 \le p_0, p_1, s_0, s_1 \le \infty$. Then for $s_0 \le p_0$ and $s_1 \le p_1$, one has*

$$\|u\|_{\theta,p_0,p_1} \le C_{\theta,r_0,r_1}\, \|u\|_{\theta,s_0,s_1}$$

for all $u \in (X_0, X_1)_{\theta,s_0,s_1}$, where the constant

$$C_{\theta,r_0,r_1} := \begin{cases} 1 & \text{if } s_0 = p_0 \text{ and } s_1 = p_1,\\ \inf\limits_{\varphi\in D_+} \|t^{-\theta}\varphi\|^{1-\theta}_{L^{r_0}_*(\mathbb{R})} & \text{if } s_1 = p_1\\ \inf\limits_{\varphi\in D_+} \|t^{1-\theta}\varphi\|^{\theta}_{L^{r_1}_*(\mathbb{R})} & \text{if } s_0 = p_0\\ \inf\limits_{\varphi\in D_+} \|t^{-\theta}\varphi\|^{1-\theta}_{L^{r_0}_*(\mathbb{R})}\, \|t^{1-\theta}\varphi\|^{\theta}_{L^{r_1}_*(\mathbb{R})} & \text{if otherwise} \end{cases} \tag{4.8}$$

$$\text{with } \tfrac{1}{r_0} = 1 - \left[\tfrac{1}{s_0} - \tfrac{1}{p_0}\right] \quad \text{and} \quad \tfrac{1}{r_1} = 1 - \left[\tfrac{1}{s_1} - \tfrac{1}{p_1}\right],$$

and D_+ denotes the set of all test functions $\varphi \in C^\infty_c((0,\infty))$ satisfying $\varphi \ge 0$ and $\int_0^\infty \varphi(\frac{1}{t})\frac{dt}{t} = 1$.

Due to the result [76, Théorème 3.1] by Peetre, for every $0 < \theta < 1$ and $1 \le p_0, p_1, p \le \infty$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, the mean space $(X_0, X_1)_{\theta,p_0,p_1}$ coincides with the (classical) *real interpolation space* $(X_0, X_1)_{\theta,p}$ with equivalent norms. For the definition of the interpolation space $(X_0, X_1)_{\theta,p}$ we refer, for instance, to [27, Definition 3.2.4]. Combining this together with the density result [87, Theorem 1.6.2], we can state the following extended version of the density result [65, Théorème 2.1].

Lemma 4.5. *Let (X_0, X_1) be an interpolation couple and suppose that one of the following cases holds:*

(i) $1 \le p_0, p_1 < \infty$
(ii) $1 \le p_0 < \infty$ *and* $p_1 = \infty$
(iii) $1 \le p_1 < \infty$ *and* $p_0 = \infty$.

Then, for every $0 < \theta < 1$, the intersection space $X_1 \cap X_2$ is dense in $(X_0, X_1)_{\theta,p_0,p_1}$.

Now, we are in a position to state our first nonlinear interpolation theorem.

Theorem 4.6. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples and T be a mapping from $X_0 + X_1$ into $Y_0 + Y_1$ with domain containing $X_0 \cap X_1$. Suppose there are exponents $0 < \alpha_0, \alpha_1 < \infty$ and constants $M_0, M_1 \ge 0$ such that*

$$\|Tu - T\hat{u}\|_{Y_0} \le M_0\, \|u - \hat{u}\|^{\alpha_0}_{X_0} \tag{4.9}$$

for all $u, \hat{u} \in X_0 \cap X_1$ and

$$\|Tu - T\hat{u}\|_{Y_1} \le M_1\, \|u - \hat{u}\|^{\alpha_1}_{X_1} \tag{4.10}$$

for all $u, \hat{u} \in X_0 \cap X_1$. For every $0 < \theta < 1$ and $1 \le q_0, q_1 \le \infty$ (excluding $q_0 = q_1 = \infty$) satisfying $q_0 \ge \frac{1}{\alpha_0}$ and $q_1 \ge \frac{1}{\alpha_1}$, let $1 \le q, p_0, p_1 \le \infty$, $0 < \eta < 1$, $0 < \alpha < \infty$ be given by

$$\begin{aligned} \tfrac{1}{q} &= \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1}, & p_0 &= \alpha_0 q_0, \quad p_1 = \alpha_1 q_1,\\ \eta &= \tfrac{\theta\alpha_1}{(1-\theta)\alpha_0 + \theta\alpha_1}, & \alpha &= (1-\theta)\alpha_0 + \theta\alpha_1 \end{aligned} \tag{4.11}$$

and let $1 \le s_0 \le p_0$ and $1 \le s_1 \le p_1$. Then the following statements hold.

(1) *One has*

$$\|Tu - T\hat{u}\|_{(Y_0,Y_1)_{\theta,q_0,q_1}} \le \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta}\, M_1^{\theta}\, C^{\alpha}_{\eta,r_0,r_1} \|u - \hat{u}\|^{\alpha}_{(X_0,X_1)_{\eta,s_0,s_1}} \tag{4.12}$$

for every u, $\hat{u} \in X_0 \cap X_1$, *where the constant* C_{η,r_0,r_1} *is given by* (4.8).

(2) *If there is a* $u_0 \in X_0 \cap X_1$ *such that* $Tu_0 \in (Y_0, Y_1)_{\theta,q_0,q_1}$, *then* T *can be uniquely extended to a mapping* $T : (X_0, X_1)_{\eta,s_0,s_1} \to (Y_0, Y_1)_{\theta,q_0,q_1}$ *satisfying inequality* (4.12) *for all* u, $\hat{u} \in (X_0, X_1)_{\eta,s_0,s_1}$.

By using the preliminaries of this subsection, we can now outline the proof of this nonlinear interpolation theorem.

Proof of Theorem 4.6. First, we fix $\hat{u} \in X_0 \cap X_1$ and show that

$$\|T(u+\hat{u}) - T\hat{u}\|_{(Y_0,Y_1)_{\theta,q_0,q_1}} \le \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta}\, M_1^{\theta}\, \|u\|^{\alpha}_{(X_0,X_1)_{\eta,p_0,p_1}} \tag{4.13}$$

for all $u \in X_0 \cap X_1$. To do so, let $u \in X_0 \cap X_1$. Since $X_0 \cap X_1$ is continuously injected into $(X_0, X_1)_{\eta,p_0,p_1}$, there is a pair (v_0, v_1) of measurable functions satisfying (4.6). Since $u \in X_0 \cap X_1$ and $u = v_0 + v_1$, it follows that $v_i(t) \in X_0 \cap X_1$ for a.e. $t \in (0,\infty)$ and every $i = 0, 1$. For $\lambda := \frac{\theta}{\eta\,\alpha_0} > 0$, we set

$$w_0(t) = T(v_0(t^{\lambda}) + \hat{u}) - T\hat{u} \quad \text{and} \quad w_1(t) = T(u+\hat{u}) - T\hat{u} - w_0(t)$$

for a.e. $t \in (0,\infty)$. Then, $T(u+\hat{u}) - T\hat{u} = w_0(t) + w_1(t)$ for a.e. $t \in (0,\infty)$, and by using (4.9) and (4.10), one sees that the functions $w_i : (0,\infty) \to Y_i$ are measurable and satisfy

$$\|w_i(t)\|_{Y_i} \le M_i\, \|v_i(t^{\lambda})\|^{\alpha_i}_{X_i} \tag{4.14}$$

for a.e. $t \in (0,\infty)$ and each $i = 0, 1$. Since we have chosen $\lambda = \frac{\theta}{\eta\,\alpha_0}$ and $p_0 = q_0\alpha_0$, we obtain by applying inequality (4.14) and substituting $s = t^{\lambda}$ that

$$\|t^{-\theta} w_0\|_{L^{q_0}_*(Y_0)} \le M_0 \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q_0}} \|s^{-\eta} v_0\|^{\alpha_0}_{L^{p_0}(X_0)}.$$

On the other hand, $\eta = \frac{\theta\,\alpha_1}{(1-\theta)\alpha_0 + \theta\alpha_1}$ is equivalent to $\frac{1-\eta}{\eta} = \frac{(1-\theta)\alpha_0}{\theta\alpha_1}$ hence $\lambda = \frac{1-\theta}{(1-\eta)\alpha_1}$. Using this together with inequality (4.14), the fact that $p_1 = q_1\alpha_1$, and applying the substitution $s = t^{\lambda}$, we see that

$$\|t^{1-\theta} w_1\|_{L^{q_1}_*(Y_1)} \le M_1 \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q_1}} \|s^{1-\eta} v_1\|^{\alpha_1}_{L^{p_1}(X_1)}.$$

Thus $T(u+\hat{u}) - T\hat{u} \in (Y_1, Y_2)_{\theta,q_0,q_1}$. Combining the last two estimates together with (4.7) yields

$$\begin{aligned}
&\|T(u+\hat{u}) - T\hat{u}\|_{(Y_0,Y_1)_{\theta,q_0,q_1}} \\
&\quad\le M_0^{1-\theta}\, M_1^{\theta} \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} \|s^{-\eta} v_0\|^{(1-\theta)\alpha_0}_{L^{p_0}(X_0)}\, \|s^{1-\eta} v_1\|^{\theta\alpha_1}_{L^{p_1}(X_1)} \\
&\quad\le M_0^{1-\theta}\, M_1^{\theta} \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} \max\left\{ \|s^{-\eta} v_0\|_{L^{p_0}(X_0)}, \|s^{1-\eta} v_1\|_{L^{p_1}(X_1)} \right\}^{\alpha}.
\end{aligned}$$

Taking the infimum over all representation pairs (v_0, v_1) satisfying (4.6) shows that inequality (4.13) holds. Now, for every $u, \hat{u} \in X_0 \cap X_1$, replacing u by $u - \hat{u} \in X_0 \cap X_1$ in (4.13) gives

$$\|Tu - T\hat{u}\|_{(Y_0,Y_1)_{\theta,q_0,q_1}} \le \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta}\, M_1^{\theta}\, \|u - \hat{u}\|^{\alpha}_{(X_0,X_1)_{\eta,p_0,p_1}} \tag{4.15}$$

for all u, $\hat{u} \in X_0 \cap X_1$. Applying Lemma 4.4 yields inequality (4.12) for every u, $\hat{u} \in X_0 \cap X_1$, proving that the first statement of this theorem holds.

Under the assumption, there is a $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in (Y_0, Y_1)_{\theta,q_0,q_1}$, inequality (4.12) implies that the mapping T maps $X_0 \cap X_1$ equipped with the $(X_0, X_1)_{\eta,p_0,p_1}$-norm into $(Y_0, Y_1)_{\theta,q_0,q_1}$. Thus by Lemma (4.5) and since the spaces $(X_0, X_1)_{\eta,p_0,p_1}$ and $(Y_0, Y_1)_{\theta,q_0,q_1}$ are complete, we can conclude that T admits a unique Hölder-continuous extension from $(X_0, X_1)_{\eta,p_0,p_1}$ to $(Y_0, Y_1)_{\theta,q_0,q_1}$ satisfying (4.15) for all u, $\hat{u} \in (X_0, X_1)_{\eta,p_0,p_1}$. This completes the proof of this theorem. □

In our second nonlinear interpolation theorem, we consider the situation when the mapping T admits an element $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in Y_0 \cap Y_1$.

Theorem 4.7. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples and T a mapping from $X_0 + X_1$ into $Y_0 + Y_1$ with domain containing $X_0 \cap X_1$. Suppose T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm to Y_0 and there are $u_0 \in X_0 \cap X_1$ satisfying $Tu_0 \in Y_0 \cap Y_1$, exponents $0 < \alpha_0, \alpha_1 < \infty$, and constants $M_0, M_1 \ge 0$ such that*

$$\|Tu - Tu_0\|_{Y_0} \le M_0\, \|u - u_0\|^{\alpha_0}_{X_0} \qquad \text{for all } u \in X_0 \cap X_1 \tag{4.16}$$

and

$$\|Tu - T\hat{u}\|_{Y_1} \le M_1\, \|u - \hat{u}\|^{\alpha_1}_{X_1} \qquad \text{for all } u, \hat{u} \in X_0 \cap X_1. \tag{4.17}$$

For every $0 < \theta < 1$ and $1 \le q_0, q_1 \le \infty$ (excluding $q_0 = q_1 = \infty$) satisfying $q_0 \ge \frac{1}{\alpha_0}$ and $q_1 \ge \frac{1}{\alpha_1}$, let $1 \le q, p_0, p_1 \le \infty$, $0 < \eta < 1$, $0 < \alpha < \infty$ given by (4.11), and let $1 \le s_0 \le p_0$ and $1 \le s_1 \le p_1$. Then one has

$$\|Tu - Tu_0\|_{(Y_0,Y_1)_{\theta,q_0,q_1}} \le \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta}\, M_1^{\theta}\, C^{\alpha}_{\eta,r_0,r_1} \|u - u_0\|^{\alpha}_{(X_0,X_1)_{\eta,s_0,s_1}} \tag{4.18}$$

for every $u \in X_0 \cap X_1$, where the constant C_{η,r_0,r_1} is given by (4.8).

Proof of Theorem 4.7. Let $u \in X_0 \cap X_1$. Since $X_0 \cap X_1$ is continuously injected into $(X_0, X_1)_{\eta,p_0,p_1}$, there are measurable functions $v_i : (0, \infty) \to X_i$ for $i = 0, 1$ satisfying $u - u_0 = v_0(t) + v_1(t)$ in $X_0 + X_1$ for a.e. $t \in (0, \infty)$,

$$t^{-\theta} v_0 \in L^{p_0}_*(X_0) \qquad \text{and} \qquad t^{1-\theta} v_1 \in L^{p_1}_*(X_1). \tag{4.19}$$

For $\lambda := \frac{\theta}{\eta\,\alpha_0} > 0$, we set

$$w_0(t) = T(v_0(t^{\lambda}) + u_0) - Tu_0 \quad \text{and} \quad w_1(t) = Tu - Tu_0 - w_0(t)$$

for a.e. $t \in (0, \infty)$. By construction, $Tu - Tu_0 = w_0(t) + w_1(t)$ for a.e. $t \in (0, \infty)$. Since by assumption, T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm to Y_0, the function $w_0 : (0, \infty) \to Y_0$ is strongly measurable. By (4.10), T is Hölder-continuous from $X_0 \cap X_1$ equipped with the X_1-norm to Y_1. Thus, the function $w_1 : (0, \infty) \to Y_1$ is strongly measurable. Moreover, by (4.16) and (4.17), we have that the inequalities (4.14) hold for $i = 0, 1$. Now, we can proceed as in the proof of Theorem 4.6 to conclude that inequality (4.18) holds for all $u \in X_0 \cap X_1$. $\square$

In some applications, the assumption that *the mapping T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm topology to Y_0* in Theorem 4.7 is too strong. This can be circumvented, for instance, by the following result.

Theorem 4.8. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples, Y_0 being a separable Banach space. Let T be a mapping from $X_0 + X_1$ into $Y_0 + Y_1$ with domain containing $X_0 \cap X_1$. Suppose there is some $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in Y_0 \cap Y_1$ and T satisfies the following three conditions.*

- *T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm to Y_0 equipped with the weak topology,*
- *there are exponents $0 < \alpha_0$, $\alpha_1 < \infty$ and constants M_0, $M_1 \geq 0$ such that T satisfies (4.16) and (4.17).*

For every $0 < \theta < 1$ and $1 \leq q_0$, $q_1 \leq \infty$ (excluding $q_0 = q_1 = \infty$) satisfying $q_0 \geq \frac{1}{\alpha_0}$ and $q_1 \geq \frac{1}{\alpha_1}$, let $1 \leq q, p_0, p_1 \leq \infty$, $0 < \eta < 1$, $0 < \alpha < \infty$ given by (4.11), and let $1 \leq s_0 \leq p_0$ and $1 \leq s_1 \leq p_1$. Then T satisfies inequality (4.18) for every $u \in X_0 \cap X_1$, where the constant C_{η,r_0,r_1} is given by (4.8).

Remark 4.9. Consider the following situation: For $1 \leq q$, $r < \infty$, let $X_0 = L^q(\Sigma, \mu)$, $X_1 = L^\infty(\Sigma, \mu)$, $Y_0 = L^r(\Sigma, \mu)$ and $Y_1 = L^\infty(\Sigma, \mu)$, where one assumes that (Σ, μ) is a *separable measure space* (cf, [25, Definition on p.98]). Suppose T satisfy the assumptions of Theorem 4.8 and we choose

$$\begin{aligned} &q_0 = r, && q_1 = \infty, && p_0 = \beta q_0 = \beta r > q \geq 1, \\ &p_1 = q_1 = \infty, && s_0 = q < \beta r = p_0, && s_1 = \infty. \end{aligned}$$

Then, by Corollary B.2,

$$(X_0, X_1)_{\eta,s_0,s_1} = L^{\frac{q}{(1-\eta)}}(\Sigma, \mu) \qquad \text{and} \qquad (Y_0, Y_1)_{\theta,q_0,q_1} = L^{\frac{r}{(1-\theta)}}(\Sigma, \mu)$$

with equal norms for every $0 < \theta, \eta < 1$ and so Theorem 4.8 yields

$$\begin{aligned} &\|Tu - u_0\|_{\frac{r}{1-\theta}} \\ &\leq \left[\tfrac{\beta}{(1-\theta)\beta+\theta}\right]^{\frac{1-\theta}{r}} M_0^{1-\theta}\, M_1^{\theta}\, C_{\eta,r_0,1}^{(1-\theta)\beta+\theta}\, \|u - u_0\|_{\frac{q}{1-\eta(\theta)}}^{(1-\theta)\beta+\theta} \end{aligned} \tag{4.20}$$

for every $u \in L^q(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)$ and every $0 < \theta < 1$, where $r_0 = \frac{q\beta r}{\beta r(q-1)+q}$. In addition, to the above assumptions, we suppose

$$T \text{ is continuous from } L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu) \text{ to } L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu).$$

Since $L^q(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)$ is dense in $L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu)$, for every $u \in L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu)$, there is a sequence (u_n) in $L^q(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)$ such that u_n converges to u in $L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu)$ and so Tu_n converges to Tu in $L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu)$. By (4.20), (Tu_n) is bounded in $L^{\frac{r}{1-\theta}}(\Sigma, \mu)$ and hence, after eventually passing to a subsequence of (u_n), we may assume that Tu_n converges weakly to v in $L^{\frac{r}{1-\theta}}(\Sigma, \mu)$ for some $v \in L^{\frac{r}{1-\theta}}(\Sigma, \mu)$. Since $L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu)$ and $L^{\frac{r}{1-\theta}}(\Sigma, \mu)$ are both continuously embedded into $L^m_{loc}(\Sigma, \mu)$, with $m := \min\{\frac{q}{1-\eta(\theta)}, \frac{r}{1-\theta}\}$, we obtain $v = Tu$ a.e. on Σ and so, sending $n \to \infty$ in (4.20) for $u = u_n$ and using Fatou's lemma shows that (4.20) holds for all $u \in L^{\frac{q}{1-\eta(\theta)}}(\Sigma, \mu)$.

Proof of Theorem 4.8. Let $u \in X_0 \cap X_1$ and for $i = 0, 1$, let $v_i : (0,\infty) \to X_i$ be measurable such that $u - u_0 = v_0(t) + v_1(t)$ in $X_0 + X_1$ for a.e. $t \in (0,\infty)$ and (4.19) holds. For $\lambda := \frac{\theta}{\eta\,\alpha_0} > 0$, we set

$$w_0(t) = T(v_0(t^{\lambda}) + u_0) - Tu_0 \quad \text{and} \quad w_1(t) = Tu - Tu_0 - w_0(t)$$

for a.e. $t \in (0,\infty)$. By construction, $Tu - Tu_0 = w_0(t) + w_1(t)$ for a.e. $t \in (0,\infty)$. By assumption, T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm topology to Y_0 equipped with the weak-topology. Hence w_0 is weakly measurable. But since by assumption, Y_0 is separable, the function $w_0 : (0,\infty) \to Y_0$ is strongly measurable due to Pettis's theorem ([58, Theorem 3.5.3]). By (4.10), T is Hölder-continuous from $X_0 \cap X_1$ equipped with the X_1-norm to Y_1. Thus, the function $w_1 : (0,\infty) \to Y_1$ is strongly measurable. Moreover, by (4.16) and (4.17), we have that the inequalities (4.14) hold for $i = 0, 1$. Now, we can proceed as in the proof of Theorem 4.6 and see that the statement of this theorem holds. □

4.3. **Extrapolation towards L^∞.** To the best of our knowledge, first extrapolation results towards L^∞ in the context of *linear semigroups* and employing Riesz-Thorin's or Stein's linear interpolation theorems go back to the pioneering work [83] by Simon and Høegh-Krohn (see also [44, Theorem 3.3]). An alternative approach using a duality argument has been given in [35, Lemme 1]. However, in this article, we are confronted with a much more difficult situation, since the family of operators $\{T_t\}_{t\ge 0}$ are (in general) nonlinear. Hence neither a duality argument or a linear Riesz-Thorin interpolation theorem can be used.

Our extrapolation result towards L^∞ is a nonlinear generalisation of the techniques developed in [83, 44, 35]. Our proof relies essentially on the nonlinear interpolation results Theorem 4.6 and Theorem 4.8, as well as the fact that the mean spaces involving $L^{p_0}(\Sigma,\mu)$ and $L^{p_1}(\Sigma,\mu)$ spaces are isometrically isomorphic to an appropriate $L^p(\Sigma,\mu)$ space (cf. Corollary B.2).

Here, we shall use the notation $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \le Cv$.

Our first extrapolation result towards L^∞ is adapted to semigroups generated by completely accretive operators (Section 2.2) satisfying the L^q-L^r-regularisation effect (1.13) for *differences* and $1 \le q, r < \infty$.

Theorem 4.10. *Let $1 \le q, r < \infty$ and $\{T_t\}_{t\ge 0}$ be a semigroup acting on $L^q \cap L^\infty(\Sigma,\mu)$. Suppose $\{T_t\}_{t\ge 0}$ satisfies exponential growth (2.30) for $\tilde{q} = \infty$ and some $\omega \ge 0$, and there exist $C > 0$ and exponents α, β, $\gamma > 0$ such that the estimate*

$$\|T_t u - T_t\hat{u}\|_r \le \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, e^{\omega\beta t}\, \|u - \hat{u}\|_q^{\gamma} \tag{1.13}$$

holds for every $t > 0$ and u, $\hat{u} \in L^q(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$. If

$$\gamma\, r > q \tag{4.21}$$

then

$$\|T_t u - T_t\hat{u}\|_\infty \lesssim t^{-\alpha^*}\, e^{\omega\beta^* t}\, \|u - \hat{u}\|_{\gamma\, r\, q^{-1} m_0}^{\gamma^*} \tag{4.22}$$

for every $t > 0$ *and* $u, \hat{u} \in L^{\gamma\, r\, q^{-1}\, m_0}(\Sigma,\mu)$, *with exponents*

$$\alpha^* = \frac{\alpha\, q\, \gamma^{-1}}{(\frac{\gamma\, r}{q} - 1)\, m_0 + q(\frac{1}{\gamma} - 1)}, \qquad \gamma^* = \frac{(\frac{\gamma\, r}{q} - 1)\, m_0}{(\frac{\gamma\, r}{q} - 1)\, m_0 + q(\frac{1}{\gamma} - 1)},$$
$$\beta^* = \frac{(\beta - 1)\gamma\, r\, q^{-1} + \gamma - \beta}{(\frac{\gamma\, r}{q} - 1)\, m_0 + q(\frac{1}{\gamma} - 1)} + 1, \tag{4.23}$$

and $m_0 \geq q\,\gamma^{-1}$ *such that*

$$(\tfrac{\gamma\, r}{q} - 1)\, m_0 + q(\tfrac{1}{\gamma} - 1) > 0. \tag{4.24}$$

Remark 4.11. The two conditions (4.21) and (4.24) are heavily involved in the recursive construction

$$m_{n+1} = m_n\, \kappa - r\, \kappa^{-1}\, (\gamma - 1), \qquad (n \geq 1), \tag{4.25}$$

of a strictly increasing sequence $(m_n)_{n\geq 0} \subseteq (1, +\infty)$ satisfying $\lim_{n\to+\infty} m_n = +\infty$. If one chooses κ by

$$\kappa = \frac{\gamma\, r}{q} \tag{4.26}$$

then condition (4.21) yields $\kappa > 1$. If, in addition, m_0 satisfies (4.24) then $(m_n)_{n\geq 0}$ is strictly increasing and $\lim_{n\to+\infty} m_n = +\infty$. Inserting the sequence $(m_n)_{n\geq 0}$ into inequality (4.28) and using the semigroup property of $\{T_t\}_{t\geq 0}$, one obtains an $L^{\tilde{q}}$-L^∞ regularisation effect of the $\{T_t\}_{t\geq 0}$ for some $\tilde{q} = \gamma\, r\, q^{-1}\, m_0 \in [1, \infty)$.

Proof of Theorem 4.10. We intend to apply Theorem 4.6 to the following situation: let $X_0 = L^q(\Sigma,\mu)$, $X_1 = L^\infty(\Sigma,\mu)$, $Y_0 = L^r(\Sigma,\mu)$, $Y_1 = L^\infty(\Sigma,\mu)$, and for any fixed $t > 0$, let $T = T_t$. By assumption, T_t satisfies (1.13) and has exponential growth (2.30) for $\tilde{q} = \infty$ and some $\omega \geq 0$. Hence the mapping T satisfies inequality (4.9) with $\alpha_0 = \gamma > 0$, $M_0 = C\, e^{\omega\beta t}\, t^{-\alpha}$ and inequality (4.10) with $\alpha_1 = 1$, $M_1 = e^{\omega t}$. Further, we choose

$$\begin{aligned} &q_0 = r, && q_1 = \infty, && p_0 = \gamma\, q_0 = \gamma\, r > q \geq 1,\\ &p_1 = q_1 = \infty, && s_0 = q < \gamma\, r = p_0, && s_1 = \infty. \end{aligned}$$

Then, by Corollary B.2,

$$(X_0, X_1)_{\eta, s_0, s_1} = L^{\frac{q}{(1-\eta)}}(\Sigma,\mu) \qquad \text{and} \qquad (Y_0, Y_1)_{\theta, q_0, q_1} = L^{\frac{r}{(1-\theta)}}(\Sigma,\mu)$$

with equal norms for every $0 < \theta, \eta < 1$. Thus, Theorem 4.6 yields

$$\begin{aligned} \|T_t u - T_t \hat{u}\|_{\frac{r}{1-\theta}} \leq \left[\tfrac{\gamma}{(1-\theta)\gamma + \theta}\right]^{\frac{1-\theta}{r}} \left[C\, e^{\omega\beta t}\, t^{-\alpha}\right]^{1-\theta} e^{\omega\theta t} \times&\\ \times \Big[\inf_{\varphi\in D_+} \|s^{-\theta}\varphi\|^{1-\theta}_{L^{r_0}_*(\mathbb{R})}\Big]^{(1-\theta)\gamma+\theta} \|u - \hat{u}\|^{(1-\theta)\gamma+\theta}_{\frac{q}{1-\eta(\theta)}}& \end{aligned}$$

for every $t > 0$, $u, \hat{u} \in L^q(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$ and every $0 < \theta < 1$, where

$$r_0 = \frac{q\,\gamma\, r}{\gamma r(q-1)r + q}.$$

Next, we choose a test function $\rho \in C^\infty_c((0,\infty))$ with $\rho \geq 0$ and support $\mathrm{supp}(\varphi)$ in the closed interval $[1,3]$ satisfying

$$e^{-\frac{4}{3}} \log \tfrac{5}{3} \leq \int_0^\infty \rho(\tfrac{1}{s})\, \tfrac{ds}{s} \leq e^{-1} \log 3.$$

Then $\varphi^* := \left(\int_0^\infty \rho(\frac{1}{t})\,\frac{\mathrm{d}t}{t}\right)^{-1}\rho \in D_+$ and there are $C_{\varphi^*,1}, C_{\varphi^*,2} > 0$ such that

$$C_{\varphi^*,1} \le \|t^{-\theta}\varphi^*\|_{L^{r_0}_*(\mathbb{R})} \le C_{\varphi^*,2}$$

hence

$$\begin{aligned}\|T_tu - T_t\hat{u}\|_{\frac{r}{1-\theta}} \le \left[\tfrac{\gamma}{(1-\theta)\gamma+\theta}\right]^{\frac{1-\theta}{r}} \left[C\,e^{\omega\beta t}\,t^{-\alpha}\right]^{1-\theta} e^{\omega\theta t}\times\\ \times\, \|s^{-\theta}\varphi^*\|^{(1-\theta)((1-\theta)\gamma+\theta)}_{L^{r_0}_*(\mathbb{R})}\,\|u-\hat{u}\|^{(1-\theta)\gamma+\theta}_{\frac{q}{1-\eta(\theta)}}\end{aligned}$$

for every $t > 0$, $u, \hat{u} \in L^q(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$ and every $0 < \theta < 1$.

Next, we choose κ by (4.26) and set

$$\theta_m = 1 - \frac{1}{m}\frac{r}{\kappa} \qquad \text{for every } m > r\,\kappa^{-1} = \frac{q}{\gamma}.$$

Then by hypothesis (4.21), $\kappa > 1$ and for all $m > r\,\kappa^{-1}$, one has

$$0 < \theta_m < 1, \qquad\qquad 1-\theta_m = \frac{1}{m}\frac{r}{\kappa},$$

$$1-\eta(\theta_m) = \frac{r\,\kappa^{-1}\gamma}{m + r\kappa^{-1}(\gamma-1)}, \quad \frac{\gamma}{(1-\theta_m)\gamma+\theta_m} = \frac{\gamma\,m}{m+r\kappa^{-1}(\gamma-1)} > 0.$$

Further, we set for all $m > r\,\kappa^{-1}$,

$$C_{\varphi^*,m} := \|s^{\frac{1}{m}\frac{r}{\kappa}-1}\varphi^*\|_{L^{r_0}_*(\mathbb{R})}.$$

With this setting in mind, the previous inequality reduces to inequality (4.28) below for every $t > 0$, $u, \hat{u} \in L^q(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$ and for all $m > r\,\kappa^{-1}$.

Finally, we choose $m_0 \ge r\,\kappa^{-1}$ such that (4.24) holds (where one notes that with the setting of this proof, condition (4.24) coincides with (4.27) below) and let $m > m_0$. The condition on m_0 is sufficient to run an iteration in the time-variable. This is the contents of the next iteration lemma and from there we can conclude that the statement of this theorem holds. □

Lemma 4.12. *Suppose there are $\kappa > 1$, β, $\gamma > 0$, $1 \le r < \infty$ and $m_0 \ge r\,K^{-1}$ such that*

$$(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma) > 0. \tag{4.27}$$

Let $\{T_t\}_{t\ge 0}$ be a semigroup acting on $L^{\kappa m_0}(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$ such that

$$\begin{aligned}\|T_tu - T_t\hat{u}\|_{m\,\kappa} \le \left[\tfrac{\gamma\,m}{m+r\kappa^{-1}(\gamma-1)}\right]^{\frac{1}{m\,\kappa}} \left[C\,e^{\omega\beta t}\,t^{-\alpha}\right]^{\frac{1}{m}\frac{r}{\kappa}} e^{\omega(1-\frac{1}{m}\frac{r}{\kappa})t}\times\\ \times\, C_{\varphi^*,m}^{\frac{1}{m}\frac{r}{\kappa}(\frac{1}{m}\frac{r}{\kappa}(\gamma-1)+1)}\,\|u-\hat{u}\|^{\frac{1}{m}r\,\kappa^{-1}(\gamma-1)+1}_{m+r\,\kappa^{-1}(\gamma-1)}\end{aligned} \tag{4.28}$$

for every $u, \hat{u} \in L^{\kappa m_0}(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$, $t > 0$ and $m \ge m_0$, where $C_{\varphi^*,m}$ satisfies

$$C_{\varphi^*,1} \le C_{\varphi^*,m} \le C_{\varphi^*,2} \tag{4.29}$$

for some constants $C_{\varphi^*,1}, C_{\varphi^*,2} > 0$ independent of $m \ge m_0$. Then

$$\begin{aligned}\|T_tu - T_t\hat{u}\|_\infty \lesssim e^{\omega\left(\frac{(\beta-1)\kappa+\gamma-\beta}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}+1\right)t}\, t^{-\frac{\alpha\,r\,\kappa^{-1}}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}}\times\\ \times\,\|u-\hat{u}\|^{\frac{(\kappa-1)\,m_0}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}}_{\kappa m_0}\end{aligned} \tag{4.30}$$

for every $u, \hat{u} \in L^{\kappa m_0}(\Sigma,\mu)$ and every $t > 0$.

For the proof of this lemma, we simplify some techniques from [92] and extend them to semigroups satisfying exponential growth condition (2.30) (see also [38] in the linear case).

Proof. For $m_0 \geq r\,\kappa^{-1}$ such that (4.27) holds, we construct a sequence $(m_n)_{n\geq 0}$ recursively by (4.25). Then

$$m_{n+1} = \kappa\, m_n + r\,\kappa^{-1}(1-\gamma) \tag{4.31}$$

for every integer $n \geq 0$ and so, an induction over $n \in \mathbb{N}_0$ yields

$$m_n = \kappa^n[m_0 + r\,\kappa^{-1}\,(\gamma-1)] + r\,\kappa^{-1}(1-\gamma)\,\sum_{\nu=0}^{n}\kappa^{\nu}, \tag{4.32}$$

that is

$$m_n = \kappa^n\,\frac{(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma)}{\kappa-1} - \frac{r\,\kappa^{-1}(1-\gamma)}{\kappa-1}. \tag{4.33}$$

Using (4.32), we see that

$$m_{n+1} - m_n = \kappa^n\Big[(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma)\Big]$$

hence the sequence $(m_n)_{n\geq 0}$ is strictly increasing if and only if m_0 satisfies condition (4.27). Moreover, by (4.33), since $\kappa > 1$, and by (4.27), we see that

$$\lim_{n\to\infty} m_n = \infty \tag{4.34}$$

and

$$\lim_{n\to\infty}\frac{m_n}{\kappa^n} = \frac{(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma)}{\kappa-1}. \tag{4.35}$$

Since

$$\frac{1}{m_n}r\,\kappa^{-1}(\gamma-1)+1 = \frac{\kappa\,m_{n-1}}{m_n} \quad\text{and}\quad \frac{\gamma\,m_n}{m_n + r\kappa^{-1}(\gamma-1)} = \frac{\gamma\,m_n}{m_{n-1}\kappa},$$

inserting the sequence $(m_n)_{n\geq 0}$ into (4.28) yields

$$\|T_t u - T_t\hat{u}\|_{\kappa m_n} \leq C_{m_n}^{\frac{1}{m_n\,\kappa}}\, t^{-\frac{\alpha\,r}{m_n\,\kappa}}\, e^{\omega(\beta-1)\frac{r}{m_n\,\kappa}t}\, e^{\omega t}\, C_{\varphi^*,m_n}^{\frac{r}{m_n\kappa}\,\frac{\kappa\,m_{n-1}}{m_n}}\, \|u-\hat{u}\|_{m_{n-1}\,\kappa}^{\frac{\kappa\,m_{n-1}}{m_n}} \tag{4.36}$$

for every $t > 0$, $u, \hat{u} \in L^{\kappa m_0}(\Sigma,\mu)\cap L^{\infty}(\Sigma,\mu)$, and $n \geq 1$, where

$$C_{m_n} := \tfrac{\gamma\,m_n}{m_{n-1}\kappa}\,C^r.$$

Now, let $(t_\nu)_{\nu\geq 0}$ be a sequence in $[0,1]$ such that $\sum_{\nu=0}^{\infty} t_\nu = 1$ which we will specify below. By assumption, $\{T_t\}_{t\geq 0}$ is a semigroup and $T_t u, T_t\hat{u} \in L^{\kappa m_0}(\Sigma,\mu)\cap L^{\infty}(\Sigma,\mu)$ for every $t \geq 0$. Thus, we can iterate (4.36) and obtain

$$\begin{aligned} &\|T_{t\sum_{\nu=0}^{n}t_\nu}u - T_{t\sum_{\nu=0}^{n}t_\nu}\hat{u}\|_{\kappa m_{n+1}} \\ &\leq \prod_{\nu=1}^{n+1} C_{m_\nu}^{\frac{\kappa^{n-\nu}}{m_{n+1}}}\ \prod_{\nu=0}^{n} t_\nu^{-\frac{\alpha\,r}{\kappa}\,\frac{\kappa^{n-\nu}}{m_{n+1}}}\ e^{\omega(\beta-1)\frac{r}{\kappa m_{n+1}}\sum_{\nu=0}^{n}t_\nu\kappa^{n-\nu}t}\times \\ &\quad\times e^{\omega\frac{1}{m_{n+1}}\sum_{\nu=0}^{n}t_\nu\kappa^{n-\nu}m_{\nu+1}t}\ \prod_{\nu=1}^{n+1} C_{\varphi^*,m_\nu}^{\frac{r}{m_\nu}\,\frac{\kappa^{n+1-\nu}m_{\nu-1}}{m_{n+1}}}\ t^{-\frac{\alpha\,r}{\kappa\,m_{n+1}}\sum_{\nu=0}^{n}\kappa^{\nu}}\ \|u-\hat{u}\|_{\kappa m_0}^{m_0\frac{\kappa^{n+1}}{m_{n+1}}}. \end{aligned} \tag{4.37}$$

Since by assumption, $\kappa > 1$, by (4.33), and by (4.27), we see that

$$\lim_{n\to\infty} \frac{1}{m_{n+1}} \sum_{\nu=0}^{n} \kappa^{\nu} = \frac{1}{(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma)}. \tag{4.38}$$

Thus

$$\lim_{n\to\infty} t^{-\frac{\alpha\, r}{\kappa}\frac{1}{m_{n+1}}\sum_{\nu=0}^{n}\kappa^{n-\nu}} = t^{-\frac{\alpha\, r}{\kappa}\frac{1}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}} \qquad \text{for every } t>0. \tag{4.39}$$

If we choose, for instance, $t_\nu = 2^{-\nu-1}$, then

$$\prod_{\nu=0}^{n} t_\nu^{-\frac{\alpha r}{\kappa}\frac{\kappa^{n-\nu}}{m_{n+1}}} = 2^{\frac{\alpha r}{\kappa}\frac{\kappa^{n}}{m_{n+1}}\sum_{\nu=0}^{n}(\nu+1)\kappa^{-\nu}}.$$

Using

$$\sum_{\nu=0}^{\infty}(\nu+1)\kappa^{-\nu} = \frac{\kappa^2}{(\kappa-1)^2}$$

and (4.35), one obtains

$$\lim_{n\to\infty} \frac{\kappa^n}{m_{n+1}} \sum_{\nu=0}^{n}(\nu+1)\kappa^{-\nu} = \frac{\kappa}{\kappa-1}\,\frac{1}{(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma)}, \tag{4.40}$$

therefore

$$\lim_{n\to\infty} \prod_{\nu=0}^{n} t_\nu^{-\frac{\alpha r}{\kappa}\frac{\kappa^{n-\nu}}{m_{n+1}}} = 2^{\frac{\alpha r}{\kappa}\frac{\kappa}{\kappa-1}\frac{1}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}}. \tag{4.41}$$

Using again that $t_\nu = 2^{-\nu-1}$ together with (4.33) and (4.35), gives

$$\lim_{n\to\infty} \frac{1}{\kappa m_{n+1}} \sum_{\nu=0}^{n} t_\nu \kappa^{n-\nu} = \frac{\kappa^{-1}(\kappa-1)(2\kappa-1)^{-1}}{(\kappa-1)m_0 + r\kappa^{-1}(1-\gamma)}$$

and so

$$\lim_{n\to\infty} e^{\omega(\beta-1)\frac{r}{\kappa m_{n+1}}\sum_{\nu=0}^{n} t_\nu\kappa^{n-\nu} t} = e^{\omega(\beta-1)\frac{r\kappa^{-1}(\kappa-1)(2\kappa-1)^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}t}. \tag{4.42}$$

Similarly, we obtain that

$$\lim_{n\to\infty} \frac{1}{m_{n+1}} \sum_{\nu=0}^{n} t_\nu \kappa^{n-\nu} m_{\nu+1} = 1 - \frac{r(1-\gamma)\kappa^{-1}(2\kappa-1)^{-1}}{(\kappa-1)m_0 + r\kappa^{-1}(1-\gamma)}$$

and so

$$\lim_{n\to\infty} e^{\omega\frac{1}{m_{n+1}}\sum_{\nu=0}^{n} t_\nu\kappa^{n-\nu}m_{\nu+1} t} = e^{\omega\left(1-\frac{r(1-\gamma)\kappa^{-1}(2\kappa-1)^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}\right)t}. \tag{4.43}$$

Next, by (4.29), one has

$$C_{\varphi^*,1}^{\frac{r\kappa^{n+1}}{m_{n+1}}\sum_{\nu=1}^{n+1}\frac{m_{\nu-1}}{\kappa^{\nu}m_\nu}} \le \prod_{\nu=1}^{n+1} C_{\varphi^*,m_\nu}^{\frac{r}{m_\nu}\frac{\kappa^{n+1-\nu}m_{\nu-1}}{m_{n+1}}} \le \prod_{\nu=1}^{n+1} C_{\varphi^*,2}^{\frac{r\kappa^{n+1}}{m_{n+1}}\sum_{\nu=1}^{n+1}\frac{m_{\nu-1}}{\kappa^{\nu}m_\nu}}. \tag{4.44}$$

Since by (4.33), one has that $a_\nu := \frac{m_{\nu-1}}{\kappa^\nu m_\nu}$ satisfies $\lim_{\nu\to\infty}\lvert\frac{a_{\nu+1}}{a_\nu}\rvert = \frac{1}{\kappa}$, the ratio test implies that the series $\sum_{\nu=1}^{\infty}\frac{m_{\nu-1}}{\kappa^\nu m_\nu}$ converges. Furthermore, (4.35) yields

$$\lim_{n\to\infty} \frac{\kappa^{n+1}}{m_{n+1}} = \frac{(\kappa-1)}{(\kappa-1)m_0 + r\,\kappa^{-1}(1-\gamma)}. \tag{4.45}$$

Thus

$$\lim_{n\to\infty}\frac{r\kappa^{n+1}}{m_{n+1}}\sum_{\nu=1}^{n+1}\frac{m_{\nu-1}}{\kappa^{\nu}m_{\nu}}=\frac{r(\kappa-1)}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{m_{\nu-1}}{\kappa^{\nu}m_{\nu}}$$

so that sending $n\to\infty$ in (4.44) yields

$$\begin{aligned}C_{\varphi^*,1}^{\frac{r(\kappa-1)}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{m_{\nu-1}}{\kappa^{\nu}m_{\nu}}}&\le\liminf_{n\to\infty}\prod_{\nu=1}^{n+1}C_{\varphi^*,m_\nu}^{\frac{r}{m_\nu}\frac{\kappa^{n+1-\nu}m_{\nu-1}}{m_{n+1}}}\\&\le\limsup_{n\to\infty}\prod_{\nu=1}^{n+1}C_{\varphi^*,m_\nu}^{\frac{r}{m_\nu}\frac{\kappa^{n+1-\nu}m_{\nu-1}}{m_{n+1}}}\\&\le C_{\varphi^*,2}^{\frac{r(\kappa-1)}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{m_{\nu-1}}{\kappa^{\nu}m_{\nu}}}.\end{aligned}\tag{4.46}$$

Using again (4.45), we see that

$$\lim_{n\to\infty}\lVert u-\hat{u}\rVert_{\kappa m_0}^{m_0\frac{\kappa^{n+1}}{m_{n+1}}}=\lVert u-\hat{u}\rVert_{\kappa m_0}^{\frac{(\kappa-1)\,m_0}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}.\tag{4.47}$$

It remains to control the product

$$\prod_{\nu=1}^{n+1}C_{m_\nu}^{\frac{\kappa^{n-\nu}}{m_{n+1}}}=\prod_{\nu=1}^{n+1}\left[\frac{\beta\,m_n}{m_{n-1}\kappa}\right]^{\frac{\kappa^{n-\nu}}{m_{n+1}}}\times C^{\frac{r\sum_{\nu=1}^{n+1}\kappa^{n-\nu}}{m_{n+1}}}\tag{4.48}$$

as $n\to\infty$. Since $\kappa>1$ and by (4.35),

$$\lim_{n\to\infty}\frac{1}{m_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n-\nu}=\frac{\kappa^{-1}}{(\kappa-1)m_0+r\,\kappa^{-1}(1-\gamma)}\tag{4.49}$$

and so

$$\lim_{n\to\infty}C^{\frac{r\sum_{\nu=1}^{n+1}\kappa^{n-\nu}}{m_{n+1}}}=C^{\frac{r\kappa^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}.\tag{4.50}$$

For every $n\ge1$, the quotient $\frac{\gamma\,m_n}{m_{n-1}\kappa}=\frac{\gamma\,m_n}{m_n+r\kappa^{-1}(\gamma-1)}$ can be controlled by

$$\gamma<\frac{\gamma\,m_n}{m_{n-1}\kappa}<\frac{\gamma}{1+\frac{r}{m_0}\kappa^{-1}(\gamma-1)}\qquad\text{if }0<\gamma<1$$

and by

$$\frac{\gamma}{1+\frac{r}{m_0}\kappa^{-1}(\gamma-1)}<\frac{\gamma\,m_n}{m_{n-1}\kappa}<\gamma\qquad\text{if }\gamma\ge1.$$

Thus for general $\gamma>0$, there are constants $C_1,C_2>0$ such that

$$C_1^{\frac{1}{m_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n-\nu}}\le\prod_{\nu=1}^{n+1}\left[\frac{\gamma\,m_n}{m_{n-1}\kappa}\right]^{\frac{\kappa^{n-\nu}}{m_{n+1}}}\le C_2^{\frac{1}{m_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n-\nu}}\tag{4.51}$$

for every $n\ge0$ and so by (4.49), sending $n\to\infty$ in (4.51) yields

$$C_1^{\frac{\kappa^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}\le\liminf_{n\to\infty}\prod_{\nu=1}^{n+1}\left[\frac{\gamma\,m_n}{m_{n-1}\kappa}\right]^{\frac{\kappa^{n-\nu}}{m_{n+1}}}\le\limsup_{n\to\infty}\prod_{\nu=1}^{n+1}C_{m_\nu}^{\frac{\kappa^{n-\nu}}{m_{n+1}}}\le C_2^{\frac{\kappa^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}.$$

Thus sending $n\to\infty$ in inequality (4.37) and using (4.39), (4.41), (4.42), (4.43), (4.47), (4.50), (4.46) together with the fact that $m_n\nearrow\infty$ as $n\to\infty$ yields

$$\lVert T_tu-T_t\hat{u}\rVert_\infty\le\left[C_2\,C^r\right]^{\frac{r\kappa^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}e^{\omega\left(\frac{(\beta-1)\kappa+\gamma-\beta}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}+1\right)t}\,t^{-\frac{\alpha\,r\kappa^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}\times$$

$$\times\, C_{\varphi^*,2}^{\frac{r(\kappa-1)}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{m_\nu-1}{\kappa^\nu m_\nu}}\,\|u-\hat{u}\|_{\kappa m_0}^{\frac{(\kappa-1)\,m_0}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}$$

showing that inequality (4.30) holds for u, $\hat{u}\in L^{\kappa m_0}\cap L^\infty(\Sigma,\mu)$. By hypothesis, the semigroup $\{T_t\}$ acts on $L^{\kappa m_0}\cap L^\infty(\Sigma,\mu)$, that is, every T_t maps $L^{\kappa m_0}\cap L^\infty(\Sigma,\mu)$ to $L^{\kappa m_0}\cap L^\infty(\Sigma,\mu)$. Since $L^{\kappa m_0}\cap L^\infty(\Sigma,\mu)$ is dense in $L^{\kappa m_0}(\Sigma,\mu)$, a standard approximation argument shows that the first claim of this iteration lemma holds. This completes the proof. $\square$

Our second extrapolation result towards L^∞ is adapted to semigroups enjoying the L^q-L^r-regularising effect (1.17) for $1\le q$, $r<\infty$ and some $u_0\in L^r\cap L^{q_0}(\Sigma,\mu)$ generated by quasi m-completely accretive operators A on $L^q(\Sigma,\mu)$ (Section 2.2).

Theorem 4.13. *Let (Σ,μ) be a separable measure space, $1\le q,r<\infty$, and $\{T_t\}_{t\ge0}$ be a semigroup acting on $L^q\cap L^\infty(\Sigma,\mu)$ with exponential growth (2.30) for some $\omega\ge0$ and every $q\le\tilde{q}\le\infty$. Further, suppose there exists $u_0\in L^q\cap L^\infty(\Sigma,\mu)$ satisfying $T_tu_0=u_0$ for all $t\ge0$, and there exist $C>0$ and exponents α, β, $\gamma>0$ such that*

$$\|T_tu-u_0\|_r\le\left(\tfrac{C}{q}\right)^{1/\sigma}\,t^{-\alpha}\,e^{\omega\beta t}\,\|u-u_0\|_q^\gamma \tag{1.17}$$

holds for every $t>0$ and $u\in L^q(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)$. If the parameter γ, r, q satisfy (4.21), then

$$\|T_tu-u_0\|_\infty\lesssim e^{\omega(\gamma^*+1)t}\,t^{-\delta}\,\|u-u_0\|_{\gamma\,r\,q^{-1}\,m_0}^{\gamma}$$

for every $t>0$ and $u\in L^{\gamma\,r\,q^{-1}\,m_0}(\Sigma,\mu)$, where δ, γ and β^* are given by (4.23) and $m_0\ge q\,\gamma^{-1}$ such that (4.24) holds.

The proof of this theorem proceeds analogously as the one for Theorem 4.10, where one replaces the application of interpolation Theorem 4.6 by Theorem 4.7 or Theorem 4.8. Furthermore, one applies the extrapolation argument from Remark 4.9 and replaces Lemma 4.12 by the following one. We leave the details of the proof to the interested reader.

Lemma 4.14. *Suppose there exist $\kappa>1$, β, $\gamma>0$, $1\le r<\infty$ and $m_0\ge r\,K^{-1}$ such that (4.27) holds. Let $\{T_t\}_{t\ge0}$ be a semigroup acting on $L^{\kappa m_0}(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)$ satisfying*

$$\begin{aligned}\|T_tu-u_0\|_{m\,\kappa}\le\left[\tfrac{\gamma\,m}{m+r\kappa^{-1}(\gamma-1)}\right]^{\frac{1}{m\,\kappa}}\left[C\,e^{\omega\beta t}\,t^{-\alpha}\right]^{\frac{1}{m}\frac{r}{\kappa}}\,e^{\omega(1-\frac{1}{m}\frac{r}{\kappa})t}\,\times\\ \times\,C_{\varphi^*,m}^{\frac{1}{m}\frac{r}{\kappa}(\frac{1}{m}\frac{r}{\kappa}(\gamma-1)+1)}\,\|u-u_0\|_{m+r\,\kappa^{-1}(\gamma-1)}^{\frac{1}{m}r\,\kappa^{-1}(\gamma-1)+1}\end{aligned}$$

for every $u\in L^{\kappa m_0}(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)$, $t>0$, $m>m_0$ and some $u_0\in L^{\kappa m_0}\cap L^\infty(\Sigma,\mu)$, where $C_{\varphi^*,m}$ satisfies (4.29) for some constants $C_{\varphi^*,1}$, $C_{\varphi^*,2}>0$ independent of m. Then

$$\begin{aligned}\|T_tu-u_0\|_\infty\lesssim e^{\omega\left(\frac{(\beta-1)\kappa+\gamma-\beta}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}+1\right)t}\,t^{-\frac{\alpha\,r\,\kappa^{-1}}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}\,\times\\ \times\,\|u-u_0\|_{\kappa m_0}^{\frac{(\kappa-1)\,m_0}{(\kappa-1)m_0+r\kappa^{-1}(1-\gamma)}}.\end{aligned}$$

for every $u\in L^{\kappa m_0}(\Sigma,\mu)$ and every $t>0$.

The proof of Lemma 4.14 proceeds as the one of Lemma 4.12. We omit the details.

4.4. **An alternative approach to arrive at L^∞.** It is a fundamental fact that semigroups $\{T_t\}_{t\ge 0}$ generated by operators $-A$ in L^1 with a (c-)complete resolvent (Section 2.3) are not, in general, contractive with respect to the L^∞-norm (cf. [91, Section A.11]). Thus, if one wants to extend the L^q-L^r-regularisation effect of $\{T_t\}_{t\ge 0}$ to an L^q-L^∞-regularisation effect, one needs to proceed by an alternative approach. One possible way is the following one: firstly, show that A satisfies a *one-parameter family of Gagliardo-Nirenberg type inequalities*, and then by employing Theorem 3.10, deduce that the semigroup $\{T_t\}$ satisfies a *sequence of L^{q_n}-$L^{q_{n+1}}$-regularisation effects* for some sequence $(q_n)_{n\ge 1} \subseteq (1,\infty)$ with $q_n \nearrow \infty$. This method has been employed in the past by many authors. But to the best of our knowledge, Véron has been the first to use this method in [92] in the context of nonlinear semigroups of *contractive* mappings on $L^1(\Sigma,\mu)$ (see also [38] for another use of this type of argument in linear semigroup theory). Here, we extend and simplify this method to nonlinear semigroups $\{T_t\}_{t\ge 0}$ of *Lipschitz continuous* mappings T_t on $L^1(\Sigma,\mu)$ with constant $e^{\omega t}$, in other words, of exponential growth (2.30) for $q=1$.

Theorem 4.15. *Let $A+\omega I$ be m-accretive in $L^1(\Sigma,\mu)$ for some $\omega \ge 0$ with trace $A_{1\cap\infty}$ of A on $L^1\cap L^\infty(\Sigma,\mu)$ satisfying range condition (1.19). Suppose there exist $\kappa>1$, $m>0$, and $q_0\ge p\ge 1$ such that $\kappa m q_0 \ge 1$ and*

$$(\kappa-1)q_0+p-1-\tfrac{1}{m}>0, \tag{1.20}$$

and there exist $C>0$ and $(u_0,0)\in A_{1\cap\infty}$ such that for every $q\ge q_0$, the trace $A_{1\cap\infty}$ satisfies Sobolev type inequality

$$\|u-u_0\|_{\kappa m q}^{mq} \le \tfrac{C\,(q/p)^p}{q-p+1}\,\Big[[u-u_0,v]_{(q-p+1)m+1}+\omega\|u-u_0\|_{(q-p+1)m+1}^{(q-p+1)m+1}\Big] \tag{1.21}$$

for every $(u,v)\in A_{1\cap\infty}$, and for every $\lambda>0$ satisfying $\lambda\omega<1$, the resolvent J_λ of A satisfies (3.9) for $\tilde{q}=\kappa m q$. Then, there is a $\beta^*\ge 0$ such that the semigroup $\{T_t\}_{t\ge 0}\sim -A$ on $\overline{D(A)}^{L^1}$ satisfies

$$\|T_t u-u_0\|_\infty \lesssim e^{\omega\beta^* t}\,t^{-\frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}\,\|u-u_0\|_{\kappa m q_0}^{\frac{(\kappa-1)q_0}{(\kappa-1)q_0+p-1-\frac{1}{m}}} \tag{4.52}$$

for every $t>0$ and $u\in\overline{D(A)}^{L^1}\cap L^\infty(\Sigma,\mu)$.

Proof. From Theorem 3.10, we can conclude that the semigroup $\{T_t\}_{t\ge 0}$ satisfies inequality (4.53) below for every $t>0$, $u\in\overline{D(A)}^{L^1}\cap L^\infty(\Sigma,\mu)$ and $q\ge q_0$. Thus, we can deduce the claim of this theorem from the subsequent iteration Lemma 4.16. □

Lemma 4.16. *Suppose there exist $\kappa>1$, $m>0$, $q_0\ge p\ge 1$ such that $\kappa m q_0\ge 1$ and (1.20) hold. Furthermore, suppose, there exists $C>0$ such that the semigroup $\{T_t\}_{t\ge 0}$ on $\overline{D(A)}^{L^1}$ satisfies*

$$\begin{aligned}\|T_t u-u_0\|_{\kappa q m} \le \left[\tfrac{C\,(q/p)^p}{(q-p+1)((q-p+1)m+1)}\right]^{\frac{1}{qm}} e^{\omega(\frac{(q-p+1)m+1}{qm}+1)t}\,\times\\ \times\, t^{-\frac{1}{qm}}\,\|u-u_0\|_{(q-p+1)m+1}^{\frac{(q-p+1)m+1}{qm}}\end{aligned} \tag{4.53}$$

for every $u\in\overline{D(A)}^{L^1}\cap L^\infty(\Sigma,\mu)$ and $q\ge q_0$. Then there is a $\beta^*\ge 0$ such that the semigroup $\{T_t\}_{t\ge 0}\sim -A$ on $\overline{D(A)}^{L^1}$ satisfies inequality (4.52) for every $u\in\overline{D(A)}^{L^1}\cap L^\infty(\Sigma,\mu)$ and every $t>0$.

Proof. We fix some $q_0 \geq p$ and set

$$q_{n+1} = \kappa\, q_n + p - 1 - \frac{1}{m} \qquad \text{for every } n \in \mathbb{N}_0.$$

Then one can show by induction over $n \in \mathbb{N}_0$ that

$$q_n = \kappa^n (q_0 - ((p-1) - \tfrac{1}{m})) + ((p-1) - \tfrac{1}{m}) \sum_{\nu=0}^{n} \kappa^\nu,$$

that is

$$q_n = \frac{\kappa^n}{\kappa - 1}\left[(\kappa-1)q_0 + p - 1 - \tfrac{1}{m}\right] - \frac{p - 1 - \frac{1}{m}}{\kappa - 1}. \tag{4.54}$$

Using (4.54), we see that

$$q_{n+1} - q_n = \kappa^n \Big[(\kappa-1)q_0 + p - 1 - \tfrac{1}{m}\Big]$$

hence the sequence $(q_n)_{n\geq 0}$ is strictly increasing if and only if q_0 satisfies condition (4.27). Moreover, by (4.54), since $\kappa > 1$, and by (4.27), we see that $q_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{q_n}{\kappa^n} = \frac{(\kappa-1)q_0 + p - 1 - \frac{1}{m}}{\kappa - 1}. \tag{4.55}$$

Now, let $u \in \overline{D(A)}^{L^1} \cap L^{\kappa q_0 m}(\Sigma,\mu)$. Then, by construction of q_1, we find that $u \in L^{(q_1 - p + 1)m+1}(\Sigma,\mu)$ and so by (4.53), $T_t u \in \overline{D(A)}^{L^1} \cap L^{\kappa q_1 m}(\Sigma,\mu)$ for all $t > 0$. By construction of $(q_n)_{n\geq 1}$ and since $\{T_t\}_{t\geq 0}$ is a semigroup satisfying (4.53), we see that $T_t u \in \overline{D(A)}^{L^1} \cap L^{\kappa q_n m}(\Sigma,\mu)$ for all $n \geq 1$ and $t > 0$. Thus, inserting the sequence $(q_n)_{n\geq 0}$ into (4.53) yields

$$\|T_t u - u_0\|_{\kappa m q_{n+1}} \leq C_{q_{n+1}}\, t^{-1/m q_{n+1}}\, e^{\omega(\frac{\kappa m q_n}{m q_{n+1}} + 1)t}\, \|u - u_0\|_{\kappa m q_n}^{\kappa q_n / q_{n+1}} \tag{4.56}$$

for every $t > 0$ with

$$C_{q_{n+1}} = C^{\frac{1}{m q_{n+1}}} \left[\frac{(\frac{q_{n+1}}{p})^p}{(q_{n+1} - p + 1)(\kappa m q_n)}\right]^{\frac{1}{m q_{n+1}}}$$

for every $n \in \mathbb{N}_0$. Let $(t_\nu)_{\nu\geq 0}$ be any sequence in $[0,1]$ such that $\sum_{\nu=0}^\infty t_\nu = 1$, which will be specified below. Then by (4.56), we obtain that

$$\begin{aligned} \|T_{t \sum_{\nu=0}^n t_\nu} u - u_0\|_{\kappa q_{n+1}} \leq \prod_{\nu=1}^{n+1} C_{q_\nu}^{\kappa^{n+1-\nu} \frac{q_\nu}{q_{n+1}}} \prod_{\nu=0}^{n} t_\nu^{-\frac{\kappa^{n-\nu}}{m q_{n+1}}}\, t^{-\frac{1}{m q_{n+1}} \sum_{\nu=0}^n \kappa^\nu} \times \\ \times e^{\omega \sum_{\nu=0}^n t_\nu (\frac{\kappa q_\nu}{q_{\nu+1}} + 1) \frac{\kappa^{n-\nu} q_{\nu+1}}{q_{n+1}} t}\, \|u - u_0\|_{\kappa m q_0}^{\kappa^{n+1} q_0 / q_{n+1}}. \end{aligned} \tag{4.57}$$

Since $\kappa > 1$, by (4.55), $q_n \to \infty$ as $n \to \infty$ and since $(\kappa - 1)q_0 + p - 2 > 0$ by assumption (4.27), we see that

$$\lim_{n\to\infty} \frac{1}{q_{n+1}} \sum_{\nu=0}^n \kappa^\nu = \frac{1}{(\kappa-1)q_0 + p - 1 - \frac{1}{m}}. \tag{4.58}$$

Thus

$$\lim_{n\to\infty} t^{-\frac{1}{m q_{n+1}} \sum_{\nu=0}^n \kappa^\nu} = t^{-\frac{1}{m((\kappa-1)q_0 + p - 1 - \frac{1}{m})}} \qquad \text{for every } t > 0. \tag{4.59}$$

If we choose, for instance, $t_\nu = 2^{-\nu-1}$, then

$$\prod_{\nu=0}^{n} t_\nu^{-\frac{\kappa^{n-\nu}}{mq_{n+1}}} = 2^{\frac{\kappa^n \sum_{\nu=0}^{n}(\nu+1)\kappa^{-\nu}}{mq_{n+1}}}.$$

Using

$$\sum_{\nu=0}^{\infty}(\nu+1)\kappa^{-\nu} = \frac{\kappa^2}{(\kappa-1)^2} \tag{4.60}$$

and (4.55), one obtains

$$\lim_{n\to\infty} \frac{\kappa^n \sum_{\nu=0}^{n}(\nu+1)\kappa^{-\nu}}{mq_{n+1}} = \frac{\kappa}{m(\kappa-1)} \frac{1}{(\kappa-1)q_0+p-1-\frac{1}{m}} \tag{4.61}$$

and so

$$\lim_{n\to\infty} \prod_{\nu=0}^{n} t_\nu^{-\frac{\kappa^{n-\nu}}{mq_{n+1}}} = 2^{\frac{\kappa}{m(\kappa-1)[(\kappa-1)q_0+p-1-\frac{1}{m}]}}. \tag{4.62}$$

Next, by (4.55), we see that

$$\lim_{n\to\infty} \kappa^{n+1}\frac{q_0}{q_{n+1}} = \frac{(\kappa-1)q_0}{(\kappa-1)q_0+p-1-\frac{1}{m}},$$

thus

$$\lim_{n\to\infty} \|u-u_0\|_{\kappa m q_0}^{\kappa^{n+1}q_0/q_{n+1}} = \|u-u_0\|_{\kappa m q_0}^{(\kappa-1)q_0/((\kappa-1)q_0+p-1-\frac{1}{m})}. \tag{4.63}$$

Further, since for $a_\nu := 2^{-\nu}(\frac{\kappa q_\nu}{q_{\nu+1}}+1)\kappa^{-\nu}q_{\nu+1}$ for every $\nu \geq 0$, (4.54) yields

$$\lim_{\nu\to\infty}\left|\frac{a_{\nu+1}}{a_\nu}\right| = \frac{1}{2},$$

the ratio test implies that the series

$$\frac{1}{2}\sum_{\nu=0}^{\infty} 2^{-\nu}\left(\frac{\kappa q_\nu}{q_{\nu+1}}+1\right)\kappa^{-\nu}q_{\nu+1}$$

converges; we denote the sum of the series by $S \geq 0$. Thus, by (4.55),

$$\lim_{n\to\infty} \frac{\kappa^{n+1}}{q_{n+1}}\frac{1}{2\kappa}\sum_{\nu=0}^{\infty} 2^{-\nu}\left(\frac{\kappa q_\nu}{q_{\nu+1}}+1\right)\kappa^{-\nu}q_{\nu+1} = \frac{\kappa-1}{(\kappa-1)q_0+p-1-\frac{1}{m}}\frac{1}{2\kappa}S =: \beta^*$$

and so

$$\lim_{n\to\infty} e^{\omega\sum_{\nu=0}^{n} t_\nu(\frac{\kappa q_\nu}{q_{\nu+1}}+1)\frac{\kappa^{n-\nu}q_{\nu+1}}{q_{n+1}}t} = e^{\omega\beta^* t}. \tag{4.64}$$

It remains to show that we can control the product

$$\prod_{\nu=1}^{n+1} C_{q_\nu}^{\frac{\kappa^{n+1-\nu}q_\nu}{q_{n+1}}} = \prod_{\nu=1}^{n+1} C^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} \times \prod_{\nu=1}^{n+1}\left[\frac{(\frac{q_{n+1}}{p})}{(q_{n+1}-p+1)(\kappa m q_n)}\right]^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} \tag{4.65}$$

as $n\to\infty$. First, note that

$$\prod_{\nu=1}^{n+1} C^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} = e^{\frac{\log C}{mq_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n+1-\nu}} = e^{\frac{\log C}{mq_{n+1}}\sum_{\nu=0}^{n}\kappa^{\nu}}.$$

Thus by (4.58),

$$\lim_{n\to\infty}\prod_{\nu=1}^{n+1} C^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} = C^{\frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}. \tag{4.66}$$

By (4.54), we have that

$$q_\nu = \frac{\kappa^\nu[(\kappa-1)q_0+p-1-\frac{1}{m}]-((p-1)-\frac{1}{m})}{\kappa-1}$$

for every $\nu\in\mathbb{N}_0$. From this, we conclude that

$$q_\nu \le M\kappa^\nu \tag{4.67}$$

for every $\nu\in\mathbb{N}_0$, where

$$M := \begin{cases} \frac{[(\kappa-1)q_0+p-1-\frac{1}{m}]}{\kappa-1} & \text{if } (p-1)-\frac{1}{m}\ge 0, \text{ and}\\ q_0 & \text{if } (p-1)-\frac{1}{m}<0.\end{cases}$$

Applying (4.67) and using that $q_\nu\ge q_0\ge p>1$ and $\kappa m>1$, one sees that on the one hand

$$\prod_{\nu=1}^{n+1}\left[\frac{(q_\nu/p)^p}{(q_\nu-p+1)\kappa m q_\nu}\right]^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} \le \prod_{\nu=1}^{n+1}\left(\frac{M}{p}\right)^{\frac{p\kappa^{n+1-\nu}}{mq_{n+1}}}\prod_{\nu=1}^{n+1}\kappa^{\frac{\nu\kappa^{n+1-\nu}}{mq_{n+1}}},$$

and by (4.58), (4.60), and (4.55), one has

$$\lim_{n\to\infty}\prod_{\nu=1}^{n+1}\left(\frac{M}{p}\right)^{\frac{p\kappa^{n+1-\nu}}{mq_{n+1}}} = \left(\frac{M}{p}\right)^{\frac{p}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}$$

and

$$\lim_{n\to\infty}\prod_{\nu=1}^{n+1}\kappa^{\frac{\nu\kappa^{n+1-\nu}}{mq_{n+1}}} = \kappa^{\frac{\kappa}{(\kappa-1)m((\kappa-1)q_0+p-1-\frac{1}{m})}}.$$

On the other hand, by using that $q_\nu-p+1\le q_\nu$, $q_\nu\ge p$ and (4.67), one finds

$$\prod_{\nu=1}^{n+1}\left[\frac{(q_\nu/p)^p}{(q_\nu-p+1)\kappa m q_\nu}\right]^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} \ge \prod_{\nu=1}^{n+1}(M^3K)^{-\frac{\kappa^{n+1-\nu}}{mq_{n+1}}}\prod_{\nu=1}^{n+1}\kappa^{-2\nu\frac{\kappa^{n+1-\nu}}{mq_{n+1}}}$$

with

$$\lim_{n\to\infty}\prod_{\nu=1}^{n+1}(M^3K)^{-\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} = (M^3K)^{-\frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}$$

and

$$\lim_{n\to\infty}\prod_{\nu=1}^{n+1}\kappa^{-2\nu\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} = \kappa^{-\frac{2\kappa}{(\kappa-1)m((\kappa-1)q_0+p-1-\frac{1}{m})}}.$$

Thus, by taking

$$M_1 = \left(\frac{M}{p}\right)^{\frac{p}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}\kappa^{\frac{\kappa}{(\kappa-1)m((\kappa-1)q_0+p-1-\frac{1}{m})}}$$

and

$$M_2 = (M^3K)^{-\frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}\kappa^{-\frac{2\kappa}{(\kappa-1)m((\kappa-1)q_0+p-1-\frac{1}{m})}},$$

we have that

$$0 < M_1 \le \liminf_{n\to\infty} \prod_{\nu=1}^{n+1} \left[\frac{(q_\nu/p)^p}{(q_\nu-p+1)\kappa m q_\nu}\right]^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}}$$

$$\le \limsup_{n\to\infty} \prod_{\nu=1}^{n+1} \left[\frac{(q_\nu/p)^p}{(q_\nu-p+1)\kappa m q_\nu}\right]^{\frac{\kappa^{n+1-\nu}}{mq_{n+1}}} \le M_2 < \infty$$

hence by (4.65) and (4.66),

$$0 < C^{\frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}\, M_1 \le \liminf_{n\to\infty} \prod_{\nu=1}^{n+1} C_{q_\nu}^{\frac{\kappa^{n+1-\nu}q_\nu}{q_{n+1}}}$$

$$\le \limsup_{n\to\infty} \prod_{\nu=1}^{n+1} C_{q_\nu}^{\frac{\kappa^{n+1-\nu}q_\nu}{q_{n+1}}} \le C^{\frac{1}{m((\kappa-1)q_0+p-1-\frac{1}{m})}}\, M_2 < \infty.$$

Thus sending $n \to \infty$ in inequality (4.57) and using the limits (4.59), (4.62), (4.63) and (4.64) together with the fact that $q_n \nearrow \infty$ as $n \to \infty$ yields the desired inequality (4.52). This completes the proof of the lemma. $\square$

5. Application I: Mild solutions in L^1 are weak energy solutions

This section is concerned with illustrating a first application of the L^1-L^∞ regularisation estimate

(5.1) $$\|T_t u\|_\infty \le \tilde{C}\, t^{-\alpha}\, e^{\omega\beta t}\, \|u\|_1^\gamma, \qquad \text{holding for all } t>0,\ u \in \overline{D(A_{1\cap\infty}\phi)}^{L^1}$$

for some exponents $\alpha, \beta, \gamma > 0$, and a constant $\tilde{C} > 0$, satisfied by the semigroup $\{T_t\} \sim -(A_{1\cap\infty}\phi + F)$ on $\overline{D(A_{1\cap\infty}\phi)}^{L^1}$.

Let A be an m-completely accretive operator on $L^2(\Sigma,\mu)$ and which is the realisation in $L^2(\Sigma,\mu)$ of a monotone operator $\Psi' : V \to V'$ of a convex, Gâteaux-differentiable real-valued functional Ψ defined on a reflexive Banach space V (see the precise hypotheses on A and V below). Further,

($\mathcal{H}$a) let ϕ be a strictly increasing continuous functions on $\mathbb{R}$ with Yosida operator β_λ of $\beta = \phi^{-1}$ satisfying condition (2.43), ($\lambda > 0$),

($\mathcal{H}$b) let F be the Nemytski operator in $L^q(\Sigma,\mu)$, ($1 \le q \le \infty$), of a Carathéodory function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ satisfying Lipschitz condition (2.17) for some $\omega \ge 0$ and $f(x,0) = 0$ for a.e. $x \in \Sigma$.

Then, the aim of this section is to show that for every initial value $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1}$, the mild solution u of Cauchy problem

(5.2) $$\begin{cases} \frac{du}{dt} + \overline{A_{1\cap\infty}\phi} u + F(u) = 0 & \text{in } L^1(\Sigma,\mu) \text{ on } (0,+\infty), \\ u(0) = u_0 & \end{cases}$$

is, in fact, a *weak energy solution* of problem

(5.3) $$\begin{cases} \frac{du}{dt} + \Psi'(\phi(u)) + F(u) = 0 & \text{in } V' \text{ on } (0,\infty), \\ u(0) = u_0 & \end{cases}$$

in the sense of Definition 5.2 below.

In this section, we work in the following framework. We assume that the classical Lebesgue space $L^q(\Sigma,\mu)$, $1 \le q \le \infty$, is defined on a finite measure space (Σ,μ), and V be a reflexive Banach space such that there are $a \geq 0$ and a semi-norm $|\cdot|_V$ on V such that

$$|\cdot|_V + a\|\cdot\|_2$$

defines an equivalent norm on V. Further, suppose the continuous embedding $i : V \to L^2(\Sigma,\mu)$ has a dense image. Then, the adjoint operator i^* of i from the dual space $(L^2(\Sigma,\mu))'$ of $L^2(\Sigma,\mu)$ to the dual space V' of V is also an injective linear bounded operator. After identifying $L^2(\Sigma,\mu)$ with $(L^2(\Sigma,\mu))'$, we see that

$$V \hookrightarrow L^2(\Sigma,\mu) \hookrightarrow V',$$

where each inclusion "$\hookrightarrow$" denotes a continuous embedding with a dense image. Moreover, the duality brackets $\langle\cdot,\cdot\rangle_{V',V}$ of $V \times V'$ and the inner product $\langle\cdot,\cdot\rangle$ on $L^2(\Sigma,\mu)$ coincide whenever both make sense (cf. [25, Remark 3, Chapter 5.2]), that is

$$\langle u, v\rangle_{V',V} = \langle u, v\rangle \qquad \text{for all } u \in L^2(\Sigma,\mu) \text{ and } v \in V. \tag{5.4}$$

Thus, in order to keep our notation simple, we only employ the brackets $\langle\cdot,\cdot\rangle$.

Further, we assume that $\Psi : V \to \mathbb{R}$ is a convex, lower semicontinuous, and Gâteaux differentiable functional satisfying

($\mathcal{H}$i): *there are* $1 < p < \infty$, $\eta > 0$, $C > 0$ *such that*

$$\langle \Psi'(v), v\rangle \geq \eta |v|_V^p \qquad \textit{and} \tag{5.5}$$

$$\|\Psi'(v)\|_{V'} \leq C\,|v|_V^{p-1} \qquad \textit{for every } v \in V, \tag{5.6}$$

($\mathcal{H}$ii): $\Psi' : V \to V'$ *is hemicontinuous, that is, for every* u, v, $w \in V$, *the function* $\lambda \mapsto \langle \Psi'(v + \lambda u), w\rangle$ *is continuous on* $\mathbb{R}$,

($\mathcal{H}$iii): *there is a* $\varepsilon > 0$ *such that* $\Psi + \varepsilon\|\cdot\|_2^2$ *is weakly coercive in* V, *that is, for every* $c \in \mathbb{R}$, *the sub-level set* $E_c := \{v \in V \,|\, \Psi(v) + \varepsilon\|v\|_2^2 \leq c\}$ *is relatively compact with respect to the weak topology on* V.

($\mathcal{H}$iv): *the subgradient* $A := \partial_{L^2}\Psi^{L^2}$ *in* $L^2(\Sigma,\mu)$ *of the extended functional* Ψ^{L^2} *of* Ψ *on* $L^2(\Sigma,\mu)$ *is an m-completely accretive operator in* $L^2(\Sigma,\mu)$.

($\mathcal{H}$v): *the functional* Ψ *is related to a "Poincaré type inequality"*

$$\|u\|_p \leq C\,\Psi(u) \qquad \text{for all } u \in V, \tag{5.7}$$

where the constant $C > 0$ *is independent of* $u \in V$.

Remark 5.1. We note that hypothesis ($\mathcal{H}$iii) is needed only to ensure that that the extended functional Ψ^{L^2} of Ψ on $L^2(\Sigma,\mu)$ is lower semicontinuous on $L^2(\Sigma,\mu)$. For a more detailed discussion on this, we refer the interested reader to [31].

Definition 5.2. Let $1 < p < \infty$ with conjugate exponent $p' = \frac{p}{p-1}$, $T > 0$, the operator $\Psi' : V \to V'$ satisfy the hypotheses ($\mathcal{H}$i) and ($\mathcal{H}$ii), and ϕ be a continuous function on $\mathbb{R}$. Then for given $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma,\mu)$, we call a function $u \in C([0,T]; L^1(\Sigma,\mu))$ a *weak energy solution* of (5.3) if $u(0) = u_0$ in $L^1(\Sigma,\mu)$, and for every $0 < \delta < T$,

$$\frac{du}{dt} \in L^{p'}(\delta, T; V'), \qquad \phi(u) \in L^p(\delta, T; V),$$

and
$$\int_\delta^T \Big\{\Big\langle\frac{du}{dt},v\Big\rangle_{V',V} + \langle \Psi'\phi(u(t)),v\rangle + \langle F(u(t)),v\rangle\Big\}\,\mathrm{d}t = 0$$
for all $v \in L^p(\delta,T;V)$.

Remark 5.3. To the best of our knowledge, the notion of *weak energy solution* was introduced in [91, Section 5.3.2] in connection with the porous media operator $A\phi = \Delta\phi$. The word *energy* in this notion indicates that the solution u of (5.3) has the property
$$\int_\delta^t \Psi(\phi(u(t)))\,\mathrm{d}\mu\,\mathrm{d}s \qquad \text{is finite for every } 0<\delta<t.$$

Remark 5.4. We note that under the additional assumptions that ϕ is non-decreasing on $\mathbb{R}$ and the weak energy solution u of problem (5.3) is in $L^\infty(0,T;L^\infty(\Sigma,\mu))$, Lemma 5.11 below yields for the primitive

$$\Phi(s) := \int_0^s \phi(r)\mathrm{d}r, \qquad (s\in\mathbb{R}), \tag{5.8}$$

of ϕ that
$$\int_\Sigma \Phi(u)\,\mathrm{d}\mu \in W^{1,1}(0,1)$$
and *integration by parts rule* (5.43) from Section 5.2 holds.

Definition 5.5. For a given continuous function ϕ on $\mathbb{R}$ satisfying $\phi(0)=0$ and every $\varepsilon>0$, we call the function
$$\phi_\varepsilon(s) := \tfrac{1}{\varepsilon}\int_{\mathbb{R}} \phi(r)\,\rho(\tfrac{s-r}{\varepsilon})\,dr + \varepsilon s + c_\varepsilon \qquad \text{for every } s\in\mathbb{R},$$
the *regularisation of* ϕ. Here, the constant c_ε is chosen such that $\phi_\varepsilon(0)=0$ and $\rho\in C^\infty(\mathbb{R})$, $\rho\ge 0$, $\int_{\mathbb{R}}\rho\mathrm{d}r = 1$, $\rho\equiv 0$ on $\mathbb{R}\setminus[-1,1]$.

The following theorem is the main result of this section, which provides sufficient conditions that mild solutions are weak energy solutions.

Theorem 5.6. *Let $\Psi : V\to\mathbb{R}$ be a convex, lower semicontinuous, and Gâteaux differentiable functional satisfying the hypotheses ($\mathcal{H}$i)-($\mathcal{H}$v), ϕ be a strictly increasing continuous function on $\mathbb{R}$ satisfying ($\mathcal{H}$a) and F be an operator on $L^q(\Sigma,\mu)$ satisfying ($\mathcal{H}$b). Further, suppose that there are exponents α, β, $\gamma>0$ and a constant $\tilde{C}>0$ such that the semigroup $\{T_t\}\sim -(\overline{A_{1\cap\infty}\phi}+F)$ on $\overline{D(A_{1\cap\infty}\phi)}^{L^1}$ satisfies L^1-L^∞ regularisation estimate (5.7). Then, the following statements hold:*

(1) For every initial value $u_0\in\overline{D(A_{1\cap\infty}\phi)}^{L^1}$, the mild solution $u(t)=T_tu_0$, ($t\ge 0$), of Cauchy problem (5.2) in $L^1(\Sigma,\mu)$ is a weak energy solution of Cauchy problem (5.3) satisfying

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{\alpha(p'-1)+1}\Psi(\phi(u(s)))\,\mathrm{d}s + t^{\alpha(p'-1)+1}\int_\Sigma \Phi(u(t))\,\mathrm{d}\mu\\
&\quad\le \tfrac{(\alpha(p'-1)+1)^p\,C^{p-1}}{p'(4^{-1}p)^{p-1}}\int_0^t e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_0\|_1^{\gamma\,(p'-1)+1}\\
&\qquad + \tfrac{\omega^p\,C^{p-1}}{p'(4^{-1}p)^{p-1}}\int_0^t s\,e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_0\|_1^{\gamma\,(p'-1)+1}.
\end{aligned} \tag{5.9}$$

for every $t>0$.

(2) *If, in addition,* $\phi' \in L^\infty(\mathbb{R})$, ϕ^{-1} *is locally bounded and* $0 < \alpha \leq 1$ *in estimate* (5.7), *then for every initial value* $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1}$, *the mild solution* $u(t) = T_t u_0$, $(t \geq 0)$, *of Cauchy problem* (5.2) *in* $L^1(\Sigma, \mu)$ *is a strong solution of* (5.2) *in* $L^1(\Sigma, \mu)$ *with the following properties:*

(a) *One has*
$$u \in W^{1,2}_{loc}((0,\infty); L^2(\Sigma,\mu))$$

(b) *the function* $\phi(u) \in W^{1,2}_{loc}((0,T]; L^2(\Sigma,\mu))$ *with weak derivative*

$$\frac{d}{dt}\phi(u(t)) = \phi'(u(t))\frac{du}{dt}(t) \qquad \text{in } L^2(\Sigma,\mu) \text{ for a.e. } t > 0, \tag{5.10}$$

(c) *for a.e.* $t > 0$, *one has* $\phi(u(t)) \in D(A)$ *and*

$$\tfrac{du}{dt}(t) + A\phi(u(t)) + F(u(t)) \ni 0 \qquad \text{in } L^2(\Sigma,\mu), \tag{5.11}$$

(d) *the real-valued function* $t \mapsto \Psi(\phi(u(t)))$ *is locally absolutely continuous on* $(0,\infty)$ *satisfying for a.e.* $t > 0$,

$$\tfrac{d}{dt}\Psi(\phi(u(t))) = -\|\tfrac{du}{dt}(t)\sqrt{\phi'(u(t))}\|_2^2 - \langle F(u(t)), \tfrac{du}{dt}(t)\,\phi'(u(t))\rangle, \tag{5.12}$$

(e) *for every* $t > 0$,

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{\alpha(p'-1)+2}\int_\Sigma \phi'(u(s))\left|\frac{du}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{\alpha(p'-1)+2}\,\Psi(\phi(u(t)))\\
&\quad\leq \tfrac{(\alpha(p'-1)+2)\,2}{p'}\Bigg[\tfrac{(k+1)^p\,C^{p-1}}{(4^{-1}p)^{p-1}}\int_0^t e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_0\|_1^{\gamma\,(p'-1)+1}\\
&\qquad + \tfrac{\omega^p\,C^{p-1}}{(4^{-1}p)^{p-1}}\int_0^t s\,e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_0\|_1^{\gamma\,(p'-1)+1}\\
&\qquad + \tfrac{(\alpha(p'-1)+1)^p\,C^p}{p'(2^{-1}p)^{p-1}}\int_0^t e^{\omega\,\beta(p'-1)\,s}\,\mathrm{d}s\|u_0\|_1^{\gamma(p'-1)+1}\Bigg]\\
&\qquad + \tfrac{\omega^2\,\tilde{C}}{2}\|\phi'\|_\infty\int_0^t s^{1-\alpha}\,e^{\omega(\beta+1)\,s}\,\mathrm{d}s\,\|u_0\|_1^{\gamma+1}.
\end{aligned} \tag{5.13}$$

The proof of Theorem 5.9 is divided into three steps. The first step is to consider the *smooth case*, that is, under the assumption that ϕ and its inverse ϕ^{-1} are locally Lipschitz continuous (see Theorem 5.7). Then, in the second step, we consider a general *continuous strictly increasing* function ϕ but we take initial values $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma,\mu)$ (see Theorem 5.9). In the last and third step, one uses the estimates established in step two to conclude by using the continuous dependence of the semigroup $\{T_t\}_{t\geq 0}$ and the its L^1-L^∞ regularisation effect to conclude the statement of the main theorem (Theorem 5.6).

5.1. **The smooth case.** We begin by considering the smooth case. Here, the statement of our following theorem confirms positively a conjecture stated in [10, Remarque 2.13] and generalises the results in [10, Proposition 2.18] and partially some results in [51, Section 3] to the general subgradient setting. Our next theorem is the main results in this subsection.

Theorem 5.7. *Let* $\Psi : V \to \mathbb{R}$ *be a convex, lower semicontinuous, and Gâteaux differentiable functional satisfying* ($\mathcal{H}$iii) *and* ($\mathcal{H}$iv). *Further, let* ϕ *be a strictly increasing function on* $\mathbb{R}$ *such that* ϕ *and* ϕ^{-1} *are locally Lipschitz continuous, and the Yosida operator* β_λ *of* $\beta = \phi^{-1}$ *satisfies condition* (2.43), $(\lambda > 0)$, *and* F *be an operator on*

$L^q(\Sigma,\mu)$ satisfying ($\mathcal{H}$b). We set $\Phi(r) = \int_0^r \phi(s)\,\mathrm{d}s$ for every $r \in \mathbb{R}$. Then, for every $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma,\mu)$, the mild solution $u(t) = T_t u_0$, $(t \geq 0)$, of problem (5.2) in $L^1(\Sigma,\mu)$ is a strong solution of

$$\tfrac{du}{dt} + A\phi(u) + F(u) \ni 0 \quad \text{in } L^2(\Sigma,\mu) \text{ on } (0,T), \qquad u(0) = u_0 \tag{5.14}$$

with the regularity

$$u \in C([0,T];L^q(\Sigma,\mu)) \cap W^{1,2}_{loc}((0,T];L^2(\Sigma,\mu)) \cap L^\infty([0,T];L^\infty(\Sigma,\mu)) \tag{5.15}$$

for every $1 \leq q < \infty$ and satisfying

1. *the function $\phi(u) \in W^{1,2}_{loc}((0,T];L^2(\Sigma,\mu))$ with weak derivative (5.10),*
2. *for a.e. $t > 0$, one has $\phi(u(t)) \in D(A)$ and (5.11) holds,*
3. *the real-valued function $t \mapsto \Psi(\phi(u(t)))$ is locally absolutely continuous on $(0,\infty)$ satisfying (5.12) for a.e. $t > 0$,*
4. *for every $k \geq 0$ and $t > 0$, one has*

$$\begin{aligned}
&\int_0^t s^{k+1}\Psi(\phi(u(s)))\,\mathrm{d}s + t^{k+1}\int_\Sigma \Phi(u(t))\,\mathrm{d}\mu \\
&\qquad \leq (k+1)\int_0^t s^k \int_\Sigma \phi(u(t))\,u(t)\,\mathrm{d}\mu\mathrm{d}s \\
&\qquad\qquad - \int_0^t s^{k+1}\int_\Sigma F(u(s))\phi(u(s))\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{k+2}\int_\Sigma \phi'(u(s))\left|\frac{du}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{k+2}\,\Psi(\phi(u(t))) \\
&\qquad \leq (k+2)(k+1)\int_0^t s^k \int_\Sigma \phi(u(t))\,u(t)\,\mathrm{d}\mu\,\mathrm{d}s \\
&\qquad\qquad - (k+2)\int_0^t s^{k+1}\int_\Sigma F(u(s))\phi(u(s))\,\mathrm{d}\mu\,\mathrm{d}s \\
&\qquad\qquad\qquad + \tfrac{1}{2}\int_0^t s\int_\Sigma \phi'(u(s))\,|F(u(s))|^2\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned} \tag{5.17}$$

Before outlining the proof of Theorem 5.7, we recall the following *convergence result* (cf. [8, Proposition 4.4 & Theorem 4.14]), which we state in a version suitable for the framework of this paper.

Theorem 5.8. *For $\omega \in \mathbb{R}$ and $1 \leq q \leq \infty$, let $(A_n)_{n\geq 1}$ be a sequence of operators A_n on $L^q(\Sigma,\mu)$ such that $A_n + \omega I$ is accretive in $L^q(\Sigma,\mu)$. For given $u_{0,n} \in \overline{D(A_n)}^{L^q}$, let u_n be the unique mild solution of initial value problem*

$$\tfrac{du_n}{dt} + A_n u_n \ni 0 \quad \text{on } (0,T) \text{ and } \quad u_n(0) = u_{0,n}.$$

Further, let A be an operator on $L^q(\Sigma,\mu)$ such that $A + \omega I$ is accretive in $L^q(\Sigma,\mu)$ and for given $u \in \overline{D(A)}^{L^q}$, let u be the unique mild solution of

$$\tfrac{du}{dt} + Au \ni 0 \quad \text{on } (0,T) \text{ and } \quad u(0) = u_0.$$

Suppose that $\lim_{n\to\infty} u_{0,n} = u_0$ in $L^q(\Sigma,\mu)$ and for every $\lambda > 0$ satisfying $\lambda\omega < 1$, the resolvent J^A_λ of A and the resolvent $J^{A_n}_\lambda$ of A_n satisfy

$$\lim_{n\to\infty} J^{A_n}_\lambda x \to J^A_\lambda x \qquad \text{in } L^q(\Sigma,\mu) \text{ for every } x \in L^q(\Sigma,\mu), \tag{5.18}$$

then $u_n \to u$ *in* $C([0,T];L^q(\Sigma,\mu))$.

Our proof of Theorem 5.7 improves an idea from [10].

Proof of Theorem 5.7. By Proposition 2.19, $\overline{A_{1\cap\infty}\phi}+F$ is m-accretive in L^1 with complete resolvent and for every $\lambda>0$ such that $\omega\lambda<1$, $A_{1\cap\infty}\phi+F$ satisfies range condition (2.45). In particular, $-(\overline{A\phi}+F)$ generates a strongly continuous semigroup $\{T_t\}_{t\ge 0}$ on $\overline{D(A_{1\cap\infty}\phi)}^{L^1}$. Therefore, for $u_0\in\overline{D(A_{1\cap\infty}\phi)}^{L^1}\cap L^\infty(\Sigma,\mu)$, the function $u(t):=T_tu_0$ for every $t\ge 0$ is the unique mild solution of problem (5.2) in $L^1(\Sigma,\mu)$ and by Proposition 2.16, one has

$$\|u(t)\|_q\le e^{\omega t}\|u_0\|_q \qquad \text{for every } t\ge 0 \text{ and } 1\le q\le\infty, \tag{5.19}$$

where $\omega\ge 0$ is the Lipschitz constant of F. Fix $T>0$ and set $M:=e^{\omega T}\|u_0\|_\infty$. Then, the values of $\phi(s)\in\mathbb{R}$ for $|s|\ge M$ do not intervene if one considers the solutions merely on the time interval $[0,T]$. Thus, there is no loss of generality if we assume, ϕ and ϕ^{-1} are globally Lipschitz continuous on $\mathbb{R}$.

Now, for every $\lambda>0$, let $\Psi_\lambda : L^2(\Sigma,\mu)\to\mathbb{R}_+$ denote the *Moreau regularisation* of Ψ on $L^2(\Sigma,\mu)$ (cf. [24, Proposition 2.11]). Then, Ψ_λ is continuously Fréchet-differentiable on $L^2(\Sigma,\mu)$ and the Fréchet-derivative Ψ'_λ of Ψ coincides with the Yosida operator $A_\lambda:=\frac{1}{\lambda}(I-J_\lambda)$ of A in $L^2(\Sigma,\mu)$. Since the resolvent operator J_λ of A is contractive on $L^2(\Sigma,\mu)$ and since ϕ is globally Lipschitz continuous, the composition operator $J_\lambda\phi$ is globally Lipschitz continuous on $L^2(\Sigma,\mu)$. Hence by [24, Corollaire 1.1], for every $\lambda>0$, there is a unique strong solution

$$u_\lambda\in C^1([0,T];L^2(\Sigma,\mu))$$

of the Cauchy problem

$$\begin{cases} \frac{du_\lambda}{dt}+A_\lambda\phi(u_\lambda)+F(u_\lambda)=0 & \text{in } L^2(\Sigma,\mu) \text{ on } (0,T),\\ \hfill u_\lambda(0)=u_0. \end{cases} \tag{5.20}$$

Since A is accretive in $L^1(\Sigma,\mu)$, one easily verifies that for every $\lambda>0$, the Yosida operator A_λ is also accretive in $L^1(\Sigma,\mu)$. Moreover, for every $p\in P_0$, there is a $\theta(x)\in(0,1)$ such that

$$\begin{aligned}\int_\Sigma p(u)A_\lambda u\,\mathrm{d}\mu &= \int_\Sigma p(J_\lambda u)A_\lambda u\,\mathrm{d}\mu+\lambda\int_\Sigma p'(\theta u+(1-\theta)J_\lambda u)|A_\lambda u|^2\,\mathrm{d}\mu\\ &\ge \int_\Sigma p(J_\lambda u)A_\lambda u\,\mathrm{d}\mu.\end{aligned}$$

Since $A_\lambda u\in A(J_\lambda u)$ and since A has a complete resolvent,

$$\int_\Sigma p(J_\lambda u)A_\lambda u\,\mathrm{d}\mu\ge 0$$

yielding the Yosida operator A_λ has a complete resolvent. Thus by Proposition 2.19, the operator $A_\lambda\phi+F$ is quasi accretive in $L^1(\Sigma,\mu)$ with complete resolvent. Thus,

$$\|u_\lambda(t)\|_q\le e^{\omega t}\,\|u_0\|_q \qquad \text{for every } t\in[0,T],\ \lambda>0,\ 1\le q\le\infty. \tag{5.21}$$

and in particular, $\|u_\lambda(t)\|_\infty\le M$.

Next, let $\rho > 0$ such that $\rho\,\omega < 1$ and $x \in L^1 \cap L^\infty(\Sigma,\mu)$. Then, by range condition (2.45), there is $u_\rho \in D(A_{1\cap\infty}\phi)$ such that $u_\rho = J_\rho^{A_{1\cap\infty}\phi+F}x$ or, equivalently,

$$u_\rho = x - \rho(A\phi(u_\rho) + F(u_\rho)). \tag{5.22}$$

Since $A_\lambda\phi + F + \omega I$ is Lipschitz continuous and accretive in $L^1(\Sigma,\mu)$, $A_\lambda\phi + F + \omega I$ is m-accretive in $L^1(\Sigma,\mu)$. Thus for every $\lambda > 0$, there is $u_{\rho,\lambda} \in D(A_\lambda\phi)$ such that $u_{\rho,\lambda} = J_\rho^{A_\lambda\phi+F}x$ or, equivalently,

$$u_{\rho,\lambda} = x - \rho(A_\lambda\phi(u_{\rho,\lambda}) + F(u_{\rho,\lambda})), \tag{5.23}$$

and

$$\|u_{\rho,\lambda}\|_q \le (1-\rho\,\omega)^{-1}\|x\|_q \tag{5.24}$$

for every $1 \le q \le \infty$. Now, by the two equations (5.22) and (5.23), since

$$A_\lambda(\phi(u_{\rho,\lambda})) \in A(J_\lambda\phi(u_{\rho,\lambda})),$$

since the operators A and $F + \omega I$ are accretive in $L^2(\Sigma,\mu)$, ϕ is non-decreasing, and F Lipschitz continuous with constant $\omega \ge 0$, we see

$$\begin{aligned}
&(1-\rho\omega)\int_\Sigma (u_{\rho,\lambda}-u_\rho)(\phi(u_{\rho,\lambda})-\phi(u_\rho))\,\mathrm{d}\mu\\
&= -\rho\int_\Sigma \Big[(A_\lambda\phi(u_{\rho,\lambda}) + F(u_{\rho,\lambda}) + \omega u_{\rho,\lambda}) - (A\phi(u_\rho) + F(u_\rho) + \omega u_\rho)\Big]\times\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\ (\phi(u_{\rho,\lambda}) - \phi(u_\rho))\,\mathrm{d}\mu\\
&= -\rho\int_\Sigma \Big[A\phi(u_\rho) - A_\lambda\phi(u_{\rho,\lambda})\Big]\times\\
&\qquad\qquad\qquad\times\ (\phi(u_\rho) - J_\lambda^A\phi(u_{\rho,\lambda}) + J_\lambda^A\phi(u_{\rho,\lambda}) - \phi(u_{\rho,\lambda}))\,\mathrm{d}\mu\\
&\qquad -\rho\int_\Sigma \Big[(F(u_{\rho,\lambda}) + \omega u_{\rho,\lambda}) - (F(u_\rho) + \omega u_\rho)\Big]\,(\phi(u_{\rho,\lambda}) - \phi(u_\rho))\,\mathrm{d}\mu\\
&\le \rho\,\lambda\int_\Sigma (A\phi(u_\rho) - A_\lambda\phi(u_{\rho,\lambda}))\ A_\lambda\phi(u_{\rho,\lambda})\,\mathrm{d}\mu\\
&= -\lambda\int_\Sigma (u_\rho - u_{\rho,\lambda})\,A_\lambda\phi(u_{\rho,\lambda})\,\mathrm{d}\mu\\
&\qquad\qquad -\rho\,\lambda\int_\Sigma (F(u_\rho) - F(u_{\rho,\lambda}))\,A_\lambda\phi(u_{\rho,\lambda})\,\mathrm{d}\mu\\
&\le \lambda\,(1+\rho\,\omega)\int_\Sigma |u_\rho - u_{\rho,\lambda}|\,|A_\lambda\phi(u_{\rho,\lambda})|\,\mathrm{d}\mu\\
&\le \lambda\frac{(1+\rho\,\omega)\,2}{\rho\,(1-\rho\omega)}\,\|x\|_\infty\int_\Sigma |u_{\rho,\lambda} - x + \rho\,F(u_{\rho,\lambda})|\,\mathrm{d}\mu\\
&\le \lambda\frac{(1+\rho\,\omega)\,2}{\rho\,(1-\rho\omega)}\,\|x\|_\infty\left(\int_\Sigma |u_{\rho,\lambda}|\,\mathrm{d}\mu + \int_\Sigma |x|\,\mathrm{d}\mu + \rho\,\omega\int_\Sigma |u_{\rho,\lambda}|\,\mathrm{d}\mu\right)
\end{aligned}$$

and so by (5.24),

$$\begin{aligned}
&\int_\Sigma (u_{\rho,\lambda}-u_\rho)(\phi(u_{\rho,\lambda})-\phi(u_\rho))\,\mathrm{d}\mu\\
&\qquad \le \lambda\frac{(1+\rho\,\omega)\,2}{\rho\,(1-\rho\omega)^2}\,\|x\|_\infty\left(\frac{1}{(1-\rho\omega)} + 1 + \frac{\rho\omega}{(1-\rho\omega)}\right)\int_\Sigma |x|\,\mathrm{d}\mu
\end{aligned} \tag{5.25}$$

From this we can conclude that $\lim_{\lambda\to 0} u_{\rho,\lambda} = u_\rho$ a.e. on Σ since ϕ is continuous, strictly increasing and $\phi(s) = 0$ if and only if $s = 0$. Since (Σ, μ) is finite and by (5.24), Lebesgue's dominated convergence theorem yields

$$\lim_{\lambda\to 0+} J_\rho^{A_\lambda\phi+F} x = \lim_{\lambda\to 0+} u_{\rho,\lambda} = u_\rho = J_\rho^{A_{1\cap\phi}\phi+F} x \qquad \text{in } L^1(\Sigma, \mu)$$

for every $x \in L^1 \cap L^\infty(\Sigma, \mu)$ and $\rho > 0$. Since $L^1 \cap L^\infty(\Sigma, \mu)$ is dense in $L^1(\Sigma, \mu)$ and $J_\rho^{\overline{A_{1\cap\infty}\phi}+F}$ and $J_\rho^{A_\lambda\phi+F}$ are Lipschitz continuous, a standard density argument shows that the hypothesis (5.18) in Theorem 5.8 for $q = 1$ holds. Therefore,

$$\lim_{\lambda\to 0+} u_\lambda = u \qquad \text{in } C([0, T]; L^1(\Sigma, \mu)).$$

and by (5.19) and (5.21), for the strong solution u_λ of (5.20) and the mild solution u of (2.22), one has

$$\lim_{\lambda\to 0+} u_\lambda = u \qquad \text{in } C([0, T]; L^q(\Sigma, \mu)) \qquad \text{for all } 1 \le q < \infty. \tag{5.26}$$

Next, we show that

$$u \in W^{1,2}_{loc}((0, T]; L^2(\Sigma, \mu)) \tag{5.27}$$

By the Lipschitz continuity of the Nemytski operator F on $L^2(\Sigma, \mu)$ and since u_λ belongs to $C([0, T]; L^2(\Sigma, \mu))$, we have that $F(u_\lambda) \in C([0, T]; L^2(\Sigma, \mu))$. Further, since ϕ is Lipschitz continuous, $\phi(0) = 0$, and $u_\lambda \in C^1([0, T]; L^2(\Sigma, \mu))$, we can conclude that the function $\phi(u_\lambda) \in W^{1,2}(0, T; L^2(\Sigma, \mu))$ with weak derivative

$$\frac{d}{dt}\phi(u_\lambda(t)) = \phi'(u_\lambda(t))\frac{du_\lambda}{dt}(t) \qquad \text{for a.e. } t \in (0, T). \tag{5.28}$$

By equation (5.20) and by $A_\lambda = \Psi'_\lambda$, [24, Lemme 3.3] implies that the function $\Psi_\lambda(\phi(u_\lambda)) : [0, T] \to \mathbb{R}$ is absolutely continuous and

$$\frac{d}{dt}\Psi_\lambda(\phi(u_\lambda(t))) = \langle A_\lambda\phi(u_\lambda(t)), \frac{d}{dt}\phi(u_\lambda(t))\rangle = \langle A_\lambda\phi(u_\lambda(t)), \phi'(u_\lambda(t))\frac{du_\lambda}{dt}(t)\rangle$$

for a.e. $t \in (0, T)$. Let $k \ge 0$ and multiply equation (5.20) by $s^{k+2}\,\frac{d}{ds}\phi(u_\lambda(s))$ with respect to the L^2-inner product and integrating over $(0, t)$, for some $0 < t < T$. Then

$$\begin{aligned}
&\int_0^t s^{k+2}\int_\Sigma \phi'(u_\lambda(s))\left|\frac{du_\lambda}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{k+2}\,\Psi_\lambda(\phi(u_\lambda(t)))\\
&\quad = (k+2)\int_0^t s^{k+1}\psi_\lambda(\phi(u_\lambda(s)))\,\mathrm{d}s - \int_0^t s^{k+2}\int_\Sigma F(u_\lambda(s))\phi'(u_\lambda(s))\frac{du_\lambda}{ds}(s)\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned}$$

Since $\phi'(u_\lambda) \ge 0$, Young's inequality gives

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{k+2}\int_\Sigma \phi'(u_\lambda(s))\left|\frac{du_\lambda}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{k+2}\,\Psi_\lambda(\phi(u_\lambda(t)))\\
&\quad \le (k+2)\int_0^t s^{k+1}\psi_\lambda(\phi(u_\lambda(s)))\,\mathrm{d}s\\
&\qquad + \tfrac{1}{2}\int_0^t s^{k+2}\int_\Sigma |F(u_\lambda(s))|^2\phi'(u_\lambda(s))\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned} \tag{5.29}$$

for every $0 < t \leq T$. On the other hand, the Yosida operator A_λ is the subgradient $\partial_{L^2}\Psi_\lambda$ of Ψ_λ, and $A_\lambda(0) = 0$ and $\Psi_\lambda(0) = 0$. Thus

$$\langle A_\lambda\phi(u_\lambda(t)), 0 - \phi(u_\lambda(t))\rangle \leq \Psi_\lambda(0) - \Psi_\lambda(\phi(u_\lambda(t)))$$

for every $0 < t < T$. Multiplying this inequality by (-1) and taking advantage of equation (5.20) yields

$$\Psi_\lambda(\phi(u_\lambda(t))) \leq -\langle \tfrac{du_\lambda}{dt}(t), \phi(u_\lambda(t))\rangle - \langle F(u_\lambda(t)), \phi(u_\lambda(t))\rangle \tag{5.30}$$

for every $0 < t < T$. Since ϕ is non-decreasing on $\mathbb{R}$ and $\Phi(0) = 0$, one has $\Phi(r) \leq \phi(r)r$ for every $r \in \mathbb{R}$. Thus, by integrating inequality (5.30) over $(0,t)$ for some $t \in (0,T]$, we obtain

$$\begin{aligned}\int_0^t \Psi_\lambda(\phi(u_\lambda(s)))\,\mathrm{d}s &+ \int_\Sigma \Phi(u_\lambda(t))\,\mathrm{d}\mu\\ &\leq \int_\Sigma \Phi(u_\lambda(t))\,\mathrm{d}\mu - \int_0^t\int_\Sigma F(u_\lambda(s))\phi(u_\lambda(s))\,\mathrm{d}\mu\,\mathrm{d}s.\end{aligned}$$

Similarly, multiplying inequality (5.30) by s^{k+1} and subsequently integrating over $(0,t)$ for some $t \in (0,T]$ gives

$$\begin{aligned}\int_0^t s^{k+1}\Psi_\lambda(\phi(u_\lambda(s)))\,\mathrm{d}s &+ t^{k+1}\int_\Sigma \Phi(u_\lambda(t))\,\mathrm{d}\mu\\ &\leq (k+1)\int_0^t s^k\int_\Sigma \Phi(u_\lambda(t))\,\mathrm{d}\mu\mathrm{d}s\\ &\qquad - \int_0^t s^{k+1}\int_\Sigma F(u_\lambda(s))\phi(u_\lambda(s))\,\mathrm{d}\mu\,\mathrm{d}s\\ &\leq (k+1)\int_0^t s^k\int_\Sigma \phi(u_\lambda(t))\,u_\lambda(t)\mathrm{d}\mu\mathrm{d}s\\ &\qquad - \int_0^t s^{k+1}\int_\Sigma F(u_\lambda(s))\phi(u_\lambda(s))\,\mathrm{d}\mu\,\mathrm{d}s\end{aligned}\tag{5.31}$$

for every $0 < t \leq T$. Since $\phi(0) = 0$ and ϕ is non-decreasing on $\mathbb{R}$, one has that the function $\Phi(r) \geq 0$ for every $r \in \mathbb{R}$ and so,

$$\int_\Sigma \Phi(u_\lambda(t))\,\mathrm{d}\mu \geq 0.$$

Thus, applying estimate (5.31) to the right hand-side of (5.29) yields

$$\begin{aligned}\tfrac{1}{2}\int_0^t s^{k+2}\int_\Sigma \phi'(u_\lambda(s))\left|\frac{du_\lambda}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s &+ t^{k+2}\Psi_\lambda(\phi(u_\lambda(t)))\\ &\leq (k+2)(k+1)\int_0^t s^k\int_\Sigma \phi(u_\lambda(t))\,u_\lambda(t)\,\mathrm{d}\mu\,\mathrm{d}s\\ &\qquad - (k+2)\int_0^t s^{k+1}\int_\Sigma F(u_\lambda(s))\phi(u_\lambda(s))\,\mathrm{d}\mu\,\mathrm{d}s\\ &\qquad + \tfrac{1}{2}\int_0^t s\int_\Sigma \phi'(u_\lambda(s))\,|F(u_\lambda(s))|^2\,\mathrm{d}\mu\,\mathrm{d}s.\end{aligned}\tag{5.32}$$

By assumption, there are constants $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \leq \phi'(s) \leq \alpha_2$ for all $s \in [-M,M]$, by the boundedness of ϕ on $[-M,M]$, by the continuity of $F : L^2(\Sigma,\mu) \to L^2(\Sigma,\mu)$, and by (5.19) and (5.21), we can conclude from estimate (5.32) that the sequence $(u_\lambda)_{\lambda>0}$ is bounded in $W^{1,2}_{loc}((0,T];L^2(\Sigma,\mu))$. Thus, for every sequence $(\lambda_n) \in (0,1)$ such that $\lambda_n \to 0$ as $n \to \infty$, there is $w \in L^2_{loc}((0,T];L^2(\Sigma,\mu))$ and after eventually passing to a subsequence, $\frac{du_{\lambda_n}}{dt} \rightharpoonup w$ weakly in $L^2([\delta,T];L^2(\Sigma,\mu)$ for every $\delta \in (0,T)$. Hence, sending $n \to \infty$ in

$$u_{\lambda_n}(t) - u_{\lambda_n}(s) = \int_s^t \frac{du_{\lambda_n}}{dr}(r)\,\mathrm{d}r$$

and using (5.26) yields $w(t) = \frac{du}{dt}(t)$ in $L^2(\Sigma,\mu)$ for a.e. $t \in (0,T)$. Therefore, (5.27) holds and

$$\lim_{\lambda\to 0+} \frac{du_{\lambda_n}}{dt} = \frac{du}{dt} \qquad \text{weakly in } L^2_{loc}((0,T];L^2(\Sigma,\mu)). \tag{5.33}$$

Next, by (5.26) and since ϕ is Lipschitz continuous,

$$\lim_{\lambda\to 0+} \phi(u_\lambda) = \phi(u) \qquad \text{in } C([0,T];L^2(\Sigma,\mu)).$$

In particular, $\phi(u(t)) \in L^2(\Sigma,\mu)$ for every $t \in [0,T]$. By assumption, Ψ is densely defined on $L^2(\Sigma,\mu)$ and so by [24, Proposition 2.11 & Théorème 2.2], the resolvent operator J_λ of A satisfies

$$\lim_{\lambda\to 0+} J_\lambda\phi(u(t)) = \phi(u(t)) \qquad \text{in } L^2(\Sigma,\mu) \text{ for every } t \in [0,T].$$

Thus and since for every $t \in [0,T]$,

$$\begin{aligned}
&\|J_\lambda\phi(u_\lambda(t)) - \phi(u(t))\|_2 \\
&\qquad \le \|J_\lambda\phi(u_\lambda(t)) - J_\lambda\phi(u(t))\|_2 + \|J_\lambda\phi(u(t)) - \phi(u(t))\|_2 \\
&\qquad \le \|\phi(u_\lambda(t)) - \phi(u(t))\|_2 + \|J_\lambda\phi(u(t)) - \phi(u(t))\|_2,
\end{aligned}$$

we can conclude that

$$\lim_{\lambda\to 0+} J_\lambda\phi(u_\lambda(t)) = \phi(u(t)) \qquad \text{in } L^2(\Sigma,\mu) \text{ for every } t \in [0,T]. \tag{5.34}$$

Now, let $(\hat{w},\hat{v}) \in A$ and $t \in (0,T)$ such that $\frac{du}{dt}(t)$ exists in $L^2(\Sigma,\mu)$. Since

$$A_\lambda(\phi(u_\lambda(t))) \in A(J_\lambda\phi(u_\lambda))$$

and since A is accretive in $L^2(\Sigma,\mu)$, one has

$$\left[J_\lambda\phi(u_\lambda(t)) - \hat{w}, \left(-F(u_\lambda(t)) - \frac{du_\lambda}{dt}(t)\right) - \hat{v}\right]_2 \ge 0$$

Sending $\lambda \to 0+$ in this inequality and using (5.33) and (5.34) yields

$$\left[\phi(u_\lambda(t)) - \hat{w}, \left(-F(u(t)) - \frac{du}{dt}(t)\right) - \hat{v}\right]_2 \ge 0.$$

Since $(\hat{w},\hat{v}) \in A$ was arbitrary, A is m-accretive in $L^2(\Sigma,\mu)$ and $\frac{du}{dt}(t)$ exists in $L^2(\Sigma,\mu)$ for a.e. $t \in (0,T)$, we can conclude that for a.e. $t \in (0,T)$, one has $\phi(u(t)) \in D(A)$ satisfying inclusion (5.14), showing that u is a strong solution of (5.14) in $L^2(\Sigma,\mu)$. Now, proceeding as in the previous steps of this proof, one sees that chain rule (5.28) and estimates (5.29) and (5.31) satisfied by u_λ hold, in particular, for u, proving that $\phi(u) \in W^{1,2}_{loc}((0,T];L^2(\Sigma,\mu))$, chain rule (5.10) and the estimates (5.16) and (5.17) hold. Moreover, by (5.21), (5.26), and (5.27), we see that u has the regularity as stated in (5.15). Next, recall that A is the subgradient $\partial_{L^2}\Psi$ in $L^2(\Sigma,\mu)$ of a convex, proper, lower semicontinuous functional Ψ on $L^2(\Sigma,\mu)$. Further, for every $0 < \delta < T$, the function

$v := \phi(u) \in W^{1,2}([\delta, T]; L^2(\Sigma, \mu))$ and inclusion (5.14) yields $g := -F(u) - \frac{du}{dt} \in L^2([\delta, T]; L^2(\Sigma, \mu))$ and satisfies $g(t) \in Av(t)$ for a.e. $t \in (\delta, T)$. Thus for every $0 < \delta < T$, by [24, Lemme 3.3], the function $t \mapsto \Psi(v(t))$ is absolutely continuous on $[\delta, T]$ and for a.e. $t \in (\delta, T)$,

$$\tfrac{d}{dt}\Psi(v(t)) = \langle g(t), \tfrac{dv}{dt}(t)\rangle.$$

Combining this together with chain rule (5.10), we see that equation (5.12) holds. □

5.2. **Weak solutions for general ϕ and initial values in L^∞.** This subsection is concerned with the second major step toward the proof of Theorem 5.6. Our next results shows that mild solutions are, in fact, weak energy solutions for initial values $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma, \mu)$. This is known to be true for the homogeneous porous media equation (cf. [10, 91]), and generalises this result to general quasi-m-accretive operators in L^1 with complete resolvent of the form $\overline{A_{1\cap\infty}\phi}$.

Theorem 5.9. *Let $\Psi : V \to \mathbb{R}$ be a convex, lower semicontinuous, and Gâteaux differentiable functional satisfying the hypotheses ($\mathcal{H}$i)-($\mathcal{H}$iv), ϕ be a strictly increasing continuous function on $\mathbb{R}$ satisfying ($\mathcal{H}$a) and F be an operator on $L^q(\Sigma, \mu)$ satisfying ($\mathcal{H}$b). Then, for every $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma, \mu)$ and $T > 0$, the mild solution $u(t) = T_t u_0$, $(t \geq 0)$, of Cauchy problem (5.2) in $L^1(\Sigma, \mu)$ is a weak energy solution of Cauchy problem (5.3) satisfying*

$$u \in C([0, T]; L^q(\Sigma, \mu)) \cap L^\infty(0, T; L^\infty(\Sigma, \mu)), \tag{5.35}$$

for every $1 \leq q < \infty$,

$$\tfrac{du}{dt} \in L^{p'}(0, T; V'), \qquad \phi(u) \in L^p(0, T; V) \tag{5.36}$$

and identity

$$\int_0^T \Big\{ \Big\langle \tfrac{du}{dt}, v \Big\rangle_{V', V} + \langle \Psi' \phi(u(t)), v\rangle + \langle F(u(t)), v\rangle \Big\}\, \mathrm{d}t = 0 \tag{5.37}$$

holds for every $v \in L^p(0, T; V)$. In particular, for the function Φ given by (5.8), one has that

$$\Phi(u) \in C([0, T]; L^1(\Sigma, \mu)) \quad \text{with} \quad \int_\Sigma \Phi(u)\, \mathrm{d}\mu \in W^{1,1}(0, T), \tag{5.38}$$

"integration by parts rule" (5.43) holds, and for every $k \geq 0$ and $t > 0$, inequality (5.16) and energy estimate

$$\begin{aligned} &\int_0^t \Psi(\phi(u(s)))\, \mathrm{d}s + \int_\Sigma \Phi(u(t))\, \mathrm{d}\mu \\ &\qquad \leq \int_\Sigma \Phi(u_0)\, \mathrm{d}\mu - \int_0^t \int_\Sigma F(u(s))\phi(u(s))\, \mathrm{d}\mu\, \mathrm{d}s, \end{aligned} \tag{5.39}$$

holds.

For the proof of this result, we need the following approximation result.

Lemma 5.10. *Let A be an m-completely accretive operator in $L^2(\Sigma, \mu)$ of a finite measure space (Σ, μ), F be the Nemytski operator of a Carathéodory function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ satisfying (2.17), and ϕ be a strictly increasing continuous function on $\mathbb{R}$ such that for every $\lambda > 0$, the Yosida operator β_λ of $\beta = \phi^{-1}$ and of $\beta = \phi_\varepsilon^{-1}$ the regularisation*

(ϕ_ε) of ϕ satisfy condition (2.43). Further, for every $\lambda > 0$, let J^ε_λ denote the resolvent operator of $\overline{A_{1\cap\infty}\phi_\varepsilon} + F$ and J_λ the resolvent operator of $\overline{A_{1\cap\infty}\phi} + F$. Then, for every $\lambda > 0$ such that $\omega\lambda < 1$ and $u \in L^1 \cap L^\infty(\Sigma,\mu)$, one has

$$\lim_{\varepsilon\to 0} J^\varepsilon_\lambda u = J_\lambda u \qquad \text{in } L^q(\Sigma,\mu) \text{ for every } 1 \le q < \infty. \tag{5.40}$$

Proof of Lemma 5.10. By Proposition 2.19, $\overline{A_{1\cap\infty}\phi} + F$ is m-accretive in L^1 with complete resolvent and for every $\lambda > 0$ such that $\omega\lambda < 1$, $A_{1\cap\infty}\phi + F$ satisfies the range condition (2.45). Moreover, since one has that

$$A_{1\cap\infty}\phi_\varepsilon \subseteq \overline{A_{1\cap\infty}\phi_\varepsilon} \quad \text{and} \quad A_{1\cap\infty}\phi \subseteq \overline{A_{1\cap\infty}\phi},$$

the range condition (2.45) yields that the resolvent J^ε_λ coincides with the resolvent of $A_{1\cap\infty}\phi_\varepsilon$ on $L^1 \cap L^\infty(\Sigma,\mu)$ and the resolvent J_λ of $\overline{A_{1\cap\infty}\phi}$ coincides with the resolvent of $A_{1\cap\infty}\phi$ on $L^1 \cap L^\infty(\Sigma,\mu)$. Thus, for every $\lambda > 0$ such that $\omega\lambda < 1$ and every $u \in L^1 \cap L^\infty(\Sigma,\mu)$, $\varepsilon > 0$, there are $(u_\varepsilon, v_\varepsilon) \in A_{1\cap\infty}\phi_\varepsilon$ and $(u_0, v_0) \in A_{1\cap\infty}\phi$ satisfying

$$u_\varepsilon + \lambda(v_\varepsilon + F(u_\varepsilon)) = u \qquad \text{and} \qquad u_0 + \lambda(v_0 + F(u_0)) = u, \tag{5.41}$$

or equivalently, $u_\varepsilon = J^\varepsilon_\lambda u$ for every $\varepsilon > 0$ and $u_0 = J_\lambda u$. Now, by using (5.41) and since A and $F + \omega I$ are accretive operators in $L^2(\Sigma,\mu)$, we see that

$$\begin{aligned}
(1-\omega\lambda) & \int_\Sigma (u_\varepsilon - u_0)(\phi(u_\varepsilon) - \phi(u_0))\,\mathrm{d}\mu \\
&= (1-\omega\lambda) \int_\Sigma (u_\varepsilon - u_0)(\phi_\varepsilon(u_\varepsilon) - \phi(u_0))\,\mathrm{d}\mu \\
&\qquad\qquad - \int_\Sigma (u_\varepsilon - u_0)(\phi(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon))\,\mathrm{d}\mu \\
&= -\lambda \int_\Sigma \Big[v_\varepsilon + F(u_\varepsilon) + \omega u_\varepsilon - (v_0 + F(u_0) + \omega u_0)\Big] \times \\
&\qquad\qquad \times\; (\phi_\varepsilon(u_\varepsilon) - \phi(u_0))\,\mathrm{d}\mu \\
&\qquad\qquad - \int_\Sigma (u_\varepsilon - u_0)(\phi(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon))\,\mathrm{d}\mu \\
&= -\lambda \Big[\phi_\varepsilon(u_\varepsilon) - \phi(u_0), v_\varepsilon - v_0\Big]_2 \\
&\qquad - \lambda \Big[\phi_\varepsilon(u_\varepsilon) - \phi(u_0), (F(u_\varepsilon) + \omega u_\varepsilon) - (F(u_0) + \omega u_0)\Big]_2 \\
&\qquad\qquad - \int_\Sigma (u_\varepsilon - u_0)(\phi(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon))\,\mathrm{d}\mu \\
&\le - \int_\Sigma (u_\varepsilon - u_0)(\phi(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon))\,\mathrm{d}\mu.
\end{aligned}$$

By Proposition 2.16,

$$\|u_\varepsilon\|_q \le (1-\lambda\,\omega)^{-1}\|u\|_q =: M \tag{5.42}$$

for all $\varepsilon \ge 0$ and $1 \le q \le \infty$ and so,

$$0 \le \int_\Sigma (u_\varepsilon - u_0)(\phi(u_\varepsilon) - \phi(u_0))\,\mathrm{d}\mu \le \frac{2\|u\|_1}{(1-\lambda\,\omega)^2}\|\phi_\varepsilon - \phi\|_{L^\infty(-M,M)}.$$

Since $\phi_\varepsilon \to \phi$ uniformly on compact subsets of $\mathbb{R}$, since ϕ is strictly increasing on $\mathbb{R}$ and $\phi(s) = 0$ if and only if $s = 0$, it follows that $\lim_\varepsilon u_\varepsilon = u_0$ a.e. on Σ. Using again (5.42) and that the measure space (Σ,μ) is finite, we can conclude that (5.40) holds for $u \in L^1 \cap L^\infty(\Sigma, \mu)$. A standard density argument yields the statement of this lemma. □

The following *integration by parts rule* is an important tool in the proof of Theorem 5.9. It also appears in different versions in the literature (cf., for instance, [2, p. 366]).

Lemma 5.11. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continuous function and $u \in L^\infty(0,T;L^\infty(\Sigma,\mu)) \cap C([0,T];L^1(\Sigma,\mu))$ such that $\frac{du}{dt} \in L^{p'}(0,T;V')$ and $\phi(u) \in L^p(0,T;V)$. Set $\Phi(r) = \int_0^r \phi(s)\,\mathrm{d}s$ for every $r \in \mathbb{R}$. Then,*

$$\int_{t_1}^{t_2} \Big\langle \frac{du}{dt}, \phi(u) \Big\rangle_{V,V} \mathrm{d}t = \int_\Sigma \Phi(u(t_2))\mathrm{d}\mu - \int_\Sigma \Phi(u(t_1))\mathrm{d}\mu \tag{5.43}$$

for every $0 \le t_1 < t_2 \le T$.

Proof. By assumption, there is a constant $M = \|u\|_{L^\infty(0,T;L^\infty(\Sigma,\mu))} \ge 0$ such that ϕ is bounded on $[-M, M]$ with constant $L_\phi > 0$. From this, we easily obtain that

$$\|\Phi(u(t)) - \Phi(u(s))\|_1 \le L_\phi\, M\, \|u(t) - u(s)\|_1 \tag{5.44}$$

for every t, $s \in [0,T]$ and so $\Phi(u) \in C([0,T];L^1(\Sigma))$. Furthermore, Hölder's inequality yields $\langle \frac{du}{dt}, \phi(u)\rangle_{V';V} \in L^1(0,T)$. Thus, both sides of equation (5.43) are finite. Now, let $0 \le t_1 < t_2 \le T$. For every $h > 0$ and for $t_1 < t < t_2$ such that $h < t_{2-t}$, the *Steklov average* $[\frac{du}{dt}]_h$ of $\frac{du}{dt}$ is given by

$$\Big[\frac{du}{dt}\Big]_h(t) := \frac{1}{h}\int_t^{t+h} \frac{du}{ds}(s)\,\mathrm{d}s \qquad \text{in } V'.$$

Since $\frac{du}{dt} \in L^{p'}(0,T;V')$, one easily checks that

$$\lim_{h\to 0+} \Big[\frac{du}{dt}\Big]_h = \frac{du}{dt} \qquad \text{in } L^{p'}(t_1,t_2;V')$$

and so,

$$\lim_{h\to 0+} \int_{t_1}^{t_2} \Big\langle \Big[\frac{du}{dt}\Big]_h, \phi(u) \Big\rangle_{V',V} \mathrm{d}t = \int_{t_1}^{t_2} \Big\langle \frac{du}{dt}, \phi(u) \Big\rangle_{V,V} \mathrm{d}t. \tag{5.45}$$

Furthermore, for every $t \in (0, T-h)$, $[\frac{du}{dt}]_h(t) = h^{-1}(u(t+h) - u(t))$. Using this together with the convexity of Φ and (5.4), we see that

$$\begin{aligned} \int_{t_1}^{t_2} \Big\langle \Big[\frac{du}{dt}\Big]_h, \phi(u) \Big\rangle_{V',V} \mathrm{d}t &\le \int_{t_1}^{t_2} h^{-1} \int_\Sigma (\Phi(u(t+h)) - \Phi(u(t)))\,\mathrm{d}\mu\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_\Sigma \frac{d}{dt}\Phi_h(u(t))\,\mathrm{d}\mu\,\mathrm{d}t \end{aligned} \tag{5.46}$$

By (5.44), we can apply Fubini's Theorem, to conclude that

$$\int_{t_1}^{t_2}\int_\Sigma \frac{d}{dt}\Phi_h(u(t))\,\mathrm{d}\mu\,\mathrm{d}t = \int_\Sigma \Phi_h(u(t_2))\,\mathrm{d}\mu - \int_\Sigma \Phi_h(u(t_2))\,\mathrm{d}\mu$$

for every $h > 0$. Since $\Phi(u) \in C([0,T];L^1(\Sigma))$, one has that

$$\lim_{h\to 0+} \Phi_h(u(t_2)) \qquad \text{in } C([t_1,t_2];L^1(\Sigma))$$

(cf. [54, Lemma 3.3.4] for $V = \mathbb{R}$ and note that the general Banach space-valued case V is shown analogously). Thus, sending $h \to 0+$ in (5.46) and using (5.45) yields

$$\int_{t_1}^{t_2} \left\langle \frac{du}{dt}, \phi(u) \right\rangle_{V',V} \mathrm{d}t \le \int_\Sigma \Phi(u(t_1))\,\mathrm{d}\mu - \int_\Sigma \Phi(u(t_2))\,\mathrm{d}\mu.$$

In order to see that the reverse inequality holds as well, we take $h < 0$ such that $0 < -h < t_1$. Then by the convexity of Φ and since $h^{-1} < 0$, we obtain that

$$\begin{aligned}
\int_{t_1}^{t_2} \left\langle \left[\frac{du}{dt}\right]_h, \phi(u) \right\rangle_{V',V} \mathrm{d}t &= \int_{t_1}^{t_2} h^{-1} \langle u(t+h) - u(t), \phi(u(t)) \rangle \,\mathrm{d}t \\
&\ge \int_{t_1}^{t_2} h^{-1} \int_\Sigma (\Phi(u(t+h)) - \Phi(u(t)))\,\mathrm{d}\mu\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_\Sigma \frac{d}{dt} \Phi_h(u(t))\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Now, proceeding as in the first part of this proof, we see that (5.43) holds. □

With the above preliminaries, we can now outline the proof of Theorem 5.9.

Proof of Theorem 5.9. Let $\phi : \mathbb{R} \to \mathbb{R}$ be strictly increasing continuous and for every $\varepsilon > 0$ and ϕ_ε be the regularisation of ϕ satisfying the assumptions of this theorem. Since $\overline{A_{1\cap\infty}\phi} + F$ and $\overline{A_{1\cap\infty}\phi_\varepsilon} + F$ are quasi m-accretive in $L^1(\Sigma, \mu)$, the Crandall-Liggett theorem yields the existence of strongly continuous semigroups $\{T_t\}_{t\ge0} \sim -(\overline{A_{1\cap\infty}\phi} + F)$ on $\overline{D(A_{1\cap\infty}\phi)}$ and $\{T_t^\varepsilon\}_{t\ge0} \sim -(\overline{A_{1\cap\infty}\phi_\varepsilon} + F)$ on $\overline{D(A_{1\cap\infty}\phi_\varepsilon)}$. Since $\overline{A_{1\cap\infty}\phi} + F$ and $\overline{A_{1\cap\infty}\phi_\varepsilon} + F$ have each a complete resolvent, Proposition 2.16 yields

$$\|u_\varepsilon(t)\|_q \le e^{\omega t}\|u_0\|_q \quad \text{and} \quad \|u(t)\|_q \le e^{\omega t}\|u_0\|_q \tag{5.47}$$

for every $t \ge 0$, $\varepsilon > 0$ and $1 \le q \le \infty$. Moreover, by Lemma 5.10 and Theorem 5.8, one has for every $T > 0$ that

$$\lim_{\varepsilon\to0} u_\varepsilon = u \qquad \text{in } C([0,T]; L^1(\Sigma,\mu)). \tag{5.48}$$

Combining this with (5.47) and Hölder's inequality, we find

$$\lim_{\varepsilon\to0} u_\varepsilon = u \qquad \text{in } C([0,T]; L^q(\Sigma,\mu)), \text{ for every } 1 \le q < \infty \text{ and } T > 0. \tag{5.49}$$

Now, we fix $T > 0$ and set

$$M = e^{\omega T}\|u_0\|_\infty.$$

By (5.47), the values of u_ε and u do not exceed the interval $[-M, M]$. Since the measure space (Σ, μ) is finite, we see that

$$\begin{aligned}
\|\phi_\varepsilon(u_\varepsilon(t)) - \phi(u(t))\|_q &\le \|\phi_\varepsilon(u_\varepsilon(t)) - \phi(u_\varepsilon(t))\|_q + \|\phi(u_\varepsilon(t)) - \phi(u(t))\|_q \\
&\le \ \mu(\Sigma)^{1/q}\, \|\phi_\varepsilon - \phi\|_{L^\infty(-M,M)} + \|\phi(u_\varepsilon(t)) - \phi(u(t))\|_q
\end{aligned}$$

and so, the uniform convergence of $\phi_\varepsilon \to \phi$ on $[-M, M]$ as $\varepsilon \to 0+$, the continuity of ϕ combined with (5.47), (5.49), and Lebesgue's dominated convergence theorem imply

$$\lim_{\varepsilon\to0} \phi_\varepsilon(u_\varepsilon) = \phi(u) \qquad \text{in } C([0,T]; L^q(\Sigma,\mu)) \text{ for every } 1 \le q < \infty. \tag{5.50}$$

Since F is Lipschitz continuous in all L^q-spaces, (5.49) and (5.50) imply

$$\lim_{\varepsilon\to 0}\int_0^t s^{k+1}\int_\Sigma F(u_\varepsilon)\phi_\varepsilon(u_\varepsilon)\,\mathrm{d}\mu\,\mathrm{d}s = \int_0^t s^{k+1}\int_\Sigma F(u)\phi(u)\,\mathrm{d}\mu\,\mathrm{d}s \tag{5.51}$$

for every $t \in (0,T]$ and $k \geq -1$. We set

$$\Phi_\varepsilon(s) = \int_0^s \phi_\varepsilon(r)\,\mathrm{d}r \qquad \text{for every } s \in \mathbb{R}.$$

Since $\phi_\varepsilon \to \phi$ uniformly on $[-2M,2M]$ as $\varepsilon \to 0+$, there is some $\varepsilon_0 > 0$ such that the sequence $(\phi_\varepsilon)_{\varepsilon>\varepsilon_0}$ is bounded $L^\infty(-2M,2M)$. Thus and by the mean-value theorem, we obtain that

$$\begin{aligned}
&\|\Phi_\varepsilon(u_\varepsilon(t)) - \Phi(u(t))\|_1\\
&\quad\leq \|\Phi_\varepsilon(u_\varepsilon(t)) - \Phi_\varepsilon(u(t))\|_1 + \|\Phi_\varepsilon(u(t)) - \Phi(u(t))\|_1\\
&\quad\leq \sup_{\varepsilon\geq\varepsilon_0}\|\phi_\varepsilon\|_{L^\infty(-M,M)}\,\|u_\varepsilon(t)-u(t)\|_1 + \|\phi_\varepsilon - \phi\|_{L^\infty(-M,M)}\,e^{\omega T}\|u_0\|_1.
\end{aligned}$$

Applying to this limit (5.49) and the uniform convergence of $\phi_\varepsilon \to \phi$ on $[-M,M]$ as $\varepsilon \to 0+$, we obtain that

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) = \Phi(u) \qquad \text{in } C([0,T];L^1(\Sigma,\mu)). \tag{5.52}$$

By Theorem 5.7, every u_ε is a strong solution of Cauchy problem (5.10) for ϕ replaced by ϕ_ε and has the same regularity as stated in (5.15). Moreover, every u_ε satisfies inequality (5.39), that is,

$$\begin{aligned}
&\int_0^t \Psi(\phi_\varepsilon(u_\varepsilon(s)))\,\mathrm{d}s + \int_\Sigma \Phi_\varepsilon(u_\varepsilon(t))\,\mathrm{d}\mu\\
&\qquad\leq \int_\Sigma \Phi_\varepsilon(u_0)\,\mathrm{d}\mu - \int_0^t\int_\Sigma F(u_\varepsilon(s))\phi_\varepsilon(u_\varepsilon(s))\,\mathrm{d}\mu\,\mathrm{d}s
\end{aligned} \tag{5.53}$$

and inequality (5.16), that is,

$$\begin{aligned}
&\int_0^t s^{k+1}\Psi(\phi_\varepsilon(u_\varepsilon(s)))\,\mathrm{d}s + t^{k+1}\int_\Sigma \Phi_\varepsilon(u_\varepsilon(t))\,\mathrm{d}\mu\\
&\qquad\leq (k+1)\int_0^t s^k\int_\Sigma \phi_\varepsilon(u_\varepsilon(t))\,u_\varepsilon(t)\,\mathrm{d}\mu\,\mathrm{d}s\\
&\qquad\qquad - \int_0^t s^{k+1}\int_\Sigma F(u_\varepsilon(s))\,\phi_\varepsilon(u_\varepsilon(s))\,\mathrm{d}\mu\,\mathrm{d}s
\end{aligned} \tag{5.54}$$

for every $t \in (0,T]$ and $\varepsilon > 0$, where $k \geq 0$ is fixed. Since $\Phi_\varepsilon \geq 0$, inequality (5.53) together with (5.47) and the two limits (5.50) and (5.51) imply that the sequence $(\phi_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is bounded in $L^p(0,T;V)$ and so by (5.6), $(\Psi'\phi_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is bounded in $L^{p'}(0,T;V')$. In particular, by equation (5.11) we have that $(\frac{du_\varepsilon}{dt})_{\varepsilon>0}$ is bounded in $L^{p'}(0,T;V')$. By the continuous embedding of V into $L^2(\Sigma,\mu)$ and by (5.50), for any sequence $(\varepsilon_n)_{n\geq 1}$ of the open interval $(0,1)$ satisfying $\lim_{n\to\infty}\varepsilon_n = 0$, there are v and $\chi \in L^{p'}(0,T;V')$ and a subsequence of $(u_{\varepsilon_n})_{n\geq 1}$, which we denote again by $(u_{\varepsilon_n})_{n\geq 1}$ such that

$$\lim_{n\to\infty}\phi_{\varepsilon_n}(u_{\varepsilon_n}) = \phi(u) \qquad \text{weakly in } L^p(0,T;V), \tag{5.55}$$

$$\lim_{n\to\infty}\tfrac{du_{\varepsilon_n}}{dt} = v \qquad \text{weakly in } L^{p'}(0,T;V'), \tag{5.56}$$

$$\lim_{n\to\infty}\Psi'\phi_{\varepsilon_n}(u_{\varepsilon_n}) = \chi \qquad \text{weakly in } L^{p'}(0,T;V'). \tag{5.57}$$

By (5.49) and (5.56) combined with standard techniques employed for vector-valued distributions (see, for instance, [53, pp 36]), one sees that

$$\nu = \frac{du}{dt} \qquad \text{in } L^{p'}(0,T;V').$$

By (5.4) and Lipschitz continuity of F on $L^{p'}(0,T;L^2(\Sigma,\mu))$, we may multiply equation (5.11) by $v \in C^1_c(0,T;V)$ and subsequently integrate over $(0,T)$. Then,

$$\int_0^T \left\langle \frac{du_{\varepsilon_n}}{dt}, v\right\rangle_{V',V} \mathrm{d}t + \int_0^T \langle \Psi'\phi_{\varepsilon_n}(u_{\varepsilon_n}), v\rangle_{V',V}\,\mathrm{d}t + \int_0^T \langle F(u_{\varepsilon_n}), v\rangle\,\mathrm{d}t = 0$$

for every $n \geq 1$. Sending $n \to \infty$ in the latter equation and employing (5.49), (5.56), (5.57) and the Lipschitz continuity of F on $L^{p'}(0,T;L^2(\Sigma,\mu))$ yields

$$\frac{du}{dt} + \chi + F(u) = 0 \qquad \text{in } L^{p'}(0,T;V'). \tag{5.58}$$

It remains to show that $\chi = \Psi'(u)$. To see this, let $0 < t_1 < t_2 < T$ and take $w \in L^p(0,T;V)$. By convexity of Ψ, we have that

$$\int_{t_1}^{t_2} \langle \Psi'(\phi_{\varepsilon_n}(u_{\varepsilon_n})) - \Psi'(w), \phi_{\varepsilon_n}(u_{\varepsilon_n}) - w\rangle\,\mathrm{d}t \geq 0.$$

Now, using that u_{ε_n} is a solution of equation (5.11), the latter inequality can be rewritten as

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Sigma F(u_{\varepsilon_n})(\phi_{\varepsilon_n}(u_{\varepsilon_n}) - w)\mathrm{d}\mu\mathrm{d}t + \int_\Sigma \Phi(u_{\varepsilon_n})\mathrm{d}\mu\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\langle \tfrac{du_{\varepsilon_n}}{dt}, w\rangle\,\mathrm{d}t \\ \leq -\int_{t_1}^{t_2}\langle \Psi'(w), \phi_{\varepsilon_n}(u_{\varepsilon_n}) - w\rangle\,\mathrm{d}t.\end{aligned}$$

Sending $n \to \infty$ in this inequality and using (5.49), (5.51) for $k = -1$, (5.55), (5.56), and (5.52), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Sigma F(u)(\phi(u) - w)\mathrm{d}\mu\mathrm{d}t + \int_\Sigma \Phi(u)\mathrm{d}\mu\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\langle \tfrac{du}{dt}, w\rangle\,\mathrm{d}t \\ \leq -\int_{t_1}^{t_2}\langle \Psi'(w), \phi(u) - w\rangle\,\mathrm{d}t.\end{aligned}$$

On the other hand, if we first multiply equation (5.58) with $\phi(u)$, and then integrate over (t_1,t_2) for $0 < t_1 < t_2 < T$ and apply *integration by parts formula* (Lemma 5.11) yields

$$\int_\Sigma \Phi(u)\,\mathrm{d}\mu\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\langle \chi, \phi(u)\rangle\,\mathrm{d}t + \int_{t_1}^{t_2}\int_\Sigma F(u(t))\phi(u(t))\,\mathrm{d}\mu\,\mathrm{d}t = 0.$$

Using this, we can rewrite the latter inequality as

$$\int_{t_1}^{t_2}\langle \chi - \Psi'(w), \phi(u) - w\rangle\,\mathrm{d}t \geq 0.$$

Since $w \in L^p(0,T;V')$ was arbitrary, taking $w = \phi(u) - \lambda\xi$ for any $\lambda > 0$ and for some general $\xi \in L^p(t_1,t_2;V)$ in this inequality and applying the hemicontinuity of Ψ' (hypothesis ($\mathcal{H}$ii)) yields

$$\int_{t_1}^{t_2}\langle \chi - \Psi'(\phi(u)), \xi\rangle\,\mathrm{d}t \geq 0$$

for all $\xi \in L^p(t_1,t_2;V)$. Therefore and since $0 < t_1 < t_2 < T$ were arbitrary, $\chi = \Psi'(\phi(u))$ in V' for a.e. $t \in (0,T)$, showing that u is a weak energy solution of Cauchy problem (5.3).

It remains to show that u satisfies the energy inequalities (5.39) and (5.16). To see this, we send $\varepsilon \to 0$ in the two inequalities (5.53) and (5.54) and apply limit (5.55) together with Fatou's lemma (note $\Psi \geq 0$ by assumption) and the lower semicontinuity of ψ, limit (5.49), (5.50), (5.51) and (5.52). This completes the proof of this theorem. □

5.3. **Proof of Theorem 5.6.** This subsection is dedicated to outlining the proof of Theorem 5.6.

Proof of Theorem 5.6. Let $u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1}$ and $\{T_t\}_{t\geq 0}$ be the semigroup generated by $-(\overline{A_{1\cap\infty}\phi} + F)$ on $\overline{D(A_{1\cap\infty}\phi)}^{L^1}$. By assumption, the semigroup $\{T_t\}_{t\geq 0}$ satisfies the L^1-L^∞-regularisation estimate (1.18) (for $u_0 = 0$ and $s = 1$). Thus $T_t u_0 \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma,\mu)$ for every $t > 0$ and the strong continuity of $\{T_t\}_{t\geq 0}$ in $L^1(\Sigma,\mu)$ yields the existence of a sequence $(u_{0,n})_{n\geq 1}$ with elements $u_{0,n} \in \overline{D(A_{1\cap\infty}\phi)}^{L^1} \cap L^\infty(\Sigma,\mu)$ such that $u_{0,n} = T_{t_n} u_0$ for some sequence $(t_n)_{n\geq 1}$ satisfying $0 < t_{n+1} < t_n$, $\lim_{n\to\infty} t_n = 0$, and $\lim_{n\to\infty} u_{0,n} = u_0$ in $L^1(\Sigma,\mu)$. We set $u(t) = T_t u_0$, $(t \geq 0)$, to be the unique mild solution of problem (5.2) in L^1 with initial value u_0 and $u_n(t) = T_t u_{0,n}$, $(t \geq 0)$, the unique mild solution of problem (5.2) in L^1 with initial value $u_{0,n}$. By Theorem 5.9, the mild solution u_n of (5.2) is a weak energy solution of problem (5.3) with regularity (5.35)–(5.36), satisfying (5.37) and (5.38). Now, the semigroup property (2.24) and the exponential growth property in L^1 (that is, $\tilde{q} = 1$ in (2.30)) yield that

$$u(t+t_n) = u_n(t) \qquad \text{for all } t \geq 0 \text{ and } n \geq 1. \tag{5.59}$$

Thus, if one replaces the interval $[0,T)$ by $[t_n,T)$ for every $n \geq 1$, the mild solution u is a weak energy solution of problem (5.3) with regularity (5.35)–(5.36), satisfying (5.37) and (5.38).

It remains to show that energy estimates (5.9) holds. For this, recall that by Theorem 5.9, u_n satisfies (5.16) for every $k \geq 0$, that is,

$$\begin{aligned}
&\int_0^t s^{k+1}\Psi(\phi(u_n(s)))\,\mathrm{d}s + t^{k+1}\int_\Sigma \Phi(u_n(t))\,\mathrm{d}\mu \\
&\qquad \leq (k+1)\int_0^t s^k \int_\Sigma \phi(u_n(t))\,u_n(t)\,\mathrm{d}\mu\,\mathrm{d}s \\
&\qquad\qquad - \int_0^t s^{k+1}\int_\Sigma F(u_n(s))\phi(u_n(s))\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned} \tag{5.60}$$

By using that F is Lipschitz continuous, Hölder's and Young's inequality, and then Poincaré type inequality (5.7), we see that

$$\begin{aligned}
&\pm \int_0^t s^{k+1}\int_\Sigma F(u_n(s))\phi(u_n(s))\,\mathrm{d}\mu\,\mathrm{d}s \\
&\qquad \leq \omega \int_0^t s^{k+1}\|u_n(s)\|_{p'}\,\|\phi(u_n(s))\|_p\,\mathrm{d}s \\
&\qquad \leq \varepsilon \int_0^t s^{k+1}\Psi(\phi(u_n(s)))\,\mathrm{d}s + \tfrac{\omega^p\,C^{p-1}}{p'(\varepsilon\,p)^{p-1}}\int_0^t s^{k+1}\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s
\end{aligned} \tag{5.61}$$

for every $\varepsilon > 0$. Similarly,

$$\begin{aligned}(k+1)\int_0^t s^k \int_\Sigma \phi(u_n(s))\,u_n(s)\,\mathrm{d}\mu\,\mathrm{d}s&\\ \le (k+1)\int_0^t s^k\|u_n(s)\|_{p'}\,\|\phi(u_n(s))\|_p\,\mathrm{d}s&\\ \le \varepsilon\int_0^t s^{k+1}\Psi(\phi(u_n(s)))\,\mathrm{d}s + \tfrac{(k+1)^p\,C^{p-1}}{p'\,(\varepsilon\,p)^{p-1}}\int_0^t s^k\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s&\end{aligned} \tag{5.62}$$

Choosing $\varepsilon = \frac{1}{4}$ in these two estimates and apply them to the right hand-side of inequality (5.60), we obtain

$$\begin{aligned}\tfrac{1}{2}\int_0^t s^{k+1}\Psi(\phi(u_n(s)))\,\mathrm{d}s + t^{k+1}\int_\Sigma \Phi(u_n(t))\,\mathrm{d}\mu&\\ \le \tfrac{(k+1)^p\,C^{p-1}}{p'\,(4^{-1}\,p)^{p-1}}\int_0^t s^k\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s + \tfrac{\omega^p\,C^{p-1}}{p'\,(4^{-1}\,p)^{p-1}}\int_0^t s^{k+1}\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s.&\end{aligned} \tag{5.63}$$

By assumption, there are exponents α, β, $\gamma > 0$ and a constant $\tilde{C} > 0$ such that the semigroup $\{T_t\}_{t\ge 0}$ satisfies L^1-L^∞ regularisation estimate (5.7). Now, we choose $k = \alpha(p'-1) > 0$. Then, by Hölder's inequality, by using that

$$\|u_n(t)\|_1 \le e^{\omega t}\,\|u_{0,n}\|_1 \qquad \text{for every } t \ge 0,$$

and by L^1-L^∞ regularisation estimate (5.7), we see that

$$\begin{aligned}\int_0^t s^{\alpha(p'-1)}\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s &\le \int_0^t s^{\alpha(p'-1)}\|u_n(s)\|_\infty^{(p'-1)}\,\|u_n(s)\|_1\,\mathrm{d}s\\ &\le \int_0^t s^{\alpha(p'-1)}\|u_n(s)\|_\infty^{(p'-1)}\,e^{\omega s}\,\mathrm{d}s\,\|u_{0,n}\|_1\\ &\le \int_0^t e^{\omega(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_{0,n}\|_1^{\gamma\,(p'-1)+1}\end{aligned}$$

and

$$\int_0^t s^{\alpha(p'-1)+1}\,\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s \le \int_0^t s\,e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_{0,n}\|_1^{\gamma\,(p'-1)+1}.$$

Applying these to estimate (5.63), we obtain

$$\begin{aligned}\tfrac{1}{2}\int_0^t s^{\alpha(p'-1)+1}\Psi(\phi(u_n(s)))\,\mathrm{d}s + t^{\alpha(p'-1)+1}\int_\Sigma \Phi(u_n(t))\,\mathrm{d}\mu&\\ \le \tfrac{(\alpha(p'-1)+1)^p\,C^{p-1}}{p'\,(4^{-1}\,p)^{p-1}}\int_0^t e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_{0,n}\|_1^{\gamma\,(p'-1)+1}&\\ + \tfrac{\omega^p\,C^{p-1}}{p'\,(4^{-1}\,p)^{p-1}}\int_0^t s\,e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_{0,n}\|_1^{\gamma\,(p'-1)+1}.&\end{aligned}$$

Inserting relation (5.59) into this inequality yields

$$\begin{aligned}\tfrac{1}{2}\int_0^t s^{\alpha(p'-1)+1}\Psi(\phi(u(s+t_n)))\,\mathrm{d}s + t^{\alpha(p'-1)+1}\int_\Sigma \Phi(u(t+t_n))\,\mathrm{d}\mu&\\ \le \tfrac{(\alpha(p'-1)+1)^p\,C^{p-1}}{p'\,(4^{-1}\,p)^{p-1}}\int_0^t e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u(t_n)\|_1^{\gamma\,(p'-1)+1}&\\ + \tfrac{\omega^p\,C^{p-1}}{p'\,(4^{-1}\,p)^{p-1}}\int_0^t s\,e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u(t_n)\|_1^{\gamma\,(p'-1)+1}.&\end{aligned}$$

for every $t > 0$ and $n \geq 1$. By the continuity of ϕ, since $u \in C([0,\infty);L^1(\Sigma,\mu))$, the lower semicontinuity of Ψ, and since

$$\lim_{n\to\infty} \int_\Sigma \Phi(u(t+t_n))\mathrm{d}\mu \to \int_\Sigma \Phi(u(t))\mathrm{d}\mu$$

for every $t > 0$, sending $n \to \infty$ in the last inequality yields inequality (5.9).

Next, suppose that $\phi' \in L^\infty(\mathbb{R})$, $(\phi^{-1})'$ is locally bounded, and exponent $0 < \alpha \leq 1$ in estimate (5.1). By Theorem 5.7, the function u_n given by (5.59) is a strong solution of Cauchy problem (5.14) with initial value $u_n(0) = u_{0,n} = T_{t_n}u_0$ for some sequence $(t_n)_{n\geq 1} \subseteq (0,\infty)$ satisfying $t_n \downarrow 0+$ as $n \to \infty$. By Theorem 5.7, the function u_n has regularity (5.15) satisfying the properties (1)–(4) of in this theorem. Thus, by (5.59), the mild solution u is also a strong solution of Cauchy problem (5.14) on the interval $[t_n, T)$ admitting regularity (5.15) on $[t_n, T)$ for every $n \geq 1$ large enough.

It remains to show that u satisfies energy inequality (5.13). To see this, we use that by Theorem 5.7, u_n satisfies inequality (5.17) for every $k \geq 0$ and $t > 0$, that is,

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{k+2}\int_\Sigma \phi'(u_n(s))\left|\frac{du_n}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{k+2}\,\Psi(\phi(u_n(t)))\\
&\qquad\leq (k+2)(k+1)\int_0^t s^k\int_\Sigma \phi(u_n(t))\,u_n(t)\,\mathrm{d}\mu\,\mathrm{d}s\\
&\qquad\qquad - (k+2)\int_0^t s^{k+1}\int_\Sigma F(u_n(s))\phi(u_n(s))\,\mathrm{d}\mu\,\mathrm{d}s\\
&\qquad\qquad + \tfrac{1}{2}\int_0^t s\int_\Sigma \phi'(u_n(s))\,|F(u_n(s))|^2\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned}\tag{5.64}$$

Applying the two estimates (5.61) and (5.62) to the right hand side of inequality (5.64) yields

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{k+2}\int_\Sigma \phi'(u_n(s))\left|\frac{du_n}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{k+2}\,\Psi(\phi(u_n(t)))\\
&\quad\leq (k+2)\,2\left[\varepsilon\int_0^t s^{k+1}\Psi(\phi(u_n(s)))\,\mathrm{d}s + \tfrac{(k+1)^p\,C^{p-1}}{p'(\varepsilon\,p)^{p-1}}\int_0^t s^k\|u_n(s)\|_{p'}^{p'}\,\mathrm{d}s\right]\\
&\qquad + \tfrac{1}{2}\int_0^t s\int_\Sigma \phi'(u_n(s))\,|F(u_n(s))|^2\,\mathrm{d}\mu\,\mathrm{d}s.
\end{aligned}$$

for every $\varepsilon > 0$ and $k \geq 0$. Now, taking $k = \alpha(p'-1)$ and $\varepsilon = \frac{1}{2}$, then by inequality (5.9), the Lipschitz continuity of F, Hölder's inequality, since by assumption $\phi' \in L^\infty(\mathbb{R})$, by the L^1-L^∞ regularisation estimate (5.1), and by the exponential growth property

$$\|u(t)\|_1 \leq e^{\omega t}\,\|u_0\|_1 \qquad \text{for all } t \geq 0,$$

we see that

$$\begin{aligned}
&\tfrac{1}{2}\int_0^t s^{\alpha(p'-1)+2}\int_\Sigma \phi'(u_n(s))\left|\frac{du_n}{ds}(s)\right|^2 \mathrm{d}\mu\,\mathrm{d}s + t^{\alpha(p'-1)+2}\,\Psi(\phi(u_n(t)))\\
&\quad\leq \tfrac{(\alpha(p'-1)+2)\,2}{p'}\left[\tfrac{(\alpha(p'-1)+1)^p\,C^{p-1}}{(4^{-1}\,p)^{p-1}}\int_0^t e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_{0,n}\|_1^{\gamma\,(p'-1)+1}\right.\\
&\qquad\left. + \tfrac{\omega^p\,C^{p-1}}{(4^{-1}\,p)^{p-1}}\int_0^t s\,e^{\omega\,(\beta\,(p'-1)+1)\,s}\,\mathrm{d}s\,\|u_{0,n}\|_1^{\gamma\,(p'-1)+1}\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \tfrac{(\alpha(p'-1)+1)^p\, C^{p-1}}{(2^{-1}\,p)^{p-1}} \int_0^t s^{\alpha(p'-1)} \|u_n(s)\|_\infty^{p'-1} \, \|u_n(s)\|_1 \, \mathrm{d}s \Big] \\
&\qquad\qquad + \tfrac{\omega^2}{2} \|\phi'\|_\infty \int_0^t s \, \|u_n(s)\|_\infty \, \|u_n(s)\|_1 \, \mathrm{d}s. \\
&\le \tfrac{(\alpha(p'-1)+2)\,2}{p'} \Big[\tfrac{(k+1)^p\, C^{p-1}}{(4^{-1}\,p)^{p-1}} \int_0^t e^{\omega\,(\beta\,(p'-1)+1)\,s} \, \mathrm{d}s \, \|u_{0,n}\|_1^{\gamma\,(p'-1)+1} \\
&\qquad + \tfrac{\omega^p\, C^{p-1}}{(4^{-1}\,p)^{p-1}} \int_0^t s\, e^{\omega\,(\beta\,(p'-1)+1)\,s} \, \mathrm{d}s \, \|u_{0,n}\|_1^{\gamma\,(p'-1)+1} \\
&\qquad\qquad + \tfrac{(\alpha(p'-1)+1)^p\, C^p}{p'\,(2^{-1}\,p)^{p-1}} \int_0^t e^{\omega\,\beta(p'-1)\,s} \, \mathrm{d}s \|u_{0,n}\|_1^{\gamma(p'-1)+1} \Big] \\
&\qquad\qquad + \tfrac{\omega^2\,\tilde{C}}{2} \|\phi'\|_\infty \int_0^t s^{1-\alpha}\, e^{\omega(\beta+1)\,s} \, \mathrm{d}s \|u_{0,n}\|_1^{\gamma+1}.
\end{aligned}
$$

From this estimate, the assumptions on ϕ and by inequality (5.1), it is not difficult to deduce that inequality (5.13) holds and u admits the stated properties. This concludes the proof of this theorem. □

6. Examples

This section is devoted to illustrating the power of the theory developed in the preceding sections. By using the abstract theory of nonlinear semigroups, we show in this section that mild solutions of nonlinear parabolic initial boundary-value problems satisfy an L^1-L^∞-regularisation effect provided the involved diffusion operator satisfies a Gagliardo-Nirenberg type inequality. Comparing our first examples with the results from the known literature, one sees that the methods developed in Section 3 and Section 4 yield sharp exponents and extend these results for solutions with exponential growth.

Note that in principle our theory could work for non-linear operators on non-compact manifolds, such as the porous media operators associated with the Laplace-Beltrami operator or the p-Laplace operator (for the latter see [34]). Here the Sobolev inequalities for the gradient depend on the geometry of the manifold (see [39]), the main task is to deduce Gagliardo-Nirenberg inequalities for the operator under consideration by adapting the methods of the present section, then one applies the above machinery. We leave this for future work.

In order to keep our examples simple and to focus on the essential, namely, the regularisation effect of solutions of parabolic boundary-value problems, we shall assume that Σ is an open subset of the d-dimensional Euclidean space $\mathbb{R}^d$ for $d \ge 2$. We shall specify at the beginning of each example which further assumptions we impose on the boundary $\partial\Sigma$ of Σ. We choose μ to be the d-dimensional Lebesgue measure on Σ and denote by $\mathcal{H} = \mathcal{H}^{d-1}_{|\partial\Sigma}$ the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$ restricted to the boundary $\partial\Sigma$.

Under this assumptions, we simplify our notation and write $L^q(\Sigma)$ to denote the Lebesgue space $L^q(\Sigma,\mu)$, $L^q(\partial\Sigma)$ to denote the Lebesgue space $L^q(\partial\Sigma,\mathcal{H})$, and $L^q_0(\Sigma,\mu)$ the closed linear subspace $u \in L^q(\Sigma,\mu)$ with *mean value* $\overline{u} := \frac{1}{|\Sigma|}\int_\Sigma u \, \mathrm{d}x = 0$ for $1 \le q \le \infty$.

Here, we employ the following notation: for $1 \le p, q \le \infty$, let $W^1_{p,q}(\Sigma)$ be the linear subspace of all functions $u \in L^q(\Sigma)$ having weak partial derivatives

$\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d} \in L^p(\Sigma)$ equipped with the norm

$$\|u\|_{W^1_{p,q}} := \|u\|_q + \||\nabla u|\|_p.$$

Moreover, for $1 \le p,q < \infty$, we denote by $\dot{W}^1_{p,q}(\Sigma)$ the closure of the set of test functions $C^\infty_c(\Sigma)$ in $W^1_{p,q}(\Sigma)$, $W^1_{p,q,m}(\Sigma)$ the space $L^q_0(\Sigma) \cap W^1_{p,q}(\Sigma)$, and for $0 < s < 1$, $W^s_{p,q}(\Sigma)$ denotes the set of all $u \in L^q(\Sigma)$ with finite semi-norm

$$|u|^p_{s,p} := \int_\Sigma \int_\Sigma \tfrac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\mathrm{d}y.$$

We equip $W^s_{p,q}(\Sigma)$ with the norm $\|u\|_{s,p,q} = \|u\|_q + |u|_{s,p}$. Further, we denote by $\dot{W}^s_{p,q}(\Sigma)$ the closure of $C^\infty_c(\Sigma)$ in $W^s_{p,q}(\Sigma)$. For subsets $\partial\Sigma$ in $\mathbb{R}^{d-1}$, we denote by $W^{1-1/p,p}(\partial\Sigma)$ the Sobolev-Slobodeckij space given by the set of all $u \in L^p(\partial\Sigma)$ having finite semi-norm

$$|u|^p_p := \int_{\partial\Sigma} \int_{\partial\Sigma} \tfrac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}\,\mathrm{d}\mathcal{H}(x)\mathrm{d}\mathcal{H}(y).$$

In the following, $F : L^q(\Sigma,\mu) \to L^q(\Sigma,\mu)$ be the Nemytski operator of a Carathéodory function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ satisfying (2.17) for some constant $L > 0$ and $f(\cdot, 0) = 0$ and β be an m-accretive graph in $\mathbb{R}$ with domain $D(\beta) = \mathbb{R}$ and $(0,0) \in \beta$.

We begin to illustrate our theory on the following classical example.

6.1. **Parabolic problems involving p-Laplace type operators.** The L^q-L^r-regularisation effect for $1 \le q < r \le \infty$ of solutions of parabolic equations associated with the celebrated p-Laplace operators equipped with homogeneous Dirichlet boundary conditions has been first established by Véron [92]. The ideas in [92] were followed up and extended rapidly by Alikakos and Rostamian [1] and more recently in [30, 71]. By using the logarithmic Sobolev approach, Cipriani and Grillo [33] revisited the L^q-L^r-regularisation effect for solutions of parabolic equations involving p-Laplace operators equipped with homogeneous boundary conditions. Then many papers followed on this topic by using the same method (see, for instance, [84, 69, 47, 81], and more recently, [94] for homogeneous Robin boundary conditions with a nonlocal term).

To the best of our knowledge, our results stated in this section complement the existing literature in several ways: namely, by adding (possibly multi-valued) monotone and Lipschitz continuous perturbations and by providing a simplified approach to a L^q-L^r-regularisation effect of solutions of parabolic boundary-value problems associated with p-Laplace type operators.

Further, the examples in this subsection show that the parameter m_0 appearing in the two main theorems Theorem 1.2 and Theorem 1.4 is *optimal* if $m_0 = q\gamma^{-1}$ (cf. Remark 1.3). To be more precise, consider the case $1 < p < d$ and let $\{T_t\}_{t\ge 0}$ be the semigroup generated by the negative p-Laplace operator $-\Delta_p^{\mathbb{R}^d}$ on $L^2(\mathbb{R}^d)$. Then, we show in the proof of Theorem 6.1 below that $\{T_t\}_{t\ge 0}$ satisfies L^q-L^r regularity estimate (1.18) for $u_0 = 0$ with parameters $r = \frac{pd}{d-p}$, $q = 2$ and exponent $\gamma = \frac{2}{p}$. One easily sees that $\gamma r > q$ and so one can deduce an

L^s-L^∞ regularisation estimate for $s = \gamma\, r\, q^{-1}\, m_0 = \frac{d m_0}{d-p}$ and sufficiently large $m_0 \geq q\,\gamma^{-1} = p$. By Theorem 6.1, if $\frac{2d}{d+2} < p < d$ then $m_0 = p$ satisfies (1.15), and if $\frac{2d}{d+1} < p < d$ then for $m_0 = p$, the semigroup $\{T_t\}_{t\geq 0} \sim -\Delta_p^{\mathbb{R}^d}$ for satisfies L^1-L^∞-regularisation estimate (1.18) with exponent $\alpha_1 = \frac{d}{d(p-2)+p}$ und $u_0 = 0$. The exponent α_1 coincides with exponent $\frac{d}{\lambda}$ in the *Barenblatt solution*

$$\Gamma_p(x,t) := t^{-\frac{d}{\lambda}} \left[1 + C_p\left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}, \quad \text{for } t > 0,$$
$$\text{with } \lambda = d(p-2)+p,\, C_p = \left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} \frac{2-p}{p}, \tag{6.1}$$

to the prototype parabolic p-Laplace equation

$$\partial_t u - \Delta_p^{\mathbb{R}^d} u = 0 \qquad \text{on } \mathbb{R}^d \times (0,\infty).$$

Note, the Barenblatt solution (6.1) also holds for the *singular* range $1 < p < 2$ provided the parameter $\lambda > 0$. Moreover, $\lambda > 0$ if and only if $\frac{2d}{d+1} < p < 2$ (see [50, Chapter 7.4]). It is worth noting that for singular $1 < p < 2$, the existence of a Barenblatt solution coincides with the fact that semigroup $\{T_t\}_{t\geq 0}$ generated by the negative p-Laplace operator $-\Delta_p^{\mathbb{R}^d}$ on $L^2(\mathbb{R}^d)$ satisfies L^1-L^∞-regularisation estimate (1.18) for $u_0 = 0$ with exponent $\alpha_1 = \frac{d}{d(p-2)+p}$, but also that for the same range $\frac{2d}{d+1} < p < 2$, every positive weak energy solutions of problem (6.2) (below) satisfy a *Harnack inequality* (cf. [50, Chapter 7.4]). In the degenerated range $2 < p < \infty$, the comparison of the optimal exponents α_1 has been considered, for instance, in [21].

Throughout this section, let $1 < p < \infty$. Then for given initial value $u_0 \in L^q(\Sigma)$, we intend to establish the regularisation effect of solutions $u(t) = u(x,t)$ for $t > 0$ of the *parabolic initial value problem*

$$\begin{cases} \partial_t u - \mathrm{div}(a(x,\nabla u)) + \beta(u) + f(x,u) \ni 0 & \text{on } \Sigma \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \Sigma, \end{cases} \tag{6.2}$$

respectively equipped with one of the following types of boundary conditions:

$$u = 0 \quad \text{on } \partial\Sigma \times (0,\infty), \text{ if } \Sigma \subseteq \mathbb{R}^d, \tag{6.3}$$
$$a(x,\nabla u)\cdot\nu = 0 \quad \text{on } \partial\Sigma \times (0,\infty), \text{ if } \mu(\Sigma) < \infty, \tag{6.4}$$
$$a(x,\nabla u)\cdot\nu + b(x)|u|^{p-1}u + d\,\theta_p(u) = 0 \quad \text{on } \partial\Sigma \times (0,\infty), \text{ if } \mu(\Sigma) < \infty. \tag{6.5}$$

Here, we suppose that $a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying the following *p-coercivity*, *growth* and *monotonicity* conditions

$$a(x,\xi)\xi \geq \eta|\xi|^p \tag{6.6}$$
$$|a(x,\xi)| \leq c_1|\xi|^{p-1} + h(x) \tag{6.7}$$
$$(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) > 0 \tag{6.8}$$

for a.e. $x \in \Sigma$ and all $\xi, \xi_1, \xi_2 \in \mathbb{R}^d$ with $\xi_1 \neq \xi_2$, where $h \in L^{p'}(\Sigma)$ and $c_1, \eta > 0$ are constants independent of $x \in \Sigma$ and $\xi \in \mathbb{R}^d$. Under these assumptions, the second order quasi linear operator

$$\mathcal{B}u := -\operatorname{div}(a(x, \nabla u)) \qquad \text{in } D'(\Sigma) \tag{6.9}$$

for $u \in W^{1,p}_{loc}(\Omega)$ belongs to the class of *Leray-Lions operators* (cf. [63]), of which the *p-Laplace operator* $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a classical prototype.

In some situations, one can replace (6.8) by

$$(a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) \geq \tilde{\eta}\, |\xi_1 - \xi_2|^p \tag{6.10}$$

for a.e. $x \in \Sigma$ and all $\xi_1, \xi_2 \in \mathbb{R}^d$. In fact, it is well-known ([49]) that for $p \geq 2$, the p-Laplace operator satisfies inequality (6.10) with constant $\tilde{\eta} = 2^{2-p}$.

Regarding *homogeneous Dirichlet boundary conditions* (6.3), we assume that Σ is an open subset of $\mathbb{R}^d$ and impose no further assumptions on the boundary $\partial\Sigma$ of Σ. In the case $\Sigma = \mathbb{R}^d$, the homogeneous Dirichlet boundary conditions (6.3) become the following *vanishing at infinity condition*

$$\lim_{|x|\to\infty} u(x,t) = 0 \quad \text{for every } t > 0, \quad \text{if } \Sigma = \mathbb{R}^d. \tag{6.11}$$

Concerning *homogeneous Neumann boundary conditions* (6.4), we assume that Σ is an open bounded domain with a Lipschitz boundary $\partial\Sigma$ (in the sense of [72, Sect. 1.3]). We denote by ν the (weak) outward pointing unit normal vector on $\partial\Sigma$. Under this assumption, it is not clear whether the co-normal derivative $a(x, \nabla u) \cdot \nu$ on $\partial\Sigma$ exists. Thus, the Neumann boundary condition (6.4) needs to be understood in a *weak sense* and so, we denote by $a(x, \nabla u) \cdot \nu$ the *generalised co-normal derivative* of u at $\partial\Sigma$ associated with the operator $\mathcal{B}$ (as, for instance, described in [32]).

Considering *homogeneous Robin boundary conditions* (6.5), we assume that Σ is a open bounded domain with a Lipschitz boundary $\partial\Sigma$, $d \geq 0$ is a constant, $b \in L^\infty(\partial\Sigma)$ such that $b(x) \geq b_0 > 0$ for $\mathcal{H}$-a.e. $x \in \partial\Sigma$. The operator θ_p describes the *nonlocal term on* $\partial\Sigma$ and is given by

$$\langle \theta_p(u), v\rangle = \int_{\partial\Sigma}\int_{\partial\Sigma} \tfrac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+p-2}} (v(x) - v(y))\, \mathrm{d}\mathcal{H}(x)\mathrm{d}\mathcal{H}(y) \tag{6.12}$$

for every $u, v \in W^{1-1/p,p}(\partial\Sigma)$.

6.1.1. *Homogeneous Dirichlet boundary conditions.* Let Σ be an open subset of $\mathbb{R}^d$. It is well-known (cf. [12]), at least in the case when Σ is bounded that the Leray-Lions operator $\mathcal{B}$ given by (6.9) equipped with homogeneous Dirichlet boundary conditions can be realised as follows:

$$\begin{aligned} B^D = \Big\{ (u,v) \in L^2(\Sigma) \times L^2(\Sigma) \,\Big|\, & u \in \dot{W}^1_{p,2}(\Sigma) \text{ such that} \\ & \int_\Sigma a(x, \nabla u)\nabla\xi \mathrm{d}x = \int_\Sigma v\,\xi \mathrm{d}x \text{ for all } \xi \in \dot{W}^1_{p,2}(\Sigma) \Big\}. \end{aligned} \tag{6.13}$$

We call B^D the *Dirichlet-Leray-Lions operator* in $L^2(\Sigma)$. Note that, since the set of test functions $C_c^\infty(\Sigma)$ is contained in $\dot{W}^1_{p,2}(\Sigma)$ and dense in $L^2(\Sigma)$, B^D defines a single-valued operator on $L^2(\Sigma)$ and by using (6.6), one obtains that the domain $D(B^D)$ is dense in $L^2(\Sigma)$. Furthermore, condition (6.6) yields $a(x, 0) = 0$ a.e. on Σ hence, $(0, 0) \in B^D$.

In the case $\Sigma = \mathbb{R}^d$, the space $\dot{W}^1_{p,2}(\Sigma) = W^1_{p,2}(\mathbb{R}^d)$. Hence the operator B^D becomes a realisation in $L^2(\Sigma)$ of the Leray-Lions operator $\mathcal{B}$ equipped with vanishing conditions (6.11).

To see that B^D is completely accretive in $L^2(\Sigma)$, let $T \in C^\infty(\mathbb{R})$ be such that the derivative $0 \le T' \le 1$ with compact support $\operatorname{supp}(T')$ and $T(0) = 0$. Since for every $u, \hat{u} \in \dot{W}^1_{p,2}(\Sigma)$, $T(u-\hat{u}) \in \dot{W}^1_{p,2}(\Sigma)$ with

$$\nabla T(u-\hat{u}) = T'(u-\hat{u})\,\nabla(u-\hat{u})$$

and by monotonicity condition (6.8), one sees that

$$\begin{aligned}&\int_\Sigma T(u-\hat{u})(B^D u - B^D \hat{u})\mathrm{d}x\\ &\qquad = \int_\Sigma (a(x,\nabla u) - a(x,\nabla\hat{u}))\nabla(u-\hat{u})T'(u-\hat{u})\,\mathrm{d}x \ge 0.\end{aligned}$$

Thus, by Proposition 2.4, the operator B^D is completely accretive.

Under the assumptions (6.6)-(6.8), the restriction of the operator $I + B^D$ on the reflexive Banach space $V = \dot{W}^1_{p,2}(\Sigma)$ satisfies the hypotheses of [63, Théorème 1]. Recall that an operator $I + B$ on some Banach space V is *coercive* in V if

$$\lim_{\|u\|_V\to\infty} \frac{\langle (I+B)u, u\rangle_{V',V}}{\|u\|_V} = \infty, \tag{6.14}$$

where we denote by $\langle v', v\rangle_{V',V}$ the value of $v' \in V'$ at $v \in V$. In practice, it is often easier to verify that the following statement holds, which is equivalent to (6.14): *for every $\alpha \in R$, the set of all $u \in V$ satisfying*

$$\frac{\langle (I+B)u, u\rangle_{V',V}}{\|u\|_V} \le \alpha$$

is bounded in V. For the operator $B = B^D$, the latter statement holds since for every $\alpha \in \mathbb{R}_+ := [0,\infty)$, the set $\{(a,b) \in \mathbb{R}^2_+ \,|a^2 + b^p \le \alpha(a+b)\,\}$ is bounded in $\mathbb{R}^2$. Thus and since $\dot{W}^1_{p,2}(\Sigma)$ is continuously and densely embedded into $L^2(\Sigma)$, it follows that B^D satisfies the range condition (2.14) for $X = L^2(\Sigma)$.

By hypothesis on the m-accretive graph β on $\mathbb{R}$, one has that the domain $D(\beta_2)$ of the associated accretive operator β_2 in $L^2(\Sigma)$ contains the set of test functions $C^\infty_c(\Sigma)$. Recall, for every $\lambda > 0$, the Yosida operator $\beta_{2,\lambda}$ of β_2 is given by $(\beta_{2,\lambda}u)(x) = \beta_\lambda(u(x))$ for a.e. $x \in \Sigma$, where β_λ denotes the Yosida operator of β on $\mathbb{R}$. Since the Yosida operator $\beta_\lambda : \mathbb{R} \to \mathbb{R}$ of β is monotone, Lipschitz continuous and satisfies $\beta_\lambda(0) = 0$, one has that for every $u \in \dot{W}^1_{p,2}(\Sigma)$, $\beta_{2,\lambda}(u) \in \dot{W}^1_{p,2}(\Sigma)$ with $\nabla\beta_{2,\lambda}(u) = \beta'_\lambda(u)\,\nabla u$ a.e. on Σ for all $\lambda > 0$. Thus, by definition of B^D and by (6.6),

$$[v, \beta_{2,\lambda}(u)]_2 = \int_\Sigma a(x,\nabla u)\nabla\beta_{2,\lambda}(u)\,\mathrm{d}x \ge \eta\int_\Sigma |\nabla u|^p \beta'_\lambda(u)\,\mathrm{d}x \ge 0$$

for every $(u,v) \in B^D$. Therefore, by Proposition 2.5, the operator

$$A^D := B^D + \beta_2 + F \tag{6.15}$$

is quasi m-completely accretive in $L^2(\Sigma)$ with dense domain.

By the Crandall-Liggett theorem [40], $-A^D$ generates a strongly continuous semigroup $\{T_t\}_{t\ge0}$ on $L^2(\Sigma)$ of Lipschitz continuous mappings $T_t : L^2(\Sigma) \to$

$L^2(\Sigma)$. Since $-A^D$ is completely accretive, each mapping T_t has a unique Lipschitz continuous extension on $L^q(\Sigma)$ for all $1 \le q < \infty$ and on $\overline{L^2 \cap L^\infty(\Sigma)}^{L^\infty}$ if $q = \infty$, respectively with constant $e^{\omega t}$.

The complete description of the L^q-L^∞-regularisation effect of the semigroup $\{T_t\}_{t\ge 0} \sim -A^D$ is as follows.

Theorem 6.1. *Suppose the Carathéodory function $a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the conditions (6.7), (6.10) and $a(x,0) = 0$ for a.e. $x \in \Sigma$. Then the semigroup $\{T_t\}_{t\ge 0} \sim -A^D$ for the operator A^D given by (6.15) satisfies the following L^q-L^r regularisation estimates.*

(1) *If $1 < p < d$, then (1.14) holds for*

$$\alpha_s = \frac{\alpha^*}{1-\gamma^*\left(1-\frac{s(d-p)}{dm_0}\right)}, \quad \beta_s = \frac{\frac{\beta^*}{2}+\gamma^*\frac{s(d-p)}{dm_0}}{1-\gamma^*\left(1-\frac{s(d-p)}{dm_0}\right)}, \quad \gamma_s = \frac{\gamma^*\, s(d-p)}{dm_0\left(1-\gamma^*\left(1-\frac{s(d-p)}{dm_0}\right)\right)}$$

for every $m_0 \ge p$ satisfying $(\frac{d}{d-p} - 1)m_0 + p - 2 > 0$, and every $1 \le s \le \frac{dm_0}{d-p}$ satisfying $s > \frac{d(2-p)}{p}$, where

$$\alpha^* = \frac{d-p}{pm_0+(d-p)(p-2)},\ \beta^* = \frac{(\frac{2}{p}-1)d+p}{pm_0+(d-p)(p-2)} + 1,\ \gamma^* = \frac{pm_0}{pm_0+(d-p)(p-2)}.$$

Moreover, if $\frac{2d}{d+2} < p < d$ then one can take $m_0 = p$ and if $\frac{2d}{d+1} < p < d$, then (1.14) holds for every $1 \le s \le \frac{dp}{d-p}$.

(2) *If $p = d \ge 2$, then for every $0 < \theta < 1$, inequality (1.14) holds with exponents*

$$\alpha_s = \frac{\alpha^*_\theta}{1-\gamma^*_\theta(1-\frac{s(1-\theta)}{2})}, \quad \beta_s = \frac{\frac{\beta^*}{2}+\gamma^*_\theta\frac{s(1-\theta)}{2}}{1-\gamma^*_\theta(1-\frac{s(1-\theta)}{2})}, \quad \gamma_s = \frac{\gamma^*_\theta\frac{s(1-\theta)}{2}}{1-\gamma^*_\theta(1-\frac{s(1-\theta)}{2})}$$

for every $1 \le s \le \frac{2}{1-\theta}$, where

$$\alpha^*_\theta = \frac{[2\theta+p(1-\theta)](1-\theta)}{\theta^2},\ \beta^*_\theta = \frac{[2\theta+p(1-\theta)]^2-(1-\theta)p^2}{p^2 2\theta} + 1,\ \gamma^*_\theta = \frac{2\theta^2}{2\theta+p(1-\theta)}.$$

(3) *If $d < p < \infty$, then inequality (1.14) holds with exponents*

$$\alpha_s = \frac{\alpha^*}{1-\gamma^*(1-\frac{s}{2})}, \quad \beta_s = \frac{}{1-\gamma^*(1-\frac{s}{2})}, \quad \gamma_s = \frac{\gamma^* s}{2(1-\gamma^*(1-\frac{s}{2}))}$$

for every $1 \le s \le 2$, where

$$\alpha^* = \frac{d}{pd+2(p-d)},\ \beta^* = \gamma^* + 1,\ \gamma^* = \frac{2\theta_0+p(1-\theta_0)}{p},\ \theta_0 = \frac{pd}{pd+2(p-d)}.$$

Under the assumptions that a satisfies (6.6)-(6.8), the statements (1)-(3) remain true with (1.14) replaced by (1.18) and for $u_0 = 0$.

For the proof of this theorem, we employ the classical Gagliardo-Nirenberg inequalities ([75], see also [37]). The Gagliardo-Nirenberg inequalities are valid for functions $u \in W^1_{p,q}(\mathbb{R}^d)$ and so, in particular, for test functions $u \in C^\infty_c(\Sigma)$. Thus we can use of the following version of Gagliardo-Nirenberg inequalities.

Lemma 6.2 ([75])**.** *For $1 \le q, p \le \infty$, let $u \in \dot{W}^1_{p,q}(\Sigma)$. Then there is a constant $C > 0$ depending only on d, q, p, θ such that*

$$\|u\|_{p^*} \le C\, \||\nabla u|\|_p^\theta\, \|u\|_q^{1-\theta}, \tag{6.16}$$

where

$$\frac{1}{p^*} = \theta\left(\frac{1}{p} - \frac{1}{d}\right) + (1-\theta)\frac{1}{q}\,, \tag{6.17}$$

for all $\theta \in [0,1]$ with the following exceptional cases:

(1) If $p < d$ and $q = \infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L^{\tilde{q}}(\mathbb{R}^d)$ for some finite $\tilde{q} > 0$.

(2) If $1 < p < \infty$ and $1 - d/p$ is a non negative integer, then (6.16) holds only for $\theta \in [0,1)$.

(3) If Σ is a bounded domain with a Lipschitz boundary, then inequality (6.16) is replaced by

$$\|u\|_{p^*} \le C\left(\||\nabla u|\|_p^{\theta}\,\|u\|_q^{1-\theta} + \|u\|_{\tilde{q}}\right), \tag{6.18}$$

for every $u \in W^1_{p,q}(\Sigma) \cap L^{\tilde{q}}(\Sigma)$ and any $\tilde{q} > 0$, where p^* is given by (6.17) for every $\theta \in [0,1]$ with the exceptional cases (1) and (2), and the constant $C > 0$ also depends on the domain.

Proof of Theorem 6.1. We begin to consider the case $1 < p < d$. Then by Lemma 6.2, there is a constant $C > 0$ such that

$$\|u\|_{\frac{pd}{d-p}} \le C\||\nabla u|\|_p \tag{6.19}$$

for every $u \in \dot{W}^1_{p,2}(\Sigma)$. Thus, by definition of the operator B^D and by (6.10),

$$\begin{aligned}
\|u-\hat{u}\|^p_{\frac{pd}{d-p}} &\le C\,\||\nabla(u-\hat{u})|\|_p^p\\
&\le C\,\tilde{\eta}^{-1}\int_\Sigma (a(x,\nabla u) - a(x,\nabla\hat{u}))\nabla(u-\hat{u})\,\mathrm{d}x\\
&= C\,\tilde{\eta}^{-1}\langle u-\hat{u}, B^D u - B^D\hat{u}\rangle
\end{aligned}$$

for every $u, \hat{u} \in D(B^D)$. Now, Remark 3.5 yields the operator A^D given by (6.15) satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters

$$r = \frac{pd}{d-p}, \quad \sigma = p, \quad \varrho = 0, \quad \text{and} \quad \omega = L. \tag{6.20}$$

For $\gamma = \frac{2}{p}$, one has $\gamma r > 2$ and $m_0 = 2\gamma^{-1} = p$ satisfies (1.15) if and only if $p > 2d/(d+2)$. Thus, Theorem 1.2 yields the first statement of this theorem.

Next, consider the case $p = d \ge 2$. By Lemma 6.2,

$$\|u\|_{\frac{2}{1-\theta}} \le C\||\nabla u|\|_p^{\theta}\,\|u\|_2^{1-\theta}$$

for every $u \in \dot{W}^1_{p,2}(\Sigma)$, $0 \le \theta < 1$ and some constant $C > 0$. Let $0 < \theta < 1$. Then by definition of the operator B^D and by (6.10),

$$\begin{aligned}
\|u-\hat{u}\|^{\frac{p}{\theta}}_{\frac{2}{1-\theta}} &\le C^{\frac{p}{\theta}}\,\||\nabla(u-\hat{u})|\|_p^p\,\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}}\\
&\le C^{\frac{p}{\theta}}\,\tilde{\eta}^{-1}\int_\Sigma (a(x,\nabla u) - a(x,\nabla\hat{u}))\nabla(u-\hat{u})\,\mathrm{d}x\,\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}}\\
&= C^{\frac{p}{\theta}}\,\tilde{\eta}^{-1}\,\langle u-\hat{u}, B^D u - B^D\hat{u}\rangle\,\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}}
\end{aligned}$$

for every u, $\hat{u} \in D(B^D)$. Thus, by Remark 3.5, the operator A^D given by (6.15) satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters

$$r_\theta = \frac{2}{1-\theta},\ \sigma_\theta = \frac{p}{\theta},\ \varrho_\theta = \frac{p(1-\theta)}{\theta},\ \omega = L \quad \text{for every } 0 < \theta < 1. \tag{6.21}$$

For $0 < \theta < 1$, $\gamma_\theta := \frac{2\theta + p(1-\theta)}{p}$ satisfies $\gamma_\theta r_\theta > 2$ and by taking $m_0 = 2\gamma_\theta^{-1} = \frac{2p}{2\theta+p(1-\theta)}$, one has

$$\left(\frac{\gamma_\theta r_\theta}{2} - 1\right) m_0 + 2\left(\frac{1}{\gamma_\theta} - 1\right) = \frac{2\theta}{1-\theta} > 0$$

hence, condition (1.15) holds. Moreover, since $0 < \gamma_\theta \le 1$, one easily sees that $\gamma_\theta(1 - \frac{s}{r_\theta}) < 1$ for every $1 \le s \le 2^{-1}\gamma_\theta r_\theta m_0 = r_\theta$. Therefore by Theorem 1.2, the second statement of this theorem holds.

Finally, let $d < p < \infty$. Then there is an $0 < \theta_0 < 1$ such that $\theta_0(\frac{1}{p} - \frac{1}{d}) + (1-\theta_0)\frac{1}{2} = 0$ or equivalently, $\theta_0 = \frac{pd}{pd+2(p-d)}$. We apply Lemma 6.2 for this θ_0, to conclude that there is a constant $C > 0$ such that

$$\|u\|_\infty \le C \| |\nabla u| \|_p^{\theta_0} \|u\|_2^{1-\theta_0}$$

for every $u \in \dot{W}^1_{p,2}(\Sigma)$. Proceeding as in the previous step, we see that by (6.10) and by Remark 3.5, the operator A^D satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters

$$r = \infty, \quad \sigma = \frac{p}{\theta_0}, \quad \varrho = \frac{p(1-\theta_0)}{\theta_0}, \quad \text{and} \quad \omega = L. \tag{6.22}$$

Then, by the first statement of Theorem 1.2, $\gamma^* = \frac{2+\varrho}{\sigma} = \frac{2\theta_0 + p(1-\theta_0)}{p}$, $\alpha^* = \frac{\theta_0}{p}$ and $\beta^* = \gamma^* + 1$. Moreover, since $\frac{1}{p} < \gamma^* < \frac{2}{p} < 1$, one has for all $1 \le s \le 2$ that $\gamma(1 - \frac{s}{2}) < 1$. Thus, Theorem 4.1 implies that the third statement of this theorem holds. $\square$

6.1.2. *Homogeneous Neumann boundary conditions.* In this subsection, we assume that Σ is a bounded domain with a Lipschitz boundary.

Further, we assume that the monotone graph β on $\mathbb{R}$ either satisfies

$$(v - \hat{v})(u - \hat{u}) \ge \eta_0 |u - \hat{u}|^p \tag{6.23}$$

or

$$vu \ge \eta_0 |u|^p \tag{6.24}$$

for every (u, v), $(\hat{u}, \hat{v}) \in \beta$.

We define the realisation B^N in $L^2(\Sigma)$ of the Leray-Lions operator $\mathcal{B}$ equipped with homogeneous Neumann boundary conditions (6.4) by

$$\begin{aligned} B^N = \Big\{ (u,v) \in L^2(\Sigma) \times L^2(\Sigma) \;\Big|\; & u \in W^1_{p,2}(\Sigma) \text{ such that} \\ & \int_\Sigma a(x, \nabla u)\nabla\xi \mathrm{d}x = \int_\Sigma v\,\xi \mathrm{d}x \text{ for all } \xi \in W^1_{p,2}(\Sigma) \Big\}. \end{aligned} \tag{6.25}$$

Under the assumption that u, ξ and $a(\cdot, \nabla u)$ are smooth functions up to the boundary $\partial\Sigma$ and ν denotes the outward pointing unit normal vector on $\partial\Sigma$, the application of Green's first identity yields

$$\int_{\Sigma} a(x,\nabla u)\nabla\xi\,\mathrm{d}x = -\int_{\Sigma}\operatorname{div}\left(a(x,\nabla u)\right)\,\xi\,\mathrm{d}x + \int_{\partial\Sigma} a(x,\nabla u)\cdot\nu\;\xi\mathrm{d}\mathcal{H}.$$

Thus, if $u \in D(B^N)$, one has that $v = -\operatorname{div}(a(x,\nabla u))$ and $a(x,\nabla u)\cdot\nu = 0$ for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Sigma$, showing that our definition of the operator B^N is consistent with the *smooth* situation. We call B^N the *Neumann Leray-Lions operator* in $L^2(\Sigma)$.

In order to see that B^N is m-completely accretive in $L^2(\Sigma)$ and that the monotone graph β_2 in $L^2(\Sigma)$ satisfies the hypothesis (2.28) in Proposition 2.5 with respect to the operator B^N, one proceeds as in the previous example (for homogeneous Dirichlet boundary conditions), but here one needs to replace the space $\dot{W}^1_{p,2}(\Sigma)$ by $W^1_{p,2}(\Sigma)$. In addition, it is not difficult to check that the domain $D(B^N)$ is dense in $L^2(\Sigma)$. Therefore, the operator

$$A^N := B^N + \beta_2 + F \tag{6.26}$$

is quasi m-completely accretive in $L^2(\Sigma)$ with dense domain. By the Crandall-Liggett theorem, $-A^N$ generates a strongly continuous semigroup $\{T_t\}_{t\ge 0}$ on $L^2(\Sigma)$ of Lipschitz continuous mappings T_t on $L^2(\Sigma)$. The space $L^\infty(\Sigma)$ is continuously embedded into $L^2(\Sigma)$ since Σ is bounded. Thus, and since $T_t : L^q\cap L^2(\Sigma) \to L^q\cap L^2(\Sigma)$ is Lipschitz continuous with respect to the L^q-norm with constant $e^{\omega t}$ for $1\le q\le\infty$, T_t admits a unique Lipschitz continuous extension on $L^q(\Sigma)$ with the same Lipschitz constant $e^{\omega t}$ for every $1\le q\le\infty$.

Now, we state the complete description of the L^q-L^∞-regularisation effect of the semigroup $\{T_t\}_{t\ge0}\sim -A^N$.

Theorem 6.3. *Suppose the Carathéodory function $a : \Sigma\times\mathbb{R}^d \to \mathbb{R}^d$ satisfies growth condition (6.7), A^N_ϕ is the operator given by (6.15), and $\overline{u} := \frac{1}{\mu(\Sigma)}\int_\Sigma u\,\mathrm{d}x$ for any $u\in L^1(\Sigma)$. Then the following statements hold:*

(1) If a satisfies the strong monotonicity condition (6.10), $a(x,0) = 0$ for a.e. $x\in\Sigma$, and the monotone graph β satisfies (6.23), then the semigroup $\{T_t\}_{t\ge0}\sim -A^N_\phi$ on $L^2(\Sigma)$ satisfies the regularisation estimates (1.13) and (1.14) with the same exponents as the semigroup generated by $-A^D$.

(2) If a satisfies (6.6)-(6.8), and the monotone graph β satisfies (6.24), then the semigroup $\{T_t\}_{t\ge0}\sim -A^N_\phi$ on $L^2(\Sigma)$ satisfies the regularisation estimates (1.17) and (1.18) with the same exponents as the semigroup generated by $-A^D$. Moreover, the semigroup $\{T_t\}_{t\ge0}\sim -B^N$ on $L^2(\Sigma)$ satisfies

$$\|T_t u - \overline{u}\|_\infty \lesssim t^{-\alpha_s}\; e^{\omega\beta_s t}\,\|u-\overline{u}\|_s^{\gamma_s} \tag{6.27}$$

for every $t>0$, $u\in L^s(\Sigma)$ for $1\le s<\infty$ and exponents α_s, β_s and γ_s as given in Theorem 6.1 for the semigroup generated by $-A^D$.

For the proof of this theorem, Lemma 6.2 provides the crucial estimates.

Proof of Theorem 6.3. First, let $1<p<d$. Then by Lemma 6.2, there is a constant $C>0$ such that

$$\|u\|_{\frac{pd}{d-p}} \le C\left(\||\nabla u|\|_p + \|u\|_p\right) \tag{6.28}$$

for every $u \in W^1_{p,p}(\Sigma)$. Taking pth power on both sides of this inequality, using that for $q > 1$,

$$(a+b)^q \le 2^{q-1}(a^q + b^q) \qquad \text{for every } a, b \ge 0, \tag{6.29}$$

by definition of the operator B^N and by (6.10) and (6.23), we see that

$$\begin{aligned} \|u-\hat{u}\|^p_{\frac{pd}{d-p}} &\le C\,\left(\||\nabla(u-\hat{u})|\|_p^p + \|u-\hat{u}\|_p^p\right) \\ &\le C\,\tilde{\eta}^{-1}\int_\Sigma (a(x,\nabla u) - a(x,\nabla\hat{u}))\nabla(u-\hat{u})\,dx \\ &\qquad\qquad + C\,\eta_0^{-1}\int_\Sigma (v-\hat{v})(u-\hat{u})\,dx \\ &= C\,\max\{\tilde{\eta}^{-1},\eta_0^{-1}\}\,\langle u-\hat{u}, B^N u + v - (B^N\hat{u}+\hat{v})\rangle \end{aligned}$$

for every u, $\hat{u} \in D(B^N)$, $v \in \beta_2(u)$, $\hat{v} \in \beta_2(\hat{u})$. Now, Remark 3.5 yields that the operator A^N given by (6.26) satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters (6.20). Thus, Theorem 1.2 yields the first statement of this theorem for $1 < p < d$.

Next, let $p = d \ge 2$. Then by Lemma 6.2, there is a constant $C > 0$ such that

$$\|u\|_{\frac{2}{1-\theta}} \le C\left(\||\nabla u|\|_p^\theta\,\|u\|_2^{1-\theta} + \|u\|_{\tilde{q}}\right)$$

for every $u \in \dot{W}^1_{p,2}(\Sigma) \cap L^{\tilde{q}}(\Sigma)$, $0 \le \theta < 1$, and $\tilde{q} > 0$. Thus, if $p = d = 2$, we choose $\tilde{q} = 2$ and use that $\|u\|_2 = \|u\|_2^\theta\,\|u\|_2^{1-\theta}$ for every $0 \le \theta < 1$, and if $p = d > 2$ then we choose $\tilde{q}_\theta$ by $\frac{1}{\tilde{q}_\theta} = \frac{\theta}{p} + \frac{1-\theta}{2}$ for any given $0 < \theta < 1$ and apply Hölder's inequality. Then, in both cases, we obtain

$$\|u\|_{\frac{2}{1-\theta}} \le C\left(\||\nabla u|\|_p^\theta + \|u\|_p^\theta\right)\,\|u\|_2^{1-\theta} \tag{6.30}$$

for every $u \in W^1_{p,2}(\Sigma)$. Thus, by definition of the operator B^N, (6.10) and (6.23),

$$\begin{aligned} \|u-\hat{u}\|^{\frac{p}{\theta}}_{\frac{2}{1-\theta}} &\le C^{\frac{p}{\theta}}\,2^{\frac{p}{\theta}-1}\,\left(\||\nabla(u-\hat{u})|\|_p^p + \|u-\hat{u}\|_p^p\right)\,\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}} \\ &\le C^{\frac{p}{\theta}}\,2^{\frac{p}{\theta}-1}\left(\eta^{-1}\,\int_\Sigma (a(x,\nabla u) - a(x,\nabla\hat{u}))\nabla(u-\hat{u})\,dx\right. \\ &\qquad\qquad \left. + \tilde{\eta}_0^{-1}\,\int_\Sigma (v-\hat{v})(u-\hat{u})\,dx\right)\,\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}} \\ &\le C^{\frac{p}{\theta}}2^{\frac{p}{\theta}-1}\max\{\tilde{\eta}^{-1},\eta_0^{-1}\}\langle u-\hat{u}, B^N u + v - (B^N\hat{u}+\hat{v})\rangle\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}} \end{aligned}$$

By Remark 3.5, A^N satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters (6.21) hence, Theorem 1.2 yields the first statement of this theorem for $p = d$.

Now, let $d < p < \infty$. Then, there is an $0 < \theta_0 < 1$ such that $\theta_0(\frac{1}{p} - \frac{1}{d}) + (1-\theta_0)\frac{1}{2} = 0$ or equivalently, $\theta_0 = \frac{pd}{pd+2(p-d)}$. We apply Lemma 6.2 for θ_0, $\tilde{q}$ given by $\frac{1}{\tilde{q}} = \frac{\theta_0}{p} + \frac{1-\theta_0}{2}$ and apply Hölder's inequality. Then,

$$\|u\|_\infty \le C\left(\||\nabla u|\|_p^{\theta_0} + \|u\|_p^{\theta_0}\right)\,\|u\|_2^{1-\theta_0} \tag{6.31}$$

for every $u \in \dot{W}^1_{p,2}(\Sigma)$ and some constant $C > 0$. Proceeding as in the first step of this proof, we see that by (6.10), (6.23) and Remark 3.5, the operator A^N satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters (6.22). Therefore, by Theorem 4.1, the first statement of this theorem holds for $p > d$.

Under the assumption that merely the hypotheses (6.6)-(6.8) are satisfied, one proceeds as in the previous three steps of this proof and applies Theorem 1.4 with $u_0 = 0$. Thus, the second statement of this theorem holds.

To see that also the last claim of this theorem holds, one combines the Gagliardo-Nirenberg inequality (6.18) with the Poincaré inequality

$$\|u - \overline{u}\|_p \le C_N \, \||\nabla u|\|_p, \tag{6.32}$$

which holds for all $u \in W^1_{p,p}(\Sigma)$ and some constant $C_N > 0$ independent of u. Then, for $1 < p < d$, inequality (6.28) reduces to

$$\|u - \overline{u}\|_{\frac{pd}{d-p}} \le C\||\nabla u|\|_p \tag{6.33}$$

for every $u \in W^1_{p,p}(\Sigma)$, if $p = d \geq 2$ then inequality (6.30) reduces to

$$\|u - \overline{u}\|_{\frac{2}{1-\theta}} \le C_\theta \, \||\nabla u|\|_p^{\theta} \, \|u - \overline{u}\|_2^{1-\theta} \tag{6.34}$$

for every $u \in W^1_{p,2}(\Sigma)$ and $0 < \theta < 1$, and if $d < p < \infty$ then inequality (6.31) reduces to

$$\|u - \overline{u}\|_\infty \le C \, \||\nabla u|\|_p^{\theta_0} \, \|u - \overline{u}\|_2^{1-\theta_0}$$

for every $u \in W^1_{p,2}(\Sigma)$, where $\theta_0 = \frac{pd}{pd+2(p-d)}$ and the constant C can differ from line to line. Now, proceeding as in the first three steps of this proof and using the inequalities (6.32)-(6.34) instead of (6.28), (6.30) and (6.31), and noting that for every $u \in L^2(\Sigma)$, the element $(\overline{u}, 0) \in B^N$, then one obtains that for all $1 < p < \infty$, the operator B^N satisfies the Gagliardo-Nirenberg inequality (3.7) for the same exponents as found in the first three steps of this proof. Thus Theorem 1.4 yields the third statement of this theorem. □

6.1.3. *Homogeneous Robin boundary conditions.* In this subsection, we assume that Σ is a bounded domain with a Lipschitz boundary. Then the mapping $u \mapsto u_{|\partial\Sigma}$ from $C^{0,1}(\overline{\Sigma})$ to $C^{0,1}(\partial\Omega)$ has a unique continuous and surjective extension

$$\mathrm{Tr} : W^1_{p,p}(\Sigma) \to W^{1-1/p,p}(\partial\Sigma)$$

called *trace operator* (cf. [72, Théorème 4.2, 4.6, and Section 3.8]). For convenience, we write $u_{|\partial\Omega} := \mathrm{Tr}(u)$ for $u \in W^1_{p,p}(\Sigma)$ even if u does not belong to $C(\overline{\Sigma})$ and call $u_{|\partial\Omega}$ the *trace* of u. Thus, if θ denotes the boundary operator given by (6.12) then $\langle\theta(u), u\rangle$ is finite for every $u \in W^1_{p,p}(\Sigma)$ hence, under the assumptions of this section, we can define the realisation B^R in $L^2(\Sigma)$ of the Leray-Lions operator $\mathcal{B}$ equipped with homogeneous Robin boundary conditions (6.5) by

$$\begin{aligned} B^R = \Big\{(u,v) \in L^2 \times L^2(\Sigma) \,\Big|\, & u \in W^1_{p,p}(\Sigma) \text{ s.t. for all } \xi \in W^1_{p,p} \cap L^2(\Sigma) \\ & \int_\Sigma a(x,\nabla u)\nabla\xi \mathrm{d}x + \int_{\partial\Sigma} b|u|^{p-2}u\xi \mathrm{d}\mathcal{H} + d\langle\theta(u),\xi\rangle = \int_\Sigma v\xi \mathrm{d}x\Big\}. \end{aligned} \tag{6.35}$$

We call B^R the *Robin Leray-Lions operator* in $L^2(\Sigma)$.

Since $C^\infty(\overline{\Sigma})$ is contained in $W^1_{p,p}\cap L^2(\Sigma)$ and dense in $L^2(\Sigma)$, B^R defines a single-valued and densely defined operator on $L^2(\Sigma)$. To see that B^R is completely accretive, let $T\in P_0$. Then by definition of B^R, by (6.8) and since T is monotonically increasing and Lipschitz continuous on $\mathbb{R}$, and since $s\mapsto |s|^{p-2}s$ is monotonically increasing, we have that

$$\begin{aligned}
&\int_\Sigma T(u-\hat{u})(B^Ru-B^R\hat{u})\mathrm{d}x\\
&=\int_\Sigma \left(a(x,\nabla u)-a(x,\nabla\hat{u})\right)\nabla(u-\hat{u})T'(u-\hat{u})\mathrm{d}x\\
&\quad+\int_{\partial\Sigma} b(x)(|u|^{p-2}u-|\hat{u}|^{p-2}\hat{u})T(u-\hat{u})\mathrm{d}\mathcal{H}\\
&\quad+d\int_{\partial\Sigma}\int_{\partial\Sigma}\tfrac{|u(x)-u(y)|^{p-2}(u(x)-u(y))-|\hat{u}(x)-\hat{u}(y)|^{p-2}(\hat{u}(x)-\hat{u}(y))}{|x-y|^{d+p-2}}\times\\
&\qquad\qquad\times T((u(x)-u(y))-(\hat{u}(x)-\hat{u}(y)))\,\mathrm{d}\mathcal{H}(x)\mathrm{d}\mathcal{H}(y)\\
&\ge 0
\end{aligned}$$

for every $u,\hat{u}\in D(B^R)$. Thus, B^R is completely accretive. Since for every $\lambda>0$, the Yosida operator $\beta_\lambda:\mathbb{R}\to\mathbb{R}$ of β is monotonically increasing and Lipschitz continuous and since Σ is bounded, we may replace T by β_λ in the previous calculation, showing that the m-accretive graph β on $\mathbb{R}$ satisfies condition (2.28) in Proposition 2.5.

In order to see that B^R satisfies the range condition (2.14) for $X=L^2(\Sigma)$, we employ the following p-variant of Maz'ya's inequality

$$\|u\|_{\frac{pd}{d-1}}\le C\left(\||\nabla u|\|_p+\|u_{|\partial\Sigma}\|_p\right) \tag{6.36}$$

holding for all $u\in W^1_{p,p}(\Sigma)$ provided $1\le p<\infty$. Here the constant $C>0$ depends on p, the volume $|\Sigma|$, and the isoperimetric constant $C(d)$ (cf. [67, Cor. 3.6.3] and see also [55, Section 2.1]). Now, let $V=W^1_{p,p}(\Sigma)\cap L^2(\Sigma)$ be equipped with the sum norm. Then by (6.6), since for every $u\in V$, $\langle\theta(u),u\rangle\ge 0$, since $b(x)\ge b_0>0$ a.e. on $\partial\Sigma$ and by (6.36), we obtain that

$$\begin{aligned}
\langle (I+B^R)u,u\rangle &\ge \|u\|_2^2+\tfrac{\eta}{2}\||\nabla u|\|_p^p+C_1\|u\|_p^p\\
&\ge C_2\left(\|u\|_2^2+\||\nabla u|\|_p^p+\|u\|_p^p\right)
\end{aligned}$$

for every $u\in V$. Thus, the restriction of the operator $I+B^R$ on V satisfies condition (6.14) and so, the operator $I+B^R:V\to V'$ is surjective by [63, Théorème 1], proving that B^R satisfies the range condition (2.14) in $L^2(\Sigma)$.

Therefore, by Proposition 2.5, the operator

$$A^R:=B^R+\beta_2+F \tag{6.37}$$

is quasi m-completely accretive in $L^2(\Sigma)$ with dense domain and so by the Crandall-Liggett theorem, $-A^R$ generates a strongly continuous semigroup $\{T_t\}_{t\ge0}$ on $L^2(\Sigma)$ of Lipschitz continuous mappings T_t on $L^2(\Sigma)$, and each mapping T_t admits a unique Lipschitz continuous extension on $L^q(\Sigma)$ with constant $e^{\omega t}$ for every $1\le q\le\infty$.

Here, we state the complete description of the L^q-L^r-regularisation effect of the semigroup $\{T_t\}_{t\ge0}\sim -A^R$.

Theorem 6.4. *Suppose the Carathéodory function $a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies growth conditions (6.7). Further, suppose $b \in L^\infty(\partial\Sigma)$ such that $b(x) \geq b_0 > 0$ a.e. on $\partial\Sigma$, $d \geq 0$, and A^R is the operator given by (6.37). Then the following statements hold:*

(1) *If a satisfies the strong monotonicity condition (6.10) and $a(x, 0) = 0$ for a.e. $x \in \Sigma$, then the semigroup $\{T_t\}_{t\geq 0} \sim -A^R$ on $L^2(\Sigma)$ satisfies the regularisation estimates (1.13) and (1.14) with the same exponents as the semigroup generated by $-A^D$.*

(2) *If a satisfies (6.6)-(6.8), then the semigroup $\{T_t\}_{t\geq 0} \sim -A^R$ on $L^2(\Sigma)$ satisfies the regularisation estimates (1.17) and (1.18) with the same exponents as the semigroup generated by $-A^D$.*

Proof of Theorem 6.4. Note that, for every $q > 1$, there is a constant $C_q > 0$ such that

$$(|s|^{q-2}s - |t|^{q-2}t)(s - t) \geq C_q|s - t|^q \tag{6.38}$$

for all $s, t \in \mathbb{R}$ (cf. [29, Appendix]). Due to inequality (6.38), we can show that the semigroup $\{T_t\}_{t\geq 0} \sim -A^R$ satisfies inequality (1.14) provided the Carathéodory function a satisfies (6.10).

First, let $1 < p < d$. By Lemma 6.2, we have that inequality (6.28) holds. Applying Maz'ya's inequality (6.36) to estimate the term $\|u\|_p$ in (6.28) gives

$$\|u\|_{\frac{pd}{d-p}} \leq C\left(\||\nabla u|\|_p + \|u_{|\partial\Sigma}\|_p\right) \tag{6.39}$$

for every $u \in W^1_{p,p}(\Sigma)$, where the constant C can be different from the one in (6.28). Taking pth power on both sides of the last inequality, applying (6.29) and using the definition of the operator B^R combined with (6.10) and (6.38) shows that

$$\begin{aligned}
\|u - \hat{u}\|^p_{\frac{pd}{d-p}} &\leq C\left(\||\nabla(u - \hat{u})|\|^p_p + \|u_{|\partial\Sigma} - \hat{u}_{|\partial\Sigma}\|^p_p\right)\\
&\leq C\,\tilde{\eta}^{-1}\int_\Sigma (a(x, \nabla u) - a(x, \nabla\hat{u}))\nabla(u - \hat{u})\,\mathrm{d}x\\
&\qquad + C\,b_0^{-1}C_p^{-1}\int_{\partial\Sigma} b(x)\,(|u|^{p-2}u - |\hat{u}|^{p-2}\hat{u})(u - \hat{u})\,\mathrm{d}\mathcal{H}\\
&\leq C\,\max\{\tilde{\eta}^{-1}, (b_0C_p)^{-1}\}\,\langle u - \hat{u}, B^Ru - B^R\hat{u}\rangle
\end{aligned}$$

for every $u, \hat{u} \in D(B^R)$. Thus, Remark 3.5 yields the operator A^R given by (6.37) satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters (6.20). By Theorem 1.2, the first statement of this theorem holds for $1 < p < \infty$.

If $p = d \geq 2$, then applying Maz'ya's inequality (6.36) to (6.30) yields

$$\|u\|_{\frac{2}{1-\theta}} \leq C\left(\||\nabla u|\|^\theta_p + \|u_{|\partial\Sigma}\|^\theta_p\right)\,\|u\|_2^{1-\theta}$$

for every $u \in W^1_{2,p}(\Sigma)$ and $0 < \theta < 1$. Thus by (6.29), the definition of B^R, (6.10) and by inequality (6.38) for $q = p$ shows that

$$\begin{aligned}
&\|u - \hat{u}\|^{\frac{p}{\theta}}_{\frac{2}{1-\theta}}\\
&\leq C^{\frac{p}{\theta}}\,2^{\frac{p}{\theta}-1}\,\left(\||\nabla(u - \hat{u})|\|^p_p + \|u - \hat{u}\|^p_p\right)\,\|u - \hat{u}\|_2^{\frac{p(1-\theta)}{\theta}}\\
&\leq C^{\frac{p}{\theta}}\,2^{\frac{p}{\theta}-1}\left(\eta^{-1}\int_\Sigma (a(x, \nabla u) - a(x, \nabla\hat{u}))\nabla(u - \hat{u})\,\mathrm{d}x\right.
\end{aligned}$$

$$+(a_0\,C_p)^{-1}\int_{\partial\Sigma} a\,(|u|^{p-2}u-|\hat{u}|^{p-2}\hat{u})(u-\hat{u})\,\mathrm{d}\mathcal{H}\Big)\;\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}}$$

$$\leq C^{\frac{p}{\theta}}2^{\frac{p}{\theta}-1}\max\{\tilde{\eta}^{-1},(a_0\,C_p)^{-1}\}\langle u-\hat{u},B^Ru-B^R\hat{u}\rangle\|u-\hat{u}\|_2^{\frac{p(1-\theta)}{\theta}}$$

By Remark 3.5, the operator A^R satisfies the Gagliardo-Nirenberg inequality (3.2) with

$$r_\theta=\frac{2}{1-\theta},\quad \sigma_\theta=\frac{p}{\theta},\quad \varrho_\theta=\frac{p(1-\theta)}{\theta},\quad \omega=L\qquad\text{for every } 0<\theta<1.$$

Therefore, by Theorem 1.2, the first statement of this theorem holds for $p=d$.

Next, let $d<p<\infty$. Applying Maz'ya's inequality (6.36) to (6.31) with $\theta_0=\frac{pd}{pd+2(p-d)}$ and subsequently taking pth power and employing inequality (6.29) gives

$$\|u\|_\infty\leq C\left(\||\nabla u|\|_p^{\theta_0}+\|u_{|\partial\Sigma}\|_p^{\theta_0}\right)\;\|u\|_2^{1-\theta_0}$$

for every $u\in\dot{W}^1_{p,2}(\Sigma)$. Proceeding as above, we see that by (6.10), (6.38) and by Remark 3.5, the operator A^R satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters r, σ, ϱ and ω as given in (6.22). By Theorem 4.1, the first statement of this theorem holds for $p>d$.

Under the assumption that merely the hypotheses (6.6)-(6.8) are satisfied, one proceeds as in the previous threes steps of this proof and applies Theorem 1.4 with $u_0=0$. Thus, the second statement of this theorem holds as well, completing the proof. ☐

6.2. **Parabolic problems involving nonlocal operators.** In the following two subsections, we outline two examples currently attracting much interest. We begin in Subsection 6.2.1 by establishing the L^q-L^r-regularisation estimates for the *semigroup generated by the Dirichlet-to-Neumann operator associated with a Leray-Lions operator* (cf, for instance, [55] and the references therein). Subsection 6.2.2 is dedicated to the L^q-L^r-regularisation estimates for the *semigroup generated by the fractional p-Laplace operator* equipped with either homogeneous Dirichlet or Neumann boundary conditions (cf, for instance, [16, 68]). One can easily see in both examples that the standard construction of a *one-parameter family of Sobolev type inequalities* fails. Recall, this is an important intermediate step in the known literature to achieve an L^q-L^∞-regularisation estimates for $1\leq q<\infty$ of the semigroup (cf Section 1.1).

For instance, consider the example of the semigroup generated by the Dirichlet-to-Neumann operator associated with a Leray-Lions operator (6.9) satisfying the hypotheses (6.6)- (6.8). The construction of this Dirichlet-to-Neumann operator proceeds in two steps. First, one needs to know the solvability of *Dirichlet problem*

$$\begin{cases} -\operatorname{div}(a(x,\nabla u))=0 & \text{in }\Sigma,\\ u=\varphi & \text{on }\partial\Sigma\end{cases}\tag{6.40}$$

for every boundary function $\varphi\in W^{1-1/p,p}(\partial\Sigma)$. For given boundary-value φ, let $P\varphi:=u$ be the unique weak energy solution u of (6.40). Then, in order to construct a one-parameter family of Sobolev type inequalities, one needs that

$$P(|\varphi|^{q-p}\varphi)=|P\varphi|^{q-p}P\varphi\qquad\text{for every } q\geq p>1.$$

However, this does not hold in general. Thus, our next example demonstrates the strength of Theorem 1.2 and Theorem 1.4.

6.2.1. *The Dirichlet-to-Neumann operators associated with Leray-Lions operators.* In this subsection, we suppose that Σ is either the half space $\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times (0,\infty)$ or a bounded domain with a Lipschitz boundary.

We begin by outlining the construction of the Dirichlet-to-Neumann operator in the case Σ is a bounded domain with a Lipschitz continuous boundary. The construction of the operator on the half space $\Sigma = \mathbb{R}^d_+$ proceeds similarly (see also Remark 6.5 below). Under this assumption on Σ, the trace operator $\mathrm{Tr} : W^1_{p,p}(\Sigma) \to W^{1-1/p,p}(\partial\Sigma)$ has a linear bounded right inverse

$$Z : W^{1-1/p,p}(\partial\Sigma) \to W^{1,p}(\Sigma)$$

(cf. [72, Théorème 5.7]) and the kernel of Tr coincides with $\dot{W}^1_{p,p}(\Sigma)$. If the Carathéodory function $a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies (6.6)-(6.8), then by the classical theory of monotone operators ([63, Théorème 1]), we have that for every given boundary value $\varphi \in W^{1-1/p,p}(\partial\Omega)$, the Dirichlet problem (6.40) admits a unique *weak* solution $u \in W^1_{p,p}(\Sigma)$ in the following sense: for given boundary value $\varphi \in W^{1-1/p,p}(\partial\Sigma)$, a function $u \in W^1_{p,p}(\Sigma)$ is a *weak energy solution* of Dirichlet problem (6.40) on Σ if $u - Z\varphi \in \dot{W}^1_{p,p}(\Sigma)$ and

$$\int_\Sigma a(x,\nabla u)\nabla\xi \,\mathrm{d}x = 0$$

for all $\xi \in \dot{W}^1_{p,p}(\Sigma)$. Let $P : W^{1-1/p,p}(\partial\Sigma) \to W^1_{p,p}(\Sigma)$ be the mapping which assigns to each boundary value $\varphi \in W^{1-1/p,p}(\partial\Sigma)$ the unique weak energy solution $u \in W^1_{p,p}(\Sigma)$ of (6.40). Then P is injective and continuous. Furthermore, for every $\varphi \in W^{1-1/p,p}(\partial\Sigma)$ and $\Phi \in W^1_{p,p}(\Sigma)$ satisfying $\Phi_{|\partial\Sigma} = \varphi$, there is a unique $u_\Phi \in \dot{W}^1_{p,p}(\Sigma)$ such that

$$P\varphi = u_\Phi + \Phi \tag{6.41}$$

(cf. [55, Lemma 2.5]).

The Dirichlet-to-Neumann operator associated with the operator $\mathcal{B}$ defined in (6.9) assigns to each Dirichlet boundary data φ the corresponding co-normal derivative $a(x,\nabla P\varphi)\cdot\nu =: \Lambda\varphi$ on $\partial\Sigma$.

If $P\varphi$ and $a(\cdot,\nabla P\varphi)$ are smooth enough up to the boundary $\partial\Sigma$, Green's formula yields

$$\int_{\partial\Sigma} \Lambda\varphi\,\xi\,\mathrm{d}\mathcal{H} = \int_\Sigma a(x,\nabla P\varphi)\nabla\xi\,\mathrm{d}x$$

for every $\xi \in C^\infty(\overline{\Sigma})$ and if $\Lambda\varphi \in L^{p'}(\partial\Sigma)$, then an approximation argument shows that

$$\int_{\partial\Sigma} \Lambda\varphi\,\xi\,\mathrm{d}\mathcal{H} = \int_\Sigma a(x,\nabla P\varphi)\nabla Z\xi\,\mathrm{d}x$$

for every $\xi \in W^{1-1/p,p}(\partial\Sigma)$. Even if φ and ξ merely belong to $W^{1-1/p,p}(\partial\Sigma)$, the integral on the right-hand side of this equation exists. Thus, we can use this integral to define the operator Λ for the more general class of functions

$W^{1-1/p,p}(\partial\Sigma)$. By linearity of Z and by using Hölder's inequality together with growth condition (6.7), one easily sees that the functional

$$\psi \mapsto \int_\Omega a(x,\nabla P\varphi)\nabla Z\psi\,\mathrm{d}x$$

belongs to the dual space $W^{-(1-1/p),p'}(\partial\Omega)$. This justifies to define the *Dirichlet-to-Neumann operator associated with the quasi-linear operator* $\mathcal{B}$ as the operator $\Lambda : W^{1-1/p,p}(\partial\Sigma) \to W^{-(1-1/p),p'}(\partial\Sigma)$ defined by

$$\langle \Lambda\varphi, \xi\rangle = \int_\Sigma a(x,\nabla P\varphi)\nabla Z\xi\,\mathrm{d}x$$

for every $\varphi, \xi \in W^{1-1/p,p}(\partial\Sigma)$. The Dirichlet-to-Neumann operator Λ realised as an operator on $L^2(\partial\Sigma)$ is given by the restriction $\Lambda_2 := \Lambda \cap (L^2(\partial\Sigma)) \times L^2(\partial\Sigma))$. In fact, one can show (cf. [55, Proposition 3.9]) that

$$\begin{aligned}\Lambda_2 = \Big\{(\varphi,\psi) \in L^2(\partial\Sigma)\times L^2(\partial\Sigma)\;\Big|\;&\varphi \in W^{1-1/p,p}(\partial\Sigma)\ \text{ such that}\\ &\int_\Sigma a(x,\nabla P\varphi)\nabla Z\xi\mathrm{d}x = \int_\Sigma \psi\,\xi\mathrm{d}x\ \text{ for all }\ \xi \in W^{1-1/p,p}(\partial\Sigma)\cap L^2(\partial\Sigma)\Big\}.\end{aligned}$$

It is well-known (cf. [55, Proposition 3.9] or [4]), that Λ_2 is completely accretive. To see that Λ_2 satisfies the range condition (2.14) for $X = L^2(\partial\Sigma)$, we take $V = W^{1-1/p,p}(\partial\Sigma)\cap L^2(\partial\Sigma)$ equipped with the sum norm. Then, by (6.41),

$$\langle \psi,\varphi\rangle_{V',V} = \int_\Sigma a(x,\nabla P\varphi)\nabla P\varphi\,\mathrm{d}x$$

for every $(\varphi,\psi) \in \Lambda_2$. By using Maz'ya's inequality (6.36) and Poincaré's inequality on $W^1_{p,p}(\Sigma)$, one can deduce the following useful inequality

$$\|u\|_p \le \tilde{C}\Big(\||\nabla u|\|_p + \|u_{|\partial\Sigma}\|_{L^2(\partial\Sigma)}\Big) \tag{6.42}$$

holding for all $u \in W^1_{p,p}(\Sigma)$ with trace $u_{|\partial\Omega} \in L^2(\partial\Omega)$ (cf. [55, Section 2]). Now, let $\alpha \in \mathbb{R}$ and $\varphi \in V$. Then, by using (6.6), the boundedness of the trace operator Tr and inequality (6.42), we see that

$$\begin{aligned}\|\varphi_{|\partial\Sigma}\|_2^2 + \eta\||\nabla P\varphi|\|_p^p &\le \langle (I+\Lambda_2)\varphi,\varphi\rangle_{V',V}\\ &\le \alpha\, C(\|\varphi_{|\partial\Sigma}\| + \|P\varphi\|_p + \||\nabla P\varphi|\|_p)\\ &\le \alpha\,\tilde{C}(\|\varphi_{|\partial\Sigma}\| + \||\nabla P\varphi|\|_p)\end{aligned}$$

Thus, the restriction of the operator $I+\Lambda_2$ on V satisfies condition (6.14) hence $I+\Lambda_2 : V \to V'$ is surjective by [63, Théorème 1], proving that Λ_2 satisfies the range condition (2.14) in $X = L^2(\partial\Sigma)$.

By hypothesis on the m-accretive graph β on $\mathbb{R}$, the domain $D(\beta_2)$ of the associated accretive operator β_2 in $L^2(\partial\Sigma)$ contains the set $\{v_{|\partial\Sigma}\,|\,v \in C^\infty(\overline{\Sigma})\}$. Thus, the domain $D(\beta_2)$ is dense in $L^2(\partial\Sigma)$ (cf. [55, Lemma 2.1 2]). For every $\lambda > 0$, the Yosida operator β_λ of β is Lipschitz continuous, $\beta_\lambda(0) = 0$, and the Yosida operator $\beta_{2,\lambda}$ of the operator β_2 is given by $(\beta_{2,\lambda}\varphi)(x) = \beta_\lambda(\varphi(x))$ for a.e. $x \in \partial\Sigma$ and every $\varphi \in L^2(\partial\Sigma)$. Therefore, $\beta_{2,\lambda}(\varphi) \in W^{1-1/p,p}(\partial\Sigma)\cap L^2(\partial\Sigma)$ for every $\varphi \in W^{1-1/p,p}(\partial\Sigma)\cap L^2(\partial\Sigma)$. Moreover, by (6.41), there is a unique $u_\Phi \in \dot{W}^1_{p,p}(\Sigma)$

such that $P(\beta_\lambda(\varphi)) = u_\Phi + \beta_\lambda(P\varphi)$ for $\Phi = \beta_\lambda(P\varphi)$. Combining this with the definition of Λ_2, (6.6), and the fact that $\beta'_\lambda \geq 0$, we see that

$$[\psi, \beta_\lambda(\varphi)]_2 = \int_\Sigma a(x, \nabla P\varphi)\nabla\beta_\lambda(P\varphi)\,\mathrm{d}x \geq \eta \int_\Sigma |\nabla P\varphi|^p \beta'_\lambda(P\varphi)\,\mathrm{d}x \geq 0$$

for every $(\varphi, \psi) \in \Lambda_2$. Therefore, by Proposition 2.5, the operator

$$A^\Lambda := \Lambda_2 + \beta_2 + F \tag{6.43}$$

is quasi m-completely accretive in $L^2(\partial\Sigma)$ with dense domain.

By the Crandall-Liggett theorem [40], $-A^\Lambda$ generates a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on $L^2(\partial\Sigma)$ of Lipschitz continuous mappings T_t, which admits a unique Lipschitz continuous extension on $L^q(\partial\Sigma)$ with constant $e^{\omega t}$ for all $1 \leq q \leq \infty$.

Remark 6.5. In the case Σ is the half space $\mathbb{R}^d_+$, the construction is of Λ_2 is exactly the same. But one needs to replace the space $W^1_{p,p}(\Sigma)$ by the space $D^{1,p}(\mathbb{R}^d_+)$ which is the completion of the space of all $u \in C_c^\infty(\overline{\mathbb{R}^d_+})$ with respect to $\||\nabla u|\|_p$ and the space $W^{1-1/p,p}(\mathbb{R}^{d-1})$ needs to be replaced by the completion of the space of all $\varphi \in C_c^\infty(\mathbb{R}^{d-1})$ with respect to $|\varphi|_p$. We leave the details to the interested reader.

Here is the complete description of the L^q-L^∞-regularisation effect of the semigroup $\{T_t\}_{t\geq 0} \sim -A^\Lambda$.

Theorem 6.6. *Suppose the Carathéodory function $a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies growth conditions (6.7) and A^Λ be the operator given by (6.43). Then the following statements are true.*

(1) *Suppose Σ is a bounded domain with a Lipschitz boundary, a satisfies (6.10) with $a(x, 0) = 0$ for a.e. $x \in \Sigma$, and the monotone graph β satisfies (6.23). Then*

(i) *for $1 < p < d$, the semigroup $\{T_t\}_{t\geq 0} \sim -A^\Lambda$ on $L^2(\partial\Sigma)$ satisfies estimate (1.14) with exponents*

$$\alpha_s = \frac{\alpha^*}{1-\gamma^*\Big(1-\frac{s(d-p)}{(d-1)m_0}\Big)},\ \beta_s = \frac{\frac{\beta^*}{2}+\gamma^*\frac{s(d-p)}{(d-1)m_0}}{1-\gamma^*\Big(1-\frac{s(d-p)}{(d-1)m_0}\Big)},\ \gamma_s = \frac{\gamma^*\frac{s(d-p)}{(d-1)m_0}}{1-\gamma^*\Big(1-\frac{s(d-p)}{(d-1)m_0}\Big)},$$
$$\alpha^* = \frac{d-p}{(p-1)m_0+(d-p)(p-2)}, \qquad \beta^* = \frac{(\frac{2}{p}-1)d+p-\frac{2}{p}}{(p-1)m_0+(d-p)(p-2)}+1,$$
$$\gamma^* = \frac{(p-1)m_0}{(p-1)m_0+(d-p)(p-2)}.$$

for $m_0 \geq p$ satisfying $(\frac{d-1}{d-p}-1)m_0 + p - 2 > 0$, and for every $1 \leq s \leq \frac{(d-1)m_0}{d-p}$ satisfying $s > \frac{(2-p)(d-1)}{p-1}$. Moreover, if $\frac{2d}{d+1} < p < d$ then one can take $m_0 = p$ and if $\frac{2d-1}{d} < p < d$, then estimate (1.14) holds with the same exponents for every $1 \leq s \leq \frac{(d-1)p}{d-p}$.

(ii) *for $p = d \geq 2$, the semigroup $\{T_t\}_{t\geq 0} \sim -A^\Lambda$ on $L^2(\partial\Sigma)$ satisfies estimate (1.14) with exponents*

$$\alpha_s = \frac{\alpha^*_\theta}{1-\gamma^*_\theta(1-s(1-\theta))}, \quad \beta_s = \frac{\frac{\beta^*_\theta}{2}+\gamma^*_\theta s(1-\theta)}{1-\gamma^*_\theta(1-s(1-\theta))}, \quad \gamma_s = \frac{\gamma^*_\theta s(1-\theta)}{1-\gamma^*_\theta(1-s(1-\theta))},$$
$$\alpha^*_\theta = \frac{1}{\frac{1}{1-\theta}-2},\ \beta^*_\theta = \frac{\frac{2}{p^2}\frac{1}{1-\theta}-1}{\frac{1}{1-\theta}-2}+1,\ \gamma^*_\theta = \frac{\frac{1}{1-\theta}-p}{\frac{1}{1-\theta}-2},$$

for every $1 - \frac{1}{p} < \theta < 1$ *and* $1 \le s \le \frac{1}{1-\theta}$.

(iii) *for* $d < p < \infty$, *the semigroup* $\{T_t\}_{t\ge 0} \sim -A^{\Lambda}$ *on* $L^2(\partial\Sigma)$ *satisfies estimate* (1.14) *with exponents*

$$\alpha_s = \tfrac{1}{p-2+s}, \quad \beta_s = \tfrac{\frac{2+p}{2}+s}{p-2+s}, \quad \gamma_s = \tfrac{s}{p-2+s}$$

for every $1 \le s \le 2$.

(2) *Suppose* Σ *is a bounded domain with a Lipschitz boundary, a satisfies* (6.6)-(6.8) *and* β *satisfies* (6.24). *Then the following holds:*

(i) *The semigroup* $\{T_t\}_{t\ge 0} \sim -A^{\Lambda}$ *satisfies estimate* (1.18) *with* $u_0 = 0$ *for the same exponents as given in the statements* (1i)-(1iii).

(ii) *The semigroup* $\{T_t\}_{t\ge 0} \sim -\Lambda_2$ *satisfies*

$$\|T_t\varphi_{|\partial\Sigma} - \overline{\varphi_{|\partial\Sigma}}\|_\infty \lesssim e^{\omega\beta_q t}\, t^{-\delta_q}\, \|\varphi_{|\partial\Sigma} - \overline{\varphi_{|\partial\Sigma}}\|_q^{\gamma_q}$$

for every $t > 0$ *and* $\varphi \in L^q(\partial\Sigma)$, *where* $\overline{\varphi_{|\partial\Sigma}} := \frac{1}{\mathcal{H}(\partial\Sigma)}\int_{\partial\Sigma}\varphi\,\mathrm{d}\mathcal{H}$ *and the exponents* α_s, β_s *and* γ_s *are the same as given in the statements* (1i)-(1iii).

(3) *Suppose* Σ *is the half space* $\mathbb{R}^d_+$.

(vi) *If a satisfies* (6.10) *with* $a(x,0) = 0$ *for a.e.* $x \in \Sigma$ *and without any further assumptions on* β, *then for* $1 < p \le d$, *the semigroup* $\{T_t\}_{t\ge 0} \sim -A^{\Lambda}$ *satisfies estimate* (1.14) *with the same exponents as given in the statements* (1i)-(1iii).

(vii) *If a satisfies* (6.6)-(6.8) *and without any further assumptions on* β, *then for* $1 < p \le d$ *the semigroup* $\{T_t\}_{t\ge 0} \sim -A^{\Lambda}$ *satisfies estimate* (1.18) *with* $u_0 = 0$ *for the same exponents as given in* (2ii).

Since we are not aware about the existence of Gagliardo-Nirenberg inequalities involving the trace operator, we need to construct in each case $1 < p < d$, $p = d$ and $p > d$ the sufficient inequality from the known Sobolev-trace inequality ([72, Chapter 2, Sect. 4]).

Proof of Theorem 6.6. First, let $1 < p < d$. Then, by the Sobolev-trace inequality [72, Théorème 4.2] and by Maz'ya's inequality (6.36),

$$\|u_{|\partial\Sigma}\|_{\frac{p(d-1)}{d-p}} \le C\left(\||\nabla u|\|_p + \|u_{|\partial\Sigma}\|_p\right) \tag{6.44}$$

for every $u \in W^1_{p,p}(\Sigma)$ and some constant $C > 0$ independent of u. Taking pth power on both sides of this inequality, applying (6.29) and using the definition of the operator Λ_2 combined with (6.10) and (6.23) gives

$$\begin{aligned}
\|\varphi_{|\partial\Sigma} - \hat\varphi_{|\partial\Sigma}\|^p_{\frac{p(d-1)}{d-p}} &\le \tilde C\,\left(\||\nabla(P\varphi - P\hat\varphi)|\|_p^p + \|\varphi_{|\partial\Sigma} - \hat\varphi_{|\partial\Sigma}\|_p^p\right)\\
&\le \tilde C\,\tilde\eta^{-1}\int_\Sigma (a(x,\nabla P\varphi) - a(x,\nabla P\hat\varphi))\nabla(P\varphi - P\hat\varphi)\,\mathrm{d}x\\
&\qquad + \tilde C\,\eta_0^{-1}\int_{\partial\Sigma}(v-\hat v)(\varphi-\hat\varphi)\,\mathrm{d}\mathcal{H}\\
&\le \tilde C\,\max\{\tilde\eta^{-1},\eta_0^{-1}\}\,\langle\varphi-\hat\varphi,(\Lambda_2\varphi + v - (\Lambda_2\hat\varphi+\hat v)\rangle
\end{aligned}$$

for every φ, $\hat\varphi \in D(\Lambda_2)$ and $v \in \beta_2(\varphi)$, $\hat v \in \beta_2(\hat\varphi)$. Thus, by Remark 3.5, the operator A^R satisfies the Gagliardo-Nirenberg inequality (3.2) with

$$r = \frac{p(d-1)}{d-p}, \quad \sigma = p, \quad \varrho = 0, \quad \text{and} \quad \omega = L$$

hence the first statement of this theorem holds by Theorem 1.2.

If $p = d \geq 2$, then by the Sobolev inequality for trace operators [72, Théorème 4.6] and by Maz'ya's inequality (6.36), for every $0 \leq \theta < 1$, there is a constant $C > 0$ such that

$$\|u_{|\partial\Sigma}\|_{\frac{1}{1-\theta}} \leq C\left(\||\nabla u|\|_p + \|u_{|\partial\Sigma}\|_p\right) \tag{6.45}$$

for every $u \in W^1_{p,p}(\Sigma)$. Proceeding as in the first step of this proof yields that the operator A^Λ satisfies the Gagliardo-Nirenberg inequality (3.2) with

$$r = \frac{1}{1-\theta}, \quad \sigma = p, \quad \varrho = 0, \text{ and } \omega = L \quad \text{for every } 0 < \theta < 1.$$

Therefore by Theorem 1.2, the second statement of this theorem holds.

Next, let $d < p < \infty$. Then, by the classical Sobolev-Morrey inequality [72, Théorème 3.8] and by Maz'ya's inequality (6.36), there is a constant such that

$$\|u_{|\partial\Sigma}\|_\infty \leq C\left(\||\nabla u|\|_p + \|u_{|\partial\Sigma}\|_p\right) \tag{6.46}$$

for every $u \in \dot{W}^1_{p,p}(\Sigma)$. By proceeding as in the first step of this proof, we see that the operator A^Λ satisfies the Gagliardo-Nirenberg inequality (3.2) with

$$r = \infty, \quad \sigma = p, \quad \varrho = 0, \text{ and } \omega = L.$$

Therefore, by Theorem 1.2, the third statement of this theorem holds.

Under the assumption that the Carathéodory function a satisfies (6.6)-(6.8) and the accretive graph β satisfies (6.24), one proceeds as in the first three steps and applies Theorem 1.4 with $u_0 = 0$. Thus, statement (2i) of this theorem holds.

To see that the last statement holds, one applies Poincaré's inequality

$$\|u_{|\partial\Sigma} - \overline{u_{|\partial\Sigma}}\|_p \leq C\,\||\nabla u|\|_p$$

holding for all $u \in W^1_{p,p}(\Sigma)$ with mean value $\overline{u_{|\partial\Sigma}} := \frac{1}{\mathcal{H}(\partial\Sigma)}\int_{\partial\Sigma} u\,\mathrm{d}\mathcal{H}$, for some constant $C > 0$ (cf. [55, Lemma 2.5]) to the Sobolev-trace inequalities (6.44), (6.45) and (6.46). Then for $1 < p < d$, inequality (6.44) reduces to

$$\|u_{|\partial\Sigma} - \overline{u_{|\partial\Sigma}}\|_{\frac{p(d-1)}{d-p}} \leq C\||\nabla u|\|_p$$

for every $u \in W^1_{p,p}(\Sigma)$, if $p = d \geq 2$ then inequality (6.45) reduces to

$$\|u_{|\partial\Sigma} - \overline{u_{|\partial\Sigma}}\|_{\frac{1}{1-\theta}} \leq C_\theta\,\||\nabla u|\|_p$$

for every $u \in W^1_{p,p}(\Sigma)$ and $0 \leq \theta < 1$, and if $d < p < \infty$, inequality (6.46) reduces to

$$\|u_{|\partial\Sigma} - \overline{u_{|\partial\Sigma}}\|_\infty \leq C\,\||\nabla u|\|_p$$

for every $u \in W^1_{p,p}(\Sigma)$, where the constant C can differ from line to line. Now, by proceeding as in the first three steps of this proof, where one employs these three new Sobolev-trace inequalities involving the average value $\overline{u_{|\partial\Sigma}}$ and by noting that for every $\varphi \in L^2(\partial\Sigma)$, the element $(\overline{\varphi_{|\partial\Sigma}}, 0) \in \Lambda_2$, one sees that for all $1 < p < \infty$, the operator Λ_2 satisfies the Gagliardo-Nirenberg inequality (1.11) with the same exponents as in the statements (1i)-(1iii). This proves that statement (2ii) of this theorem holds.

If Σ is the half space $\mathbb{R}^d_+$, then one replaces the above used Sobolev-trace inequalities with the ones given in [62, Theorem 15.17 & Exercise 15.19] and proceeds as the first two steps of this proof. This completes the proof of this theorem. $\square$

6.2.2. *Parabolic problems involving the fractional p-Laplace operator.* Let Σ be an open subset of $\mathbb{R}^d$, $1 < p < \infty$ and $0 < s < 1$. Then, for given initial value $u(0) \in L^q(\Sigma)$, we intend to establish the L^q-L^r-regularisation estimates of solutions $u(t) = u(x,t)$ for $t > 0$ of the nonlocal diffusion equation

$$\partial_t u - (-\Delta_p)^s u + \beta(u) + f(x,u) \ni 0 \qquad \text{on } \Sigma \times (0,\infty), \tag{6.47}$$

equipped with either *homogeneous Dirichlet boundary conditions*

$$u = 0 \qquad \text{on } \mathbb{R}^d \setminus \Sigma \ \times (0,\infty), \tag{6.48}$$

or with *homogeneous Neumann boundary conditions*, that is, equation (6.47) without any further conditions. We refer the interested reader to [5] for a thorough discussion on Neumann boundary conditions in nonlocal diffusion problems.

Concerning homogeneous Dirichlet boundary conditions (6.48), we impose no further regularity conditions on the boundary $\partial\Sigma$ of Σ. Note that, if $\Sigma = \mathbb{R}^d$, then the homogeneous Dirichlet boundary conditions become *vanishing conditions near infinity* (6.11). In the case of homogeneous Neumann boundary conditions, we assume that Σ is a bounded domain with a Lipschitz boundary.

The operator $(-\Delta_p)^s$ in equation (6.47) denotes the *fractional p-Laplace operator* defined by

$$(-\Delta_p)^s u(x) = \text{P.V.} \int_\Sigma \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{d+sp}}\,\mathrm{d}y \tag{6.49}$$

for a.e. $x \in \Sigma$ and any sufficiently regular function $u : \Sigma \to \mathbb{R}$. The notation P.V. in (6.49) indicates that the integral at the right hand side is to be understood in the *Cauchy principal value sense*, that is, for given $x \in \Sigma$, the value $(-\Delta_p)^s u(x)$ denotes the limit

$$\lim_{\varepsilon\to 0+} \int_{\Sigma\setminus B_\varepsilon(x)} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{d+sp}}\,\mathrm{d}y$$

provided the limit exists. For every $u \in W^s_{p,q}(\Sigma)$, $(q \geq 1)$, such that

$$(x,y) \mapsto \frac{|u(y)-u(x)|^{p-1}}{|y-x|^{d+sp}} \quad \text{belongs to} \quad L^1(\Sigma\times\Sigma), \tag{6.50}$$

Fubini's theorem yields that $(-\Delta_p)^s u \in L^1(\Sigma)$ and

$$(-\Delta_p)^s u(x) = \int_\Sigma \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{d+sp}}\,\mathrm{d}y$$

for a.e. $x \in \Sigma$. In other words, the integral on the right hand side of (6.49) holds without the P.V.-symbol. Employing Fubini's theorem again, subsequently interchanging x and y, and using the symmetry of the kernel $|y-x|^{-(d+sp)}$, one sees that

$$\begin{aligned}(6.51)\qquad &\int_{\Sigma} -(-\Delta_p)^s u\,\xi\,\mathrm{d}x\\ &\quad = \frac{1}{2}\int_{\Sigma}\int_{\Sigma} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{d+sp}}\,(\xi(y)-\xi(x))\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

for every $\xi \in C_c^{\infty}(\overline{\Sigma})$. Since the double integral in the right hand side of (6.51) exists merely for $u, \xi \in W^s_{p,q}(\Sigma)$, it makes sense to employ this double integral in order to define the fractional p-Laplace operator $(-\Delta_p)^s$ in a *weak* sense. In particular, the previous calculation shows that our next definition of the fractional p-Laplace operator is consistent with the smooth case, that is, when $u \in W^s_{p,q}$ satisfies (6.50).

For any open subset Σ of $\mathbb{R}^d$, we define the realisation of the *Dirichlet-fractional p-Laplace operator* $(-\Delta_p^D)^s$ in $L^2(\Sigma)$ by

$$\begin{aligned}(-\Delta_p^D)^s = \Bigg\{&(u,v) \in L^2\times L^2(\Sigma)\;\Big|\; u \in \dot{W}^s_{p,2}(\Sigma) \text{ s.t. for all } \xi \in \dot{W}^s_{p,2}(\Sigma)\\ &-\frac{1}{2}\int_{\Sigma}\int_{\Sigma} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{d+sp}}\,(\xi(y)-\xi(x))\,\mathrm{d}y\,\mathrm{d}x = \int_{\Sigma} v\,\xi\,\mathrm{d}x\Bigg\}.\end{aligned}$$

If Σ is a bounded Lipschitz domain, then we define the realisation of the *Neumann-fractional p-Laplace operator* $(-\Delta_p^N)^s$ in $L^2(\Sigma)$ by

$$\begin{aligned}(-\Delta_p^N)^s = \Bigg\{&(u,v) \in L^2\times L^2(\Sigma)\;\Big|\; u \in W^s_{p,2}(\Sigma) \text{ s.t. for all } \xi \in W^s_{p,2}(\Sigma)\\ &-\frac{1}{2}\int_{\Sigma}\int_{\Sigma} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{d+sp}}\,(\xi(y)-\xi(x))\,\mathrm{d}y\,\mathrm{d}x = \int_{\Sigma} v\,\xi\,\mathrm{d}x\Bigg\}.\end{aligned}$$

Both operators $-(-\Delta_p^D)^s$ and $-(-\Delta_p^N)^s$ are completely accretive in $L^2(\Sigma)$ (cf. [68]). We show this only at the operator $-(-\Delta_p^D)^s$ since the proof for the operator $-(-\Delta_p^N)^s$ proceeds similarly. Let (u,v), $(\hat{u},\hat{v}) \in (-\Delta_p^D)^s$ and $T \in P_0$. By the Lipschitz property of T and since $T(0)=0$, one has that $T\circ u \in \dot{W}^s_{p,2}(\Sigma)$. Furthermore, since T and $s \mapsto |s|^{p-2}s$ are monotonically increasing on $\mathbb{R}$,

$$\begin{aligned}&\int_{\Sigma} T(u-\hat{u})((-v)-(-\hat{v}))\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\Sigma}\int_{\Sigma} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))-|\hat{u}(x)-\hat{u}(y)|^{p-2}(\hat{u}(x)-\hat{u}(y))}{|x-y|^{d+sp}}\times\\ &\qquad\qquad \times T((u(x)-u(y))-(\hat{u}(x)-\hat{u}(y)))\,\mathrm{d}x\mathrm{d}y \geq 0,\end{aligned}$$

proving that $-(-\Delta_p^N)^s$ is completely accretive by Proposition 2.4.

Further, for any given m-accretive graph β on $\mathbb{R}$ satisfying $0 \in \beta(0)$, one has for every $\lambda > 0$ that the Yosida operator $\beta_\lambda : \mathbb{R} \to \mathbb{R}$ of β is monotonically increasing, Lipschitz continuous and satisfies $\beta_\lambda(0) = 0$. Therefore, taking $T = \beta_\lambda$ in the previous calculation shows that the monotone graph β_2 in $L^2(\Sigma)$ satisfies condition (2.28) in Proposition 2.5.

To see that $-(-\Delta_p^D)^s$ satisfies the range condition (2.14) for $X = L^2(\Sigma)$, we take $V = \dot{W}^s_{p,2}(\Sigma)$. If for every $\alpha \in \mathbb{R}_+$, E_α denotes the set of all $u \in V$ satisfying

$$\frac{\langle (I - (-\Delta_p^D)^s)u, u\rangle_{V',V}}{\|u\|_V} \le \alpha\,,$$

then by

$$\langle (I - (-\Delta_p^D)^s)u, u\rangle_{V',V} = \|u\|_2^2 + |u|_{s,p}^p,$$

one has

$$\|u\|_2^2 + |u|_{s,p}^p \le \alpha(\|u\|_2 + |u|_{s,p})$$

for every $u \in E_\alpha$, implying that E_α is bounded in V. Thus, the operator $(I - (-\Delta_p^D)^s) : V \to V'$ is surjective by [63, Théorème 1], proving that $-(-\Delta_p^D)^s$ satisfies the range condition (2.14) in $X = L^2(\Sigma)$. Analogously, one shows that $-(-\Delta_p^N)^s$ is m-completely accretive in $L^2(\Sigma)$. Moreover, both operators have a dense domain in $L^2(\Sigma)$ as proven in [68].

Therefore, by Proposition 2.5, the operators

$$A^{D,s} := -(-\Delta_p^D)^s + \beta_2 + F \tag{6.52}$$

and

$$A^{N,s} := -(-\Delta_p^N)^s + \beta_2 + F \tag{6.53}$$

are quasi m-completely accretive in $L^2(\Sigma)$ with dense domain in $L^2(\Sigma)$. By the Crandall-Liggett theorem, $-A^{D,s}$ and $-A^{N,s}$ generate respectively a strongly continuous semigroup $\{T_t\}_{t\ge 0}$ on $L^2(\Sigma)$ of Lipschitz continuous mappings T_t, which admits a unique Lipschitz continuous extension on $L^q(\Sigma)$ for all $1 \le q < \infty$ and on $\overline{L^2 \cap L^\infty(\Sigma)}^{L^\infty}$ if $q = \infty$, respectively with constant $e^{\omega t}$.

We begin by giving the complete description of the L^q-L^r-regularisation estimates of the semigroup $\{T_t\}_{t\ge 0} \sim -A^{D,s}$ on $L^2(\Sigma)$.

Theorem 6.7. *Let $1 < p < \infty$, $0 < s < 1$, Σ be an open subset of $\mathbb{R}^d$ and $A^{D,s}$ given by (6.52). Then the following statements are true.*

(1) For $1 < sp < d$, the semigroup $\{T_t\}_{t\ge 0} \sim -A^{D,s}$ on $L^2(\Sigma)$ satisfies (1.14) for

$$\alpha_q = \frac{\alpha^*}{1-\gamma^*(1-\frac{q(d-sp)}{dm_0})},\quad \beta_q = \frac{\frac{\beta^*}{2}+\gamma^*\frac{q(d-sp)}{dm_0}}{1-\gamma^*(1-\frac{q(d-sp)}{dm_0})},\quad \gamma_q = \frac{\gamma^*\frac{q(d-sp)}{dm_0}}{1-\gamma^*(1-\frac{q(d-sp)}{dm_0})}$$

for every $m_0 \ge p$ satisfying $spm_0 + (p-2)(d-sp) > 0$ and $1 \le q \le \frac{dm_0}{d-sp}$ satisfying $q > \frac{sp}{d} + (p-2) - d\frac{2+p}{sp}$, where

$$\alpha^* = \frac{1}{(\frac{d}{d-sp}-1)m_0+p-2},\qquad \beta^* = \frac{\frac{2}{p}\frac{d}{d-sp}-1}{(\frac{d}{d-sp}-1)m_0+p-2}+1,$$
$$\gamma^* = \frac{(\frac{d}{d-sp}-1)m_0}{(\frac{d}{d-sp}-1)m_0+p-2}.$$

Moreover, if $\frac{2d}{d+2s} < p < d$ then one can take $m_0 = p$ and if $\frac{2d}{d+s} < p < d$, then (1.14) holds for every $1 \le q \le \frac{dp}{d-sp}$.

(2) *For $sp = d$, suppose that either Σ is unbounded and β satisfies (6.23) or Σ is bounded and no further assumptions on β. Then the semigroup $\{T_t\}_{t\geq 0} \sim -A^{D,s}$ satisfies (1.14) with exponents*

$$\alpha_q = \frac{\alpha^*_\theta}{1-\gamma^*_\theta(1-\frac{q(1-\theta)}{p})}, \quad \beta_q = \frac{\frac{\beta^*_\theta}{2}+\gamma^*_\theta\frac{q(1-\theta)}{p}}{1-\gamma^*_\theta(1-\frac{q(1-\theta)}{p})}, \quad \gamma_q = \frac{\gamma^*_\theta\frac{q(1-\theta)}{p}}{1-\gamma^*_\theta(1-\frac{q(1-\theta)}{p})}$$

for every $1 \leq q \leq \frac{p}{1-\theta}$ and $\max\{1-\frac{p}{2}, 2-p, 0\} < \theta < 1$, where

$$\alpha^*_\theta = \frac{1}{\frac{p}{1-\theta}-2}, \quad \beta^*_\theta = \frac{\frac{2}{1-\theta}-1}{\frac{p}{1-\theta}-2}+1, \quad \gamma^*_\theta = \frac{(\frac{1}{1-\theta}-1)p}{\frac{p}{1-\theta}-2}.$$

(3) *For $d < sp < \infty$, suppose that either Σ is unbounded and β satisfies (6.23) or Σ is bounded and no further assumptions on β. Then the semigroup $\{T_t\}_{t\geq 0} \sim -A^{D,s}$ satisfies (1.14) with exponents*

$$\alpha_q = \frac{1}{p-2(1-\frac{q}{2})}, \quad \beta_q = \frac{1+\frac{p}{2}+q}{p-2(1-\frac{q}{2})}, \quad \gamma_q = \frac{q}{p-2(1-\frac{q}{2})}$$

for every $1 \leq q \leq 2$.

Since we could not find an appropriate reference to Gagliardo-Nirenberg inequalities available for Sobolev or Besov spaces of fractional order in the spirit of the classical ones (cf. Lemma 6.2), we construct in each case $1 < sp < d$, $sp = d$ and $sp > d$ our sufficient inequalities from the known Sobolev inequality for fractional Sobolev spaces partially combined with Poincaré inequalities.

Proof of Theorem 6.7. First, let $1 < sp < d$. Then, by [62, Theorem 14.29], there is a constant $C > 0$ such that

$$\|u\|_{\frac{pd}{d-sp}} \leq C|u|_{s,p}$$

for every $u \in \dot{W}^s_{p,q}(\Sigma)$ and $q \geq 1$. Taking pth power on both sides of this inequality and applying (6.38) to $a = u(x) - u(y)$ and $b = \hat{u}(x) - \hat{u}(y)$ yields

$$\begin{aligned}
&\|u-\hat{u}\|^p_{\frac{pd}{d-sp}}\\
&\leq C\int_\Sigma\int_\Sigma \frac{|(u(x)-\hat{u}(x))-(u(y)-\hat{u}(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \tilde{C}\,\frac{1}{2}\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))-|\hat{u}(x)-\hat{u}(y)|^{p-2}(\hat{u}(x)-\hat{u}(y))}{|x-y|^{d+sp}}\\
&\hspace{5cm}\times\left((u(x)-u(y))-(\hat{u}(x)-\hat{u}(y))\right)\mathrm{d}x\,\mathrm{d}y\\
&= \tilde{C}\,\langle u-\hat{u}, (-(-\Delta^D_p)^s u) - (-(-\Delta^D_p)^s\hat{u})\rangle
\end{aligned}$$

for every u, $\hat{u} \in D((-\Delta^D_p)^s)$. Thus, by Remark 3.5, the operator $A^{D,s}$ given by (6.52) satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters

$$r = \frac{pd}{d-sp}, \quad \sigma = p, \quad \text{and} \quad \varrho = 0, \tag{6.54}$$

hence the first statement of this theorem holds by Theorem 1.2.

Next, let $sp = d \geq 2$. By [48, Theorem 6.9], there is a constant $C = C(d,p,s) > 0$ such that

$$\|u\|_{\frac{p}{1-\theta}} \leq C\,\left(|u|_{s,p} + \|u\|_p\right) \tag{6.55}$$

for every $u \in C_c^\infty(\Sigma)$ and by a standard approximation argument we see that this inequality holds for all $u \in \dot{W}^s_{p,p}(\Sigma)$ and every $0 \le \theta < 1$. If Σ is unbounded and β satisfies (6.23), then taking pth power on both sides of inequality (6.55) and applying (6.38) to $a = u(x) - u(y)$ and $b = \hat{u}(x) - \hat{u}(y)$ together with (6.23) yields

$$\begin{aligned}
&\|u - \hat{u}\|^p_{\frac{p}{1-\theta}} \\
&\le C \left(\int_\Sigma \int_\Sigma \tfrac{|(u(x)-\hat{u}(x))-(u(y)-\hat{u}(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y + \int_\Sigma |u - \hat{u}|^p \,\mathrm{d}x \right) \\
&\le C \left(2\frac{1}{2} \int_\Sigma \int_\Sigma \tfrac{|u(x)-u(y)|^{p-2}(u(x)-u(y)) - |\hat{u}(x)-\hat{u}(y)|^{p-2}(\hat{u}(x)-\hat{u}(y))}{|x-y|^{d+sp}} \right. \\
&\qquad\qquad \times ((u(x)-u(y)) - (\hat{u}(x)-\hat{u}(y)))\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad \left. + \eta_0^{-1} \int_\Sigma (v - \hat{v})(u - \hat{u})\,\mathrm{d}x \right) \\
&\le C \max\{2, \eta_0^{-1}\} \langle u - \hat{u}, (-(-\Delta_p^D)^s u + v) - (-(-\Delta_p^D)^s \hat{u} + \hat{v}) \rangle
\end{aligned}$$

for every u, $\hat{u} \in D((-\Delta_p^D)^s)$, $v \in \beta_2(u)$, $\hat{v} \in \beta_2(\hat{u})$ and some constant $C > 0$ which might be different from line to line. Therefore, by Remark 3.5, the operator $A^{D,s}$ given by (6.52) satisfies the Gagliardo-Nirenberg inequality (3.2) with parameters

$$r = \frac{p}{1-\theta}, \quad \sigma = p, \quad \text{and} \quad \varrho = 0, \qquad \text{for every } 0 \le \theta < 1. \tag{6.56}$$

If Σ is bounded, then by [64, Theorem 5], the first eigenvalue of $-(-\Delta_p^D)^s$ is positive hence the following Poincaré inequality

$$\|u\|_p \le C\,|u|_{s,p} \tag{6.57}$$

holds for every $u \in \dot{W}^s_{p,p}(\Sigma)$, $1 < p < \infty$ and $0 < s < 1$. Using (6.57) to estimate the term $\|u\|_p$ in (6.55) yields

$$\|u\|_{\frac{p}{1-\theta}} \le C\,|u|_{s,p}$$

for every $u \in \dot{W}^s_{p,2}(\Sigma)$. Now, proceeding as previously, we see that the operator $A^{D,s}$ satisfies the Gagliardo-Nirenberg inequality (3.2) with exponents (6.56). Therefore by Theorem 1.2, the second statement of this theorem holds.

Next, let $d < sp$. Then, by [48, Theorem 8.2], there is a constant $C = C(d,p,s) > 0$ such that

$$\|u\|_\infty \le C \left(|u|^p_{s,p} + \|u\|^p_p \right)^{1/p} \tag{6.58}$$

for every $u \in \dot{W}^s_{p,p}(\Sigma)$. If Σ is unbounded but β satisfies (6.23), then proceeding as in the case $sp = d$, we obtain that

$$\begin{aligned}
&\|u - \hat{u}\|^p_\infty \\
&\quad \le C \max\{2, \eta_0^{-1}\} \langle u - \hat{u}, (-(-\Delta_p^D)^s u + v) - (-(-\Delta_p^D)^s \hat{u} + \hat{v}) \rangle
\end{aligned}$$

for every u, $\hat{u} \in D((-\Delta_p^D)^s)$ and $v \in \beta_2(u)$, $\hat{v} \in \beta_2(\hat{u})$. Thus, Remark 3.5 implies that the operator $A^{D,s}$ satisfies the Gagliardo-Nirenberg inequality (3.2) with exponents

$$r = \infty, \quad \sigma = p, \quad \text{and} \quad \varrho = 0. \tag{6.59}$$

If Σ is bounded, then we apply Poincaré inequality (6.57) to estimate the term $\|u\|_p$ in (6.58) and obtain

$$\|u\|_\infty \le C|u|_{s,p}$$

for every $u \in \dot{W}^s_{p,2}(\Sigma)$. Proceeding as above, we see that the operator $A^{D,s}$ satisfies the Gagliardo-Nirenberg inequality (3.2) with exponents (6.59). Therefore by Theorem 1.2, the third statement of this theorem holds. This completes the proof. □

Next, we state the L^q-L^r-regularisation estimates of the semigroup $\{T_t\}_{t\ge0} \sim -A^{N,s}$ on $L^2(\Sigma)$.

Theorem 6.8. *Let Σ be a bounded domain of $\mathbb{R}^d$ with a Lipschitz continuous boundary. Then the following statements are true.*

(1) Suppose the monotone graph β satisfies (6.23) and $A^{N,s}$ is given by (6.53). Then, for every $1 < p < \infty$ and $0 < s < 1$, the semigroup $\{T_t\}_{t\ge0} \sim -A^{N,s}$ satisfies (1.14) with the same exponents as satisfied by the semigroup $\{T_t\}_{t\ge0} \sim -A^{D,s}$ in the statements (1)-(3) of Theorem 6.7.

(2) Suppose the monotone graph β satisfies (6.24) and $A^{N,s}$ is given by (6.53). Then, for every $1 < p < \infty$ and $0 < s < 1$, the semigroup $\{T_t\}_{t\ge0} \sim -A^{N,s}$ satisfies estimate (1.18) for $u_0 = 0$ with the same exponents as satisfied by the semigroup $\{T_t\}_{t\ge0} \sim -A^{D,s}$ in the statements (1)-(3) of Theorem 6.7.

(3) For every $1 < p < \infty$ and $0 < s < 1$, the semigroup $\{T_t\}_{t\ge0} \sim (-\Delta_p^N)^s$ satisfies (6.27) with the same exponents as given in the statements (1)-(3) of Theorem 6.7.

In particular, concerning Neumann boundary condition, we need to construct for each case $1 < sp < d$, $sp = d$ and $sp > d$ our sufficient inequalities from the known Sobolev inequalities for fractional Sobolev spaces partially combined with a Poincaré inequalities.

Proof of Theorem 6.8. We only derive the Sobolev inequalities in each case $1 < sp < d$, $sp = d$ and $sp > d$ needed to deduce the L^q-L^r- regularity estimates for the semigroups, since then one proceeds as in the proof of Theorem 6.7.

First, let $1 < sp < d$. Then, by [48, Theorem 6.9], there is a constant $C > 0$ such that

$$\|u\|_{\frac{pd}{d-sp}} \le C(|u|_{s,p} + \|u\|_p) \tag{6.60}$$

for every $u \in W^s_{p,p}(\Sigma)$. Now, one takes pth power on both sides of this inequality and applies assumption (6.23) and inequality (6.38) for $s = u(x) - u(y)$ and $t = \hat{u}(x) - \hat{u}(y)$. This, together with Remark 3.5 yields the operator $A^{N,s}$ satisfies Gagliardo-Nirenberg inequality (3.2) with exponents (6.54). If β satisfies (6.24), then one does not need inequality (6.38) to show that $A^{N,s}$ satisfies Gagliardo-Nirenberg inequality (1.11) with exponents (6.54).

In the case $sp = d \ge 2$, one uses the Sobolev inequality [48, Theorem 6.10]

$$\|u\|_{\frac{p}{1-\theta}} \le C(|u|_{s,p} + \|u\|_p), \tag{6.61}$$

which holds for all $u \in W^s_{p,p}(\Sigma)$ and any $\theta \in [0,1)$, where the constant C is independent of θ and u. In the case and $d < sp$, one employs the Sobolev inequality [48, Theorem 8.2])

$$\|u\|_\infty \le C(|u|^p_{s,p} + \|u\|^p_p)^{1/p} \tag{6.62}$$

holding for all $u \in W^s_{p,p}(\Sigma)$. In both cases, one proceeds analogously as in the proof of Theorem 6.7. If β satisfies (6.23), then one sees that operator $A^{N,s}$ satisfies the Gagliardo-Nirenberg inequality (3.2) with exponents (6.56) and (6.59), respectively. If β satisfies (6.24), then by proceeding as in the previous case but without using inequality (6.38) one sees that $A^{N,s}$ satisfies the Gagliardo-Nirenberg inequality (1.11) with exponents (6.56) and (6.59), respectively. This shows that the first and the second statement of this theorem holds.

In order to see that the last statement holds, one employs the following Poincaré inequality (see, for instance, [59])

$$\|u - \overline{u}\|_p \le C\,|u|_{s,p}$$

holding for all $u \in W^1_{p,p}(\Sigma)$ and some constant $C > 0$, to estimate the term $\|u\|_p$ in (6.60), (6.61) and (6.62). Then, one proceeds as in the previous steps of this proof and obtains that the operator $(-\Delta^N_p)^s$ satisfies the Gagliardo-Nirenberg inequality (3.2) with exponents (6.54) if $1 < sp < d$, (6.56) if $ps = d \ge 2$ and (6.59) if $sp > d$. Therefore by Theorem 1.2, the third statement of this theorem holds. □

6.3. **Nonlinear diffusion equations in L^1.** This subsection is concerned with the application of our theory developed in Section 3 and Section 4 to semigroups generated by quasi accretive operator in L^1. The here established L^1-L^∞-regularisation estimates are used in the subsequent Section 7 to show that mild solutions are strong.

For the sake of readability, we outline the example here only on the p-Laplace operator but we emphasise that it is clear that this example and the corresponding results hold very well for the general Leray-Lions operator considered in Section 6.1.

Let $1 < p < \infty$, $m > 0$, and $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be a non-decreasing function satisfying

$$\phi(0) = 0 \qquad \text{and} \qquad \phi'(s) \ge C\,|s|^{m-1} \quad \text{for every } s \neq 0, \tag{6.63}$$

for some $C > 0$ independent of $s \in \mathbb{R}$.

Remark 6.9. Typical examples of non-decreasing functions $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfying the two conditions in (6.63) are $\phi_1(s) = |s|^{m-1}s$, $(s \in \mathbb{R})$, for any $m > 0$ or for $m = 1$, $\phi_2(s) := a\,s^+ - b\,s^-$, $(s \in \mathbb{R})$, for $a, b > 0$. Note that for the function ϕ_1, the operator $\Delta_p\phi_1$ coincides with the celebrated *doubly nonlinear operator* $\Delta_p(\cdot^m)$ (cf. [21]).

Then, for given initial value $u_0 \in L^1(\Sigma)$, we investigate in this subsection the regularisation effect of *mild* solutions $u(t) = u(x,t)$ for $t > 0$ of the *parabolic initial value problem*

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla\phi(u)|^{p-2}\nabla\phi(u)) + f(x,u) \ni 0 & \text{on } \Sigma \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \Sigma, \end{cases} \tag{6.64}$$

respectively equipped with one of the following types of boundary conditions:

$$u = 0 \quad \text{on } \partial\Sigma \times (0,\infty), \text{ if } \Sigma \subseteq \mathbb{R}^d, \tag{6.65}$$

$$|\nabla\phi(u)|^{p-2}\nabla\phi(u)\cdot\nu = 0 \quad \text{on } \partial\Sigma \times (0,\infty), \text{ if } \mu(\Sigma) < \infty, \tag{6.66}$$

$$|\nabla\phi(u)|^{p-2}\nabla\phi(u)\cdot\nu + a|\phi(u)|^{p-1} = 0 \quad \text{on } \partial\Sigma \times (0,\infty), \text{ if } \mu(\Sigma) < \infty. \tag{6.67}$$

Concerning homogeneous Dirichlet boundary condition (6.65), we make no further assumptions on the boundary of Σ. However, regarding homogeneous Neumann or Robin boundary conditions (6.66) and (6.67), respectively, we need to ensure the validity of the Gagliardo-Nirenberg inequalities (6.16) hence, we assume that Σ is a bounded domain with a Lipschitz boundary. In addition, concerning homogeneous Neumann boundary conditions (6.66), we state the L^q-L^r regularisation effect merely for initial values $u_0 \in L^q_0(\Sigma)$ for $1 \le q \le \infty$.

Remark 6.10. If Σ is unbounded, then the Dirichlet boundary conditions (6.65) become *vanishing conditions at infinity* (6.11). It is well-known (cf. [91, Theorem 9.12] for the case $p = 2$ and the references therein) that the L^q-L^r-regularisation effect of mild (respectively, strong) solutions of problem (6.64) for $\Sigma = \mathbb{R}^d$ has been deduced from the uniform estimates obtained in the case $\Sigma_n = B(0,n)$ the open ball centred at $x = 0$ and radius $r = n \ge 1$, or for $p \ne 2$ under the assumption that the solutions have enough regularity (cf., for instance, [21, 18]). In this monograph, we show that we do not need to proceed in this way. We treat the case of Dirichlet boundary condition (6.65) for general open subsets Σ of $\mathbb{R}^d$ *at once*. This simplifies essentially the known approaches in the literature and has the great advantage that we know the infinitesimal generator of the nonlinear semigroup.

Now, let A denote either the negative Dirichlet p-Laplace operator $-\Delta_p^D$ on $L^2(\Sigma)$, the negative Neumann p-Laplace operator $-\Delta_p^N$ on $L^2(\Sigma)$ or on $L^2_m(\Sigma)$, or the negative Robin p-Laplace operator $-\Delta_p^R$ realised on $L^2(\Sigma)$. Then, A is a single-valued, m-completely accretive operators in $L^2(\Sigma)$ satisfying $A0 = 0$ (cf. Section 6.1.1 and 6.1.3). Furthermore, let $A_{1\cap\infty}$ be the trace of A on $L^1 \cap L^\infty(\Sigma,\mu)$. Then, by Proposition 2.8, the closure $\overline{A_{1\cap\infty}}$ in $L^1(\Sigma)$ of the trace $A_{1\cap\infty}$ is m-completely accretive in $L^1(\Sigma)$ with dense domain. In the specific case $A = -\Delta_p^N$ on $L^2_m(\Sigma)$, Proposition 2.14 yields that $\overline{A_{1\cap\infty}}$ is m-completely accretive in $L^1_m(\Sigma)$ with dense domain and with c-complete resolvent.

Next, we first consider the case when $\phi : \mathbb{R} \to \mathbb{R}$ is a general continuous, non-decreasing function satisfying $\phi(0) = 0$. For every $\lambda > 0$, let $\beta_\lambda(s) = (1+\lambda\beta)^{-1}(s)$, $(s \in \mathbb{R})$, denote the Yosida operator of $\beta := \phi^{-1}$. Then, by the Lipschitz continuity of $\beta_\lambda : \mathbb{R} \to \mathbb{R}$, since $\beta_\lambda(0) = 0$ and since β_λ monotonically increasing,

$$\begin{aligned}
[\beta_\lambda(u), A(u)]_2 &= \int_\Sigma |\nabla u|^{p-2}\nabla u \nabla\beta_\lambda(u)\,\mathrm{d}x + a\int_{\partial\Sigma}|u|^{p-2}u\beta_\lambda(u)\,\mathrm{d}\mathcal{H}\\
&= \int_\Sigma |\nabla u|^{p}\beta_\lambda'(u)\,\mathrm{d}x + a\int_{\partial\Sigma}|u|^{p-2}u\beta_\lambda(u)\,\mathrm{d}\mathcal{H}\\
&\ge 0,
\end{aligned}$$

for every $u \in D(A)$ and $\lambda > 0$ provided A is one of the three operators $-\Delta_p^D$, $-\Delta_p^N$ or $-\Delta_p^R$, where $a = 0$ if A is not $-\Delta_p^R$. Thus, condition (2.28) holds for

$q = 2$. To see that the trace $A_{1\cap\infty}$ of A in $L^1 \cap L^\infty(\Sigma, \mu)$ satisfies (2.28) for $q = 1$, let $(\gamma_\varepsilon)_{\varepsilon>0}$ be the sequence given by (2.6), then by the Lipschitz continuity of γ_ε and β_λ on $\mathbb{R}$, since $\gamma_\varepsilon(0) = 0$ and $\beta_\lambda(0) = 0$, and by the monotonicity of γ_ε and β_λ on $\mathbb{R}$,

$$\begin{aligned}
&\int_\Sigma \gamma_\varepsilon(\beta_\lambda(u))\, A_{1\cap\infty}(u)\,\mathrm{d}\mu \\
&\quad= \int_\Sigma |\nabla u|^{p-2}\nabla u \nabla\gamma_\varepsilon(\beta_\lambda(u))\,\mathrm{d}x + a\int_{\partial\Sigma}|u|^{p-2}u\gamma_\varepsilon(\beta_\lambda(u))\,\mathrm{d}\mathcal{H} \\
&\quad= \int_\Sigma |\nabla u|^{p}\gamma_\varepsilon'(\beta_\lambda(u))\beta_\lambda'(u)\,\mathrm{d}x + a\int_{\partial\Sigma}|u|^{p-2}u\gamma_\varepsilon(\beta_\lambda(u))\,\mathrm{d}\mathcal{H} \\
&\quad\geq 0,
\end{aligned}$$

for every $u \in D(A_{1\cap\infty})$, $\lambda > 0$ and $\varepsilon > 0$. Since

$$\lim_{\varepsilon\to0+} \gamma_\varepsilon(\beta_\lambda(u(x))) = \mathrm{sign}_0(\beta_\lambda(u(x))) \qquad \text{for a.e. } x \in \Sigma,$$

and $|\gamma_\varepsilon(\beta_\lambda(u))\, A_{1\cap\infty}(u)| \leq |A_{1\cap\infty}(u)| \in L^1(\Sigma)$, Lebesgue's dominated convergence theorem yields

$$\lim_{\varepsilon\to0+}\int_\Sigma \gamma_\varepsilon(\beta_\lambda(u))\, A_{1\cap\infty}(u)\,\mathrm{d}\mu = \int_\Sigma \mathrm{sign}_0(\beta_\lambda(u(x)))A_{1\cap\infty}(u)\,\mathrm{d}\mu.$$

Thus,

$$[\beta_\lambda(u), A_{1\cap\infty}(u)]_1 \geq 0 \qquad \text{for all } u \in D(A_{1\cap\infty}) \text{ and } \lambda > 0.$$

Therefore, if A is one of the three operators $-\Delta_p^D$, $-\Delta_p^N$ or $-\Delta_p^R$ then for every continuous non-decreasing function ϕ on $\mathbb{R}$ satisfying $\phi(0) = 0$, one has that condition (2.43) of Proposition 2.18 holds and so, under either hypothesis (i), hypothesis (ii) or hypothesis (iii) of Proposition 2.18, we can conclude that the closure $\overline{A_{1\cap\infty}\phi}$ of $A_{1\cap\infty}\phi$ in $L^1(\Sigma, \mu)$ is an m-accretive operator in $L^1(\Sigma)$ with complete resolvent. In particular, for $A = -\Delta_p^N$, the operator $\overline{A_{1\cap\infty}\phi}$ is m-accretive in $L^1_m(\Sigma)$ with c-complete resolvent. If ϕ satisfies either hypothesis (ii) or hypothesis (iii) holds, then $A_{1\cap\infty}\phi$ satisfies the range condition (2.44), which is important in order to apply Theorem 1.5. Therefore, if either $\phi(s)$ is locally Lipschitz continuous or A is defined on $L^2(\Sigma)$ with Σ an open subset of $\mathbb{R}^d$ of finite Lebesgue measure, then $A_{1\cap\infty}\phi$ satisfies the range condition (2.44). Moreover, we can state the following result.

Lemma 6.11. *Let ϕ be a continuous non-decreasing function satisfying $\phi(0) = 0$ and Σ an open subset of $\mathbb{R}^d$. Let A be the negative Dirichlet-p-Laplace operator $-\Delta_p^D$ on $L^2(\Sigma)$ and $A_{1\cap\infty}$ the trace of A on $L^1 \cap L^\infty(\Sigma)$. Then, $A_{1\cap\infty}\phi$ satisfies range condition (2.44).*

By using that $A_{1\cap\infty}\phi$ satisfies the range condition (2.44) and under the assumption that ϕ is a continuous strictly increasing function satisfying $\phi(0) = 0$ and Σ be either an open bounded subset of $\mathbb{R}^d$ or $\mathbb{R}^d$, it is not difficult to see that the domain $D(A_{1\cap\infty}\phi)$ is dense in $L^1(\Sigma)$.

We briefly outline the proof of Lemma (6.11).

Proof of Lemma 6.11. Let $(\Sigma_n)_{n\geq1}$ be a sequence of subsets $\Sigma_n \subseteq \Sigma$ satisfying $\Sigma_n \subseteq \Sigma_{n+1}$ and $\bigcup_{n\geq1}\Sigma_n = \Sigma$. Let $\Delta_p^{D,n}$ be the Dirichlet-p-Laplace operator on $L^2(\Sigma_n)$ and A_n the trace of $\Delta_p^{D,n}$ on $L^1 \cap L^\infty(\Sigma_n)$. Since Σ_n has finite Lebesgue measure, Proposition 2.18 implies that the operator $A_n\phi$ satisfies range condition (2.44). For every $\lambda > 0$, let J_λ^n be the resolvent operator of $A_{1\cap\infty}\phi$.

Now, let $f \in L^1 \cap L^\infty(\Sigma)$, $\lambda > 0$ and for every $n \geq 1$, set $f_n = f\mathbb{1}_{\Sigma_n}$, $u_n = J_\lambda^n[f_{|\Sigma_n}]$ and $\tilde{u}_n$ the extension of u_n on $\mathbb{R}^d$ by zero. Then, our first aim is to show that there is $u \in D(A_{1\cap\infty}\phi)$ satisfying $u + \lambda A_{1\cap\infty}\phi(u) \ni f$ and after eventually passing to a subsequence,

$$\lim_{n\to\infty} \tilde{u}_n = u \qquad \text{in } L^1(\Sigma). \tag{6.68}$$

Since $f \in L^1 \cap L^\infty(\Sigma)$ and $A_{1\cap\infty}\phi$ has a complete resolvent, it follows that

$$\|\tilde{u}_n\|_q \leq \|f\|_q \qquad \text{for every } n \geq 1 \tag{6.69}$$

and all $1 \leq q \leq \infty$. By reflexivity of $L^q(\Sigma)$ for $q > 1$, there is $u \in L^q(\Sigma)$ such that after eventually passing to a subsequence $\tilde{u}_n$ converges to u weakly in $L^q(\Sigma)$ and $\|u\|_q \leq \|f\|_q$. Moreover, since for every $\varepsilon > 0$ and for every $A \subseteq \Sigma$ satisfying $|A| < \delta(\varepsilon) := \|f\|_\infty^{-1}\,\varepsilon$, one sees that $\int_A |\tilde{u}_n|\,\mathrm{d}x \leq \varepsilon$ for all $n \geq 1$. Thus the Dunford-Pettis theorem implies $u \in L^1(\Sigma)$, $\|u\|_1 \leq \|f\|_1$ and $\tilde{u}_n$ converges to u weakly in $L^1(\Sigma)$ after passing eventually to a subsequence of $(\tilde{u}_n)_{n\geq 1}$. If $f \geq 0$, then by the T-accretivity of A_n, we have that $0 \leq \tilde{u}_n \leq \tilde{u}_{n+1}$ a.e. on Σ for every $n \geq 1$. Moreover, by (6.69) for $q = 1$, Beppo-Levi's monotone convergence theorem yields (6.68) for some $u \in L^1(\Sigma)$ satisfying $u \geq 0$ provided $f \geq 0$. Similar arguments show that $\tilde{u}_{n+1} \leq \tilde{u}_n \leq 0$ and (6.68) holds for some $u \in L^1(\Sigma)$ satisfying $u \leq 0$ provided $f \leq 0$. Since $-f^- \leq f \leq f^+$, the T-accretivity of A_n yields $J_\lambda^n[-f^-_{|\Sigma_n}] \leq u_n \leq J_\lambda^n[f^+_{|\Sigma_n}]$ a.e. on Σ. Let $\tilde{u}_{-,n}$ denote the extension on Σ of $J_\lambda^n[-f^-_{|\Sigma_n}]$ by zero and $\tilde{u}_{+,n}$ denote the extension on Σ of $J_\lambda^n[f^+_{|\Sigma_n}]$ by zero. Then, there are u_- and $u_+ \in L^1(\Sigma)$ such that $\lim_{n\to\infty} \tilde{u}_{-,n} = u_-$ and $\lim_{n\to\infty} \tilde{u}_{+,n} = u_+$ in $L^1(\Sigma)$. By the monotonicity of $(\tilde{u}_{-,n})_{n\geq 1}$ and $(\tilde{u}_{+,n})_{n\geq 1}$, we obtain $u_- \leq \tilde{u}_n \leq u_+$ a.e. on Σ for all $n \geq 1$. Thus, by Lebesgue's dominated convergence theorem, (6.68) holds provided $\tilde{u}_n$ converges to u a.e. on Σ.

We multiply equation $u_n + \lambda A_n\phi = f_n$ by $\phi(u_n)$ with respect to the L^2-inner product. Then coercivity condition (6.6) yields $(\phi(\tilde{u}_n))_{n\geq 1}$ is bounded in $W^{1,p}_0(\Sigma)$. Hence and by using Rellich-Kondrachov's compactness result combined with a diagonal-sequence argument yields the existence of a subsequence of $(\tilde{u}_n)_{n\geq 1}$, which we denote again by $(\tilde{u}_n)_{n\geq 1}$ and some $v \in W^{1,p}_0(\Sigma)$ such that $\phi(\tilde{u}_n)$ converges weakly to v in $W^{1,p}_0(\Sigma)$ and strongly in $L^p_{loc}(\Sigma)$. By the continuity of ϕ^{-1} on $\mathbb{R}$, it follows that $\tilde{u}_n = \phi^{-1}(\phi(\tilde{u}_n))$ converges to $\phi^{-1}(v)$ in $L^p_{loc}(\Sigma)$ and a.e. on Σ after passing again to a subsequence. Comparing this with the weak limit u of $(\tilde{u}_n)$ in $L^q(\Sigma)$, it follows that $\phi^{-1}(v) = u$ and that limit (6.68) holds. Now, by using classical monotonicity arguments due to Leray-Lions [63] (as employed, for instance, in [55, Lemma 2.5]) yields $u \in D(A_{1\cap\infty}\phi)$ with $u + \lambda A_{1\cap\infty}\phi(u) \ni f$. $\square$

Remark 6.12. Note that, for $A = -\Delta_p^D$, the operator $\overline{A_{1\cap\infty}\phi}$ coincides with the associated *entropy solution* operator of the composition $-\Delta_p^D\phi$. This has been investigate in the celebrated paper [13].

Let F denote the Nemytski operator on $L^1(\Sigma)$ associated with a Carathéodory function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ satisfying (2.17) for some Lipschitz constant $L > 0$. If $\Delta^D_{p,1}$, $\Delta^N_{p,1}$ and $\Delta^R_{p,1}$ are respectively the traces on $L^1 \cap L^\infty(\Sigma, \mu)$ of the operators

Δ_p^D, Δ_p^N or Δ_p^R, and if $\phi \in C(\mathbb{R})$ is a non-decreasing function satisfying $\phi(0) = 0$ then, by Proposition 2.19 and Lemma 6.11, the operators

$$A_\phi^D := \overline{(-\Delta_{p,1}^D)}\phi + F \text{ on } L^1(\Sigma), \qquad A_\phi^N := \overline{(-\Delta_{p,1}^N)}\phi + F \text{ on } L^1_m(\Sigma),$$
$$A_\phi^R := \overline{(-\Delta_{p,1}^R)}\phi + F \text{ on } L^1(\Sigma),$$

are quasi-m-accretive in L^1 with complete resolvent. By the Crandall-Liggett theorem, the operators $-A_\phi^D$, $-A_\phi^N$ and $-A_\phi^R$ generate a strongly continuous semigroup $\{T_t\}_{t\ge 0}$ on $\overline{D(A_{1\cap\infty}\phi)}^{L^1}$ of Lipschitz continuous mappings T_t on the set $\overline{D(A_{1\cap\infty}\phi)}^{L^1}$ with constant $e^{\omega t}$ and satisfying exponential growth (2.39) with respect to the $L^{\tilde{q}}$-norm for all $1 \le \tilde{q} \le \infty$.

Here, we state the complete description of the L^q-L^r-regularisation effect of the Dirichlet-semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^D$ on $\overline{D(A_\phi^D)}^{L^1}$ for a non-decreasing function $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfying (6.63) for some $m > 0$ and $C > 0$.

Theorem 6.13. *Let $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be non-decreasing function satisfying (6.63) for some $m > 0$ and $C > 0$, and let Σ be an arbitrary open set of $\mathbb{R}^d$. Then, the semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^D$ on $\overline{D(A_\phi^D)}^{L^1}$ satisfies the following regularisation estimates.*

(1) *If $1 < p < d$, then there is $\beta^* \ge 0$ such that the semigroup $\{T_t\}_{t\ge 0}$ satisfies estimate (1.18) with $u_0 = 0$ for every $u \in \overline{D(A_\phi^D)}^{L^1} \cap L^\infty(\Sigma)$ with exponents*

$$\alpha_s = \frac{\alpha^*}{1-\gamma^*\left(1-\frac{s(d-p)}{dmq_0}\right)}, \quad \beta_s = \frac{\frac{\beta^*}{2}+\gamma^*\frac{s(d-p)}{dmq_0}}{1-\gamma^*(1-\frac{s(d-p)}{dmq_0})}, \quad \gamma_s = \frac{\gamma^* s}{\frac{dmq_0}{(d-p)}\left(1-\gamma^*(1-\frac{s(d-p)}{dmq_0})\right)}$$

for every $q_0 \ge p$ satisfying $\frac{pq_0}{d-p} + p - 1 - \frac{1}{m} > 0$ and $1 \le s \le \frac{dm\,q_0}{d-p}$ satisfying $\gamma^*(1 - \frac{s(d-p)}{dmq_0}) < 1$, where

$$\alpha^* = \frac{1}{\frac{mp}{d-p}q_0+mp-m-1}, \quad \text{and} \quad \gamma^* = \frac{p\,q_0}{p\,q_0+(d-p)(p-1-\frac{1}{m})}.$$

Moreover, for $\frac{d(1+\frac{1}{m})}{1+d+\frac{1}{m}} < p < d$, one can take $q_0 = p$ and for $\frac{d(m+1)}{dm+1} < p < d$, the semigroup $\{T_t\}_{t\ge 0}$ satisfies (1.18) for every $u \in \overline{D(A_\phi^D)}^{L^1} \cap L^\infty(\Sigma)$ and $1 \le s \le \frac{dm\,q_0}{d-p}$.

(2) *If $p = d \ge 2$ and Σ has finite Lebesgue measure, then for every $\theta \in (0,1)$, there is a $\beta_\theta^* \ge 0$ such that the semigroup $\{T_t\}_{t\ge 0}$ satisfies estimate (1.18) with $u_0 = 0$ for every $u \in \overline{D(A_\phi^D)}^{L^1} \cap L^\infty(\Sigma)$ with exponents*

$$\alpha_s = \frac{\alpha^*}{1-\gamma_\theta^*\left(1-\frac{s(1-\theta)}{mq_0}\right)}, \quad \beta_s = \frac{\frac{\beta^*}{2}+\gamma_\theta^*\frac{s(1-\theta)}{mq_0}}{1-\gamma_\theta^*(1-\frac{s(\theta-1)}{mq_0})}, \quad \gamma_s = \frac{\gamma_\theta^* s}{\frac{mq_0}{(1-\theta)}(1-\gamma_\theta^*(1-\frac{s(1-\theta)}{mq_0}))}$$

for every $q_0 \ge p$ satisfying $\frac{\theta q_0}{1-\theta} + p - 1 - \frac{1}{m} > 0$ and $1 \le s \le \frac{m\,q_0}{1-\theta}$, where

$$\alpha_\theta^* = \frac{1}{m(\frac{\theta q_0}{1-\theta}+p-1-\frac{1}{m})}, \quad \text{and} \quad \gamma_\theta^* = \frac{\frac{\theta}{1-\theta}\,q_0}{\frac{\theta q_0}{1-\theta}+p-1-\frac{1}{m}}.$$

If one takes $\max\left\{0, \frac{1+m(1-p)}{m+1}\right\} < \theta < 1$*, then one can take* $q_0 = p$ *and the semigroup* $\{T_t\}_{t\geq 0}$ *satisfies estimate* (1.18) *with* $s = 1$ *for every* $u \in \overline{D(A_\phi^D)}^{L^1}$.

(3) *If* $p > d$*, then the semigroup* $\{T_t\}_{t\geq 0}$ *satisfies estimate* (1.18) *for every* $u \in \overline{D(A_\phi^D)}^{L^1} \cap L^\infty(\Sigma)$ *with exponents*

$$\alpha_s = \frac{\alpha^*}{1-\gamma^*(1-\frac{s}{m+1})}, \quad \beta_s = \frac{\frac{\beta^*}{2}+\gamma^*\frac{s}{m+1}}{1-\gamma^*(1-\frac{s}{m+1})}, \quad \gamma_s = \frac{\gamma^*\frac{s}{m+1}}{1-\gamma^*(1-\frac{s}{m+1})},$$

for every $1 \leq s \leq m+1$*, where*

$$\alpha^* = \frac{1}{pm(1-\frac{m+1}{mp}+\frac{m+1}{md})}, \quad \beta^* = \gamma^*+1 \quad \textit{and} \quad \gamma^* = \frac{m+1}{dm(1-\frac{m+1}{mp}+\frac{m+1}{md})}. \tag{6.70}$$

We outline the proof of Theorem 6.13.

Proof of Theorem 6.13. By Lemma 6.11, the operator $(-\Delta_{p,1}^D)\phi$ satisfies range condition (2.44) in Proposition 2.18. Hence, we intend to apply Theorem 1.5.

We begin by considering the case $1 < p < d$. Then by Lemma 6.2, there is a constant $C > 0$ such that inequality (6.19) holds for every $u \in \dot{W}^1_{p,2}(\Sigma)$. For every $(u,v) \in (-\Delta_{p,1}^D)\phi$, one has $\phi(u) \in \dot{W}^1_{p,\infty}(\Sigma)$. By classical interior regularity results (see [86]) and since $\phi'(r) > 0$ for all $r \neq 0$, one has $u \in C(\Sigma) \cap C^{1,\alpha}(\{u \neq 0\})$ and $\nabla u \equiv 0$ on the level set $\{u = 0\}$. Combining this with coercivity condition (6.63) and Gagliardo-Nirenberg inequality (6.16) for $1 < p < d$, we see that

$$\begin{aligned}
[u,v]_{(q-p+1)m+1} &= \int_\Sigma |\nabla\phi(u)|^{p-2}\nabla\phi(u)\nabla(|u|^{((q-p+1)m+1)-2}u)\,\mathrm{d}x\\
&= (q-p+1)\,m\int_{\{u\neq 0\}} |u|^{(q-p+1)m-1}|\nabla u|^p\,[\phi']^{p-1}(u)\,\mathrm{d}x\\
&\geq C^{p-1}\,(q-p+1)\,m\int_{\{u\neq 0\}}|u|^{qm-p}|\nabla u|^p\,\mathrm{d}x\\
&= \left[\frac{C}{m}\right]^{p-1}\frac{(q-p+1)p^p}{q^p}\,\||\nabla(|u|^{\frac{qm+p}{p}-2}u)|\|_p^p\\
&\geq \left[\frac{C}{m}\right]^{p-1}\frac{(q-p+1)p^p}{q^p}\,\tilde{C}^{-p}\,\||u|^{\frac{qm+p}{p}-2}u\|^p_{\frac{pd}{d-p}}\\
&= \left[\frac{C}{m}\right]^{p-1}\frac{(q-p+1)p^p}{q^p}\,\tilde{C}^{-p}\,\|u\|^{qm}_{\frac{qmd}{d-p}}
\end{aligned}$$

for every $q \geq p$. Here, the constant $\tilde{C} > 0$ is the one given by Gagliardo-Nirenberg inequality (6.16) and is independent of Σ. Remark 3.5 yields that the operator A_ϕ^D satisfies the one-parameter family of Gagliardo-Nirenberg type inequalities (1.21) with $u_0 = 0$ and $\kappa = \frac{d}{d-p} > 1$ and so Theorem 1.5 yields the first statement of this theorem.

Next, we consider the case $p = d$ and suppose that Σ is a general open subset of $\mathbb{R}^d$ with finite Lebesgue measure. By Lemma 6.2, for every $1 \leq q \leq \infty$ and every $\theta \in [0,1)$, there is a constant $\tilde{C} = \tilde{C}(q,d,\theta) > 0$ such that

$$\|u\|^{\frac{d}{\theta}}_{\frac{q}{1-\theta}} \leq \tilde{C}\,\||\nabla u|\|_d^d\,\|u\|_q^{d\frac{1-\theta}{\theta}}$$

for every $u \in \dot{W}^1_{p,q}(\Sigma)$. For functions $u \in C^\infty_c(\Sigma)$, Maz'ya's inequality (6.36) reduces to a Poincaré inequality, which we apply to estimate $\|u\|_q^{d\frac{1-\theta}{\theta}}$ for $q = d$ in the last inequality. Then for every $\theta \in [0,1)$, there is a constant $\tilde{C} > 0$, which might be different to the one given in the previous inequality, such that

$$\|u\|^d_{\frac{d}{(1-\theta)}} \le \tilde{C}\, \||\nabla u|\|^d_d \tag{6.71}$$

for every $u \in C^\infty_c(\Sigma)$. Since for $1 \le q < \infty$, $\dot{W}^1_{p,q}(\Sigma)$ is the closure of $C^\infty_c(\Sigma)$ in $W^1_{p,q}(\Sigma)$, an approximation argument shows that (6.71) holds also for functions $u \in \dot{W}^1_{p,q}(\Sigma)$. Now, proceeding as in the case $1 < p < d$ and using (6.71), yields

$$\begin{aligned}
[u,v]_{(q-d+1)m+1} &\ge \left[\frac{C}{m}\right]^{p-1} \frac{(q-d+1)d^d}{q^d} \tilde{C}^{-p}\, \||u|^{\frac{qm+d}{d}-2}u\|^d_{\frac{d}{(1-\theta)}}\\
&= \left[\frac{C}{m}\right]^{p-1} \frac{(q-d+1)d^d}{q^d} C^{-p}\, \|u\|^{qm}_{\frac{qm}{(1-\theta)}}
\end{aligned}$$

for every $(u,v) \in ((-\Delta^D_{p,1})\phi)_1$ and $q \ge p = d$, where for every $\theta \in [0,1)$, the constant $C > 0$ depends on the measure of Σ, θ and $p = d$. Remark 3.5 yields that the operator A^D_ϕ satisfies the one-parameter family of Gagliardo-Nirenberg type inequalities (1.21) with $u_0 = 0$ and $\kappa = \frac{1}{1-\theta} > 1$ and so Theorem 1.5 yields the third statement of this theorem.

Now, let $p > d$. Then by Lemma 6.2 there is a $\theta_0 \in (0,1)$ satisfying

$$\theta_0(\tfrac{1}{p} - \tfrac{1}{d}) + (1-\theta_0)\tfrac{m}{m+1} = 0$$

and a constant $\tilde{C} > 0$ such that

$$\|u\|_\infty \le \tilde{C}\, \||\nabla u|\|_p^{\theta_0}\, \|u\|^{1-\theta_0}_{\frac{m+1}{m}} \tag{6.72}$$

for every $u \in \dot{W}^1_{p,\frac{m+1}{m}}(\Sigma)$. By applying (6.72) and the coercivity condition (6.63) of ϕ, we see that

$$\begin{aligned}
[u,v]_{m+1}\, \|u\|_{m+1}^{mp\frac{1-\theta_0}{\theta_0}} &= m \int_{\{u\neq 0\}} |\nabla u|^p [\phi'(u)]^{p-1}\, |u|^{m-1} \mathrm{d}x\, \|u_{m+1}\|^{p\frac{1-\theta_0}{\theta_0}}_{\frac{m+1}{m}}\\
&\ge m\, C^{p-1} \int_{\{u\neq 0\}} |\nabla u|^p\, |u|^{p(m-1)} \mathrm{d}x\, \|u_{m+1}\|^{p\frac{1-\theta_0}{\theta_0}}_{\frac{m+1}{m}}\\
&= \left[\frac{C}{m}\right]^{p-1} \int_\Sigma |\nabla u_{m+1}|^p \mathrm{d}x\, \|u_{m+1}\|^{p\frac{1-\theta_0}{\theta_0}}_{\frac{m+1}{m}}\\
&\ge \left[\frac{C}{m}\right]^{p-1} \tilde{C}^{-\frac{p}{\theta_0}}\, \|u_{m+1}\|_\infty^{\frac{p}{\theta_0}}\\
&= \tilde{C}^{-\frac{p}{\theta_0}}\, \|u\|_\infty^{\frac{pm}{\theta_0}}
\end{aligned}$$

for every $(u,v) \in ((-\Delta^D_{p,1})\phi)_1$. Since $\theta_0 = (1 - \frac{m+1}{mp} + \frac{m+1}{md})^{-1}$ and by Remark 3.5, A^D_ϕ satisfies Gagliardo-Nirenberg type inequality (1.11) with

$$r = \infty, \qquad \sigma = pm(1 - \tfrac{m+1}{mp} + \tfrac{m+1}{md}), \qquad q = m+1, \qquad \varrho = mp\frac{1-\theta_0}{\theta_0}$$

and so by Theorem 1.5 and Theorem 4.3, the semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^D$ satisfies inequality (1.18) with $u_0 = 0$, $r = \infty$, $q = m+1$ and α^*, β^* and γ^* given by (6.70). Since for $m \ge 1$, $\gamma^*(1 - \frac{1}{m+1}) < 1$, Theorem 4.3 completes the proof of the last claim of this theorem. □

Next, we state the complete description of the L^q-L^r-regularisation estimates of the semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^N$ on $L^1_m(\Sigma)$. Here, we denote by $L^1_m(\Sigma)$ the space of all functions $u \in L^1(\Sigma)$ with mean value $\overline{u} := \frac{1}{|\Sigma|}\int_\Sigma u\,\mathrm{d}x = 0$.

Theorem 6.14. *Let Σ be a bounded domain with Lipschitz boundary and $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be a non-decreasing function satisfying (6.63) for some $m > 0$ and $C > 0$. Then, for $1 < p < \infty$, the semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^N$ on $L^1_m(\Sigma)$ satisfies the L^q-L^r-regularisation estimate (1.18) with $u_0 = 0$ for every $u \in L^1_m(\Sigma) \cap L^\infty(\Sigma)$ with the same exponents and conclusions as for the semigroup generated by $-A_\phi^D$ on $\overline{D(A_\phi^D)}^{L^1}$ stated in Theorem 6.13.*

For the proof, we proceed similarly as in the proof of Theorem 6.13.

Proof of Theorem 6.14. If $1 < p < d$, then inequality (6.33) reduces to Sobolev inequality (6.19) by using functions $u \in W^1_{p,p,m}(\Sigma)$. If $p \ge d$, then applying Poincaré inequality (6.32) for functions $u \in W^1_{p,p,m}(\Sigma)$ to Gagliardo-Nirenberg inequality (6.18) yields inequality (6.71) and (6.72). Thus, proceeding as in the proof of Theorem 6.13, we see that the statement of this theorem holds. □

To complete this subsection, we state the complete description of the L^q-L^r-regularisation effect of the semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^R$ on $L^1(\Sigma)$ and $\phi(s) = |s|^{m-1}s$ for $m > 0$.

Theorem 6.15. *Let Σ be a bounded domain with Lipschitz boundary and $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be a non-decreasing function satisfying (6.63) for some $m > 0$ and $C > 0$. Then, for $1 < p < \infty$, the semigroup $\{T_t\}_{t\ge 0} \sim -A_\phi^R$ on $L^1(\Sigma)$ satisfies the L^q-L^r-regularisation estimate (1.18) with $u_0 = 0$ for every $u \in L^1(\Sigma) \cap L^\infty(\Sigma)$ with the same exponents and conclusions and as for the semigroup generated by $-A_\phi^D$ on $\overline{D(A_\phi^D)}^{L^1}$ stated in Theorem 6.13.*

We proceed as in the proof of Theorem 6.13.

Proof of Theorem 6.15. We begin by considering the case $1 < p < d$. Note that by coercivity condition (6.63) of ϕ, one has

$$\phi(s) \ge \tfrac{C}{m}|s|^{m-1}s \qquad \text{for all } s \in \mathbb{R}. \tag{6.73}$$

Combining this with (6.63) and Sobolev inequality (6.39), we see that

$$\begin{aligned}
[u,v]_{(q-p+1)m+1} &\ge \left[\frac{C}{m}\right]^{p-1} \frac{(q-p+1)p^p}{q^p} \| |\nabla(|u|^{\frac{qm+p}{p}-2}u)| \|_p^p \\
&\qquad + a\left[\frac{C}{m}\right]^{p-1} \int_{\partial\Sigma} u_{(q-p+1)m+1} |u_{m+1}|^{p-2} u_{m+1}\,\mathrm{d}\mathcal{H} \\
&= \left[\frac{C}{m}\right]^{p-1} \frac{(q-p+1)p^p}{q^p} \| |\nabla(|u|^{\frac{qm+p}{p}-2}u)| \|_p^p
\end{aligned}$$

$$
\begin{aligned}
&\qquad + a \left[\frac{C}{m}\right]^{p-1} \int_{\partial\Sigma} ||u|^{\frac{qm+p}{p}-2}u|^p \,\mathrm{d}\mathcal{H}\\
&\geq \left[\frac{C}{m}\right]^{p-1} \min\left\{\tfrac{(q-p+1)p^p}{q^p}, a\right\} \tilde{C}^{-p} \| |u|^{\frac{qm}{p}} \|^p_{\frac{pd}{d-p}}\\
&= \left[\frac{C}{m}\right]^{p-1} \min\left\{\tfrac{(q-p+1)p^p}{q^p}, a\right\} \tilde{C}^{-p} \, \|u\|^{qm}_{\frac{qmd}{d-p}}
\end{aligned}
$$

for every $q \geq p$ and $(u,v) \in (-\Delta^R_{p,1})\phi$. Remark 3.5 yields that the operator A^R_ϕ satisfies the one-parameter family of Gagliardo-Nirenberg type inequalities (1.21) with $u_0 = 0$ and $\kappa = \frac{d}{d-p} > 1$ and so by Theorem 1.5, the statement of this theorem holds for $1 < p < d$.

Next, for $p = d$, then by Lemma 6.2, for every $1 \leq q \leq \infty$ and every $\theta \in [0,1)$, there is a constant $C = C(d,\theta) > 0$ such that

$$\|u\|_{\frac{d}{1-\theta}} \leq C \left(\||\nabla u|\|_d^\theta \, \|u\|_d^{1-\theta} + \|u\|_d \right)$$

for every $u \in W^1_{d,d}(\Sigma)$. Applying Maz'ya's inequality (6.36) and Young's inequality to the latter inequality and subsequently raising to the dth power yields

$$\|u\|^d_{\frac{d}{1-\theta}} \leq C \left(\||\nabla u|\|_d^d + \|u_{|\partial\Sigma}\|_d \right)^d$$

for every $u \in W^1_{d,d}(\Sigma)$, where the constant $C > 0$ can differ from the previous one. By this Sobolev type inequality, we can proceed as above and see that also for $p = d$, the statement of this theorem holds.

Now, let $p > d$. Then by Lemma 6.2 there is a $\theta_0 \in (0,1)$ satisfying

$$\theta_0\left(\tfrac{1}{p} - \tfrac{1}{d}\right) + (1-\theta_0)\tfrac{m}{m+1} = 0$$

and for every $\tilde{q} > 0$, there is a constant $C := C(\theta_0, p, d, \tilde{q}) > 0$ such that

$$\|u\|_\infty \leq C \left(\||\nabla u|\|_p^{\theta_0} \, \|u\|^{1-\theta_0}_{\frac{m+1}{m}} + \|u\|_{\tilde{q}} \right)$$

for every $u \in W^1_{p,\frac{m+1}{m}}(\Sigma) \cap L^{\tilde{q}}(\Sigma)$. Taking $\tilde{q}$ such that $\frac{1}{\tilde{q}} = \frac{\theta_0}{p} + \frac{1-\theta_0}{\frac{m+1}{m}}$ yields

$$\|u\|_\infty \leq C \left(\||\nabla u|\|_p^{\theta_0} + \|u\|_p^{\theta_0} \right) \, \|u\|^{1-\theta_0}_{\frac{m+1}{m}}$$

and so by Maz'ya's inequality (6.36), and subsequently raising to the $\frac{p}{\theta_0}$th power, we obtain that

$$\|u\|_\infty^{\frac{p}{\theta_0}} \leq C \left(\||\nabla u|\|_p^p + \|u_{|\partial\Sigma}\|_p^p \right) \, \|u\|^{p\frac{1-\theta_0}{\theta_0}}_{\frac{m+1}{m}}$$

for every $u \in W^1_{p,\frac{m+1}{m}}(\Sigma)$, where the constant can differ from the previous one. By using this Gagliardo-Nirenberg type inequality together with (6.73), we see that

$$
\begin{aligned}
[u,v]_{m+1} \, \|u\|^{m\,p\frac{1-\theta_0}{\theta_0}}_{m+1} \geq \Bigg(m\, C^{p-1} \int_{\{u\neq 0\}} |\nabla u|^p \, |u|^{p(m-1)} \mathrm{d}x &\\
+ \left[\frac{C}{m}\right]^{p-1} a \int_{\partial\Sigma} |u|^{pm} \,\mathrm{d}\mathcal{H} \Bigg) \, \|u_{m+1}\|^{p\frac{1-\theta_0}{\theta_0}}_{\frac{m+1}{m}}&
\end{aligned}
$$

$$= \left[\frac{C}{m}\right]^{p-1} \left(\|\nabla u_{m+1}\|_p^p + a\,\|u_{m+1\,|\partial\Sigma}\|_p^p \right) \|u_{m+1}\|_{\frac{m+1}{m}}^{p\frac{1-\theta_0}{\theta_0}}$$
$$\geq \left[\frac{C}{m}\right]^{p-1} \tilde{C}^{-1}\, \min\{1,a\}\, \|u\|_\infty^{\frac{mp}{\theta_0}}$$

$(u,v) \in (-\Delta_{p,1}^R)\phi$. Thus, the statement of this theorem holds in the case $p > d$, completing the proof. □

7. Application II: Mild solutions in L^1 are strong

Let $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be a strictly increasing function satisfying (6.63) and Σ be an open bounded subset of $\mathbb{R}^d$ satisfying the same assumption as in the previous Section 6.3. Then the aim of this section is to show that *mild solutions* in L^1 of the nonlinear parabolic initial value problem (6.64) equipped with one of the boundary conditions (6.65), (6.66), (6.67) on a bounded open set Σ of $\mathbb{R}^d$ are *weak energy solutions* (see Definition 7.2 below) which are globally bounded. This property implies global Hölder continuity of is mild solutions of the parabolic problem (6.64) (see [50, 80, 93]). Moreover, if ϕ is either given by

$$\phi(s) = |s|^{m-1}s \qquad \text{for every } s \in \mathbb{R}, \text{ and some } m > 0, \tag{7.1}$$

or ϕ is locally bi-Lipschitz continuous, then every mild solution in L^1 of the nonlinear parabolic initial value problem (6.64) is a *strong energy* solution (see Definition 7.2 below).

In this section, we denote by

$$V \textit{ either the space } \dot{W}^1_{p,2}(\Sigma),\ W^1_{p,2,m}(\Sigma) \textit{ or } W^1_{p,2}(\Sigma)$$

and $L^2(\Sigma,\mu)$ is either the classical $L^2(\Sigma)$ space equipped with the d-dimensional Lebesgue measure if we consider Dirichlet or Robin boundary conditions or $L^2_m(\Sigma)$ if we consider Neumann boundary conditions. Note that in each case the space V is embedded into the Hilbert space $L^2(\Sigma,\mu)$ by a continuous injection with a dense image.

Remark 7.1. Note that our approach given here is quite general and can easily be adapted to other nonlinear parabolic boundary-value problems. For instance, to problems involving the fractional p-Laplace operator as

$$\partial_t u - (-\Delta_p)^s \phi(u) + \beta(u) + f(x,u) \ni 0 \qquad \text{on } \Sigma \times (0,\infty),$$

or to problems associated with the $p(x)$-Laplace operator as

$$\partial_t u - \mathrm{div}(|\nabla\phi(u)|^{p(x)-2}\nabla\phi(u)) + \beta(u) + f(x,u) \ni 0 \qquad \text{on } \Sigma \times (0,\infty).$$

each equipped with some boundary conditions. Concerning the latter problem, we refer the interested reader to [57].

In order to conclude that the milds solution of problem (6.64) with initial value $u_0 \in L^1(\Sigma)$ is, in fact, a weak energy solution, we will take advantage of the following two properties: the negative p-Laplace operator $-\Delta_p$ equipped with one of the above given boundary conditions (6.65)-(6.67) can be realised

(i) as the first derivative $\Psi' : V \to V'$ of a continuously differentiable functional $\Psi : V \to \mathbb{R}_+$ given by

$$\Psi(u) = \frac{1}{p}\int_{\Sigma}|\nabla u|^p \,\mathrm{d}x + \tfrac{a}{p}\int_{\partial\Sigma}|u|^p \mathrm{d}\mathcal{H} \tag{7.2}$$

for very $u \in V$, where $a = 0$ if one considers Dirichlet or Neumann boundary conditions, and $a > 0$ if one considers purely *Robin boundary conditions*,

(ii) as an operator A in $L^2(\Sigma,\mu)$ by taking the *part* of Ψ' in $L^2(\Sigma,\mu)$, that is,

$$A = \Big\{(u,v) \in V \times L^2(\Sigma,\mu) \,|\, \langle \Psi'(u), v\rangle_{V';V} = \langle h, v\rangle \text{ for all } v \in V\Big\}.$$

Note, the part A of Ψ' in $L^2(\Sigma,\mu)$ coincides with the subgradient $\partial_{L^2}\Psi^{L^2}$ in $L^2(\Sigma,\mu)$ of the convex, proper, densely defined, and lower semicontinuous functional $\Psi^{L^2} : L^2(\Sigma,\mu) \to \mathbb{R}\cup\{+\infty\}$ given by

$$\Psi^{L^2}(u) = \begin{cases} \Psi(u) & \text{if } u \in V, \\ +\infty & \text{if otherwise} \end{cases}$$

for every $u \in L^2(\Sigma,\mu)$. This is well-known, but if the reader is interested in a more thorough explanation, then we refer him to [31].

One easily verifies that the functional Ψ defined in (7.2) satisfies the hypotheses ($\mathcal{H}$i)-($\mathcal{H}$v). Moreover, in this framework, the notion of *weak energy solutions* given in Definition 5.2 concerning solutions of problem (6.64) equipped with one of the boundary condition (6.65)-(6.67) makes sense, we also in this section we use the function

$$\Phi(s) := \int_0^s \phi(r)\,\mathrm{d}r \qquad \text{for every } s \in \mathbb{R}.$$

We still need to clarify the notion of *strong solutions* of such problems.

Definition 7.2. For given $u_0 \in L^1(\Sigma)$, we a function $u \in C([0,\infty);L^1(\Sigma))$ a *strong energy solution in L^1 of problem* (6.64) if u is a weak energy solution of problem (6.64) in the sense of Definition 5.2 and for every $T > 0$, one has

$$u \in W^{1,1}((0,T];L^1(\Sigma)).$$

The following theorem is the main result of this section, where we take the measure $\mathrm{d}\mu = \mathrm{d}x$ the d-dimensional Lebesgue-measure.

Theorem 7.3. *Let $1 < p < \infty$ with the restriction that*

$$\frac{d(1+\frac{1}{m})}{dm+1} < p \qquad \text{if } 1 < p < d,$$

and $\phi \in C(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$ be a strictly increasing function satisfying (6.63) for some $m > 0$ and Σ be an open bounded subset of $\mathbb{R}^d$ satisfying the same assumption as in the previous Section 6.3. Further, let $\{T_t\}_{t\geq 0}$ be the semigroup either generated by $-\overline{((-\Delta_{p,1}^D)\phi)_1 + F}$ on $L^1(\Sigma)$, $-\overline{((-\Delta_{p,1}^N)\phi)_1 + F}$ on $L^1_m(\Sigma)$ or by $-\overline{((-\Delta_{p,1}^R)\phi)_1 + F}$ on $L^1(\Sigma)$. Then, for every $u_0 \in L^1(\Sigma)$ (respectively, for every $u_0 \in L^1_m(\Sigma)$), the following statements hold.

(1) *The mild solution* $u(t) := T_t u_0$, $t \geq 0$ *of problem* (6.64) *equipped with either homogeneous Dirichlet boundary conditions* (6.65)*, homogeneous Neumann boundary conditions* (6.66)*, or homogeneous Robin boundary conditions* (6.67) *is a weak energy solution of* (6.64) *satisfying energy inequality* (5.9).
(2) *If, in addition,* ϕ *satisfies one of the following conditions*
   (i) ϕ *is homogeneous of degree* $\alpha > 0$, $\alpha \neq 1$*, that is,* $\phi(\lambda s) = \lambda^{\alpha}\phi(s)$ *for every* $s \in \mathbb{R}$ *and* $\lambda > 0$,
   (ii) ϕ *and* ϕ^{-1} *are locally Lipschitz continuous on* $\mathbb{R}$,
   then the mild solution $u(t) := T_t u_0$, $t \geq 0$*, is a strong energy solution.*

For the proof of Theorem 7.3, the main ingredients are the L^1-L^∞-regularisation estimates established in Section 6.3.

Proof of Theorem 7.3. The first statement of this theorem follows immediately from Theorem 5.6 due to the global L^1-L^∞ regularisation estimates holding uniformly for all $t > 0$ given by Theorem 6.13 concerning Dirichlet boundary conditions, Theorem 6.14 concerning Neumann boundary conditions, and Theorem 6.15 regarding Robin boundary conditions. Here, we chose in the case $p = d$, the parameter θ appearing in Theorem 6.13 such that

$$\max\left\{0, \tfrac{1+m(1-p)}{m+1}\right\} < \theta < 1.$$

The second statement follows from [14, Theorem 7] if ϕ is homogeneous of order $\alpha > 0$, $\alpha \neq 1$, and from Theorem 5.7 if ϕ and ϕ^{-1} are locally Lipschitz continuous on $\mathbb{R}$. Here, we note that if one wants to conclude from Lipschitz continuity of the mild solution u with values in $L^1(\Sigma)$ that the function $u \in W^{1,1}((0,T];L^1(\Sigma))$, one needs to apply a classical result from measure theory (cf. [91, Lemma A.1], wherein the continuity assumption of u can be omitted due to the chain rule given by Ambrosio and Dal Maso [3]). □

Appendix A. More on accretive operators in L^1

We begin this section by outlining the proof of Proposition 2.10.

Proof of Proposition 2.10. Let $u, v \in L^1(\Sigma,\mu)$ and suppose (2.35) holds for all $j \in \mathcal{J}$ and $\lambda > 0$. For every $T \in P$, one has either $T < 0$ on $\mathbb{R}$ or $T \in P_0$ or $T > 0$ on $\mathbb{R}$. If $T \in P_0$, then inequality (2.36) follows from Proposition 2.4. If $T > 0$ on $\mathbb{R}$, then the function $j(s) := \int_0^s T(r)\,\mathrm{d}r$ for every $s \in \mathbb{R}$ belongs to $\mathcal{J}$. Since the support of the derivative T' of T is a compact subset of $\mathbb{R}$, the function T is bounded on $\mathbb{R}$. Thus, there is a constant $M \geq 0$ such that

$$|j(u+\lambda v)| \leq |j(0)| + \int_0^1 |T((u+\lambda v)s)|\,\mathrm{d}s|u+\lambda v| \leq |j(0)| + M\,|u+\lambda v|$$

for a.e. $x \in \Sigma$, showing that $j(u+\lambda v) \in L^1(\Sigma,\mu)$. By (2.35) and since $j(u) \geq 0$, we have that $j(u) \in L^1(\Sigma,\mu)$ satisfies

$$0 \leq \int_\Sigma (j(u+\lambda v) - j(u))\,\mathrm{d}\mu$$

for all $\lambda > 0$. By convexity of j and since $j \in C^1(\mathbb{R})$, one has that $\frac{j(u+\lambda v)-j(u)}{\lambda}$ decreases to $T(u)v$ a.e. on Σ. Thus and since $T(u)v \in L^1(\Sigma,\mu)$, it follows that (2.36) holds for $T > 0$. In the case that $T < 0$, we first truncate u and v at hight n. More precisely, for every $n > 1$, let $u_n = u$ if $|u| \le n$ and $u_n = 0$ if otherwise and analogously, define v_n. Further, define $j_n \in \mathcal{J}$ by

$$j_n(s) = \begin{cases} \int_{-n}^{n} T(r)\,\mathrm{d}r - 2\,n\,T(-n) & \text{if } s \ge n,\\ \int_{-n}^{s} T(r)\,\mathrm{d}r - 2\,n\,T(-n) & \text{if } |s| \le n,\\ -2\,n\,T(-n) & \text{if } s \le -n \end{cases}$$

for every $s \in \mathbb{R}$. Now, proceeding as above yields

$$\int_\Sigma T(u_n)\,v_n\,\mathrm{d}\mu \ge 0$$

for every $n > 1$. By dominated convergence, $T(u_n)\,v_n$ converges to $T(u)\,v$ in $L^1(\Sigma,\mu)$ hence we can conclude that (2.36) holds as well for $T > 0$.

It remains to show that the other inclusion holds as well. To see this, let u, $v \in L^1(\Sigma,\mu)$ satisfy (2.36) for every $T \in P$. For given $j \in \mathcal{J}$, let $j_\nu(s) := \inf_{r\in\mathbb{R}}\{j(r) + \nu|s-r|\}$ for every $s \in \mathbb{R}$ and $\nu \ge 0$. Then the sequence $(j_\nu)_{\nu\ge 0}$ consists of Lipschitz continuous, convex functions $j_\nu \in \mathcal{J}$ such that for every $h \in L^1(\Sigma,\mu)$, $j_\nu(h)$ converges monotone increasingly to $j(h)$ a.e. on Σ and $\int_\Sigma j_\nu(h)\mathrm{d}\mu \uparrow \int_\Sigma j(h)\mathrm{d}\mu$ as $\nu \to \infty$. Next, for every $n \ge 1$, let $j_{\nu,n} \in \mathcal{J}$ be given by

$$j_{\nu,n}(r) = \begin{cases} j_\nu'(n)(r-n) + j_\nu(n) & \text{if } r \ge n,\\ j_\nu(r) & \text{if } |s| \le n,\\ j_\nu'(-n)(r+n) + j_\nu(-n) & \text{if } s \le -n \end{cases}$$

for every $r \in \mathbb{R}$. By construction, the a.e. derivative $j_{\nu,n}'$ is positive and bounded by the same Lipschitz constant $L_\nu > 0$ of j_ν. Thus, for every $h \in L^1(\Sigma,\mu)$,

$$|j_{\nu,n}(h)| \le |j_{\nu,n}(0)| + \int_0^1 |j_{\nu,n}'(hs)||h|\,\mathrm{d}s \le |j_\nu(0)| + L_\nu|h| \tag{A.1}$$

a.e. on Σ hence $j_{\nu,n}(h) \in L^1(\Sigma,\mu)$. Since, the function $j_{\nu,n}$ is convex,

$$\inf_{\lambda>0} \frac{j_{\nu,n}(u+\lambda v) - j_{\nu,n}(u)}{\lambda} = j_{\nu,n}'(u)v$$

for a.e. $x \in \Sigma$ and by the Lipschitz continuity of $j_{\nu,n}$,

$$\left|\frac{j_{\nu,n}(u+\lambda v) - j_{\nu,n}(u)}{\lambda}\right| \le L_\nu|v|.$$

Therefore, by the dominated convergence theorem and since $L_\nu^{-1}j_{\nu,n}' \in P$, it follows by (2.36) that

$$\begin{aligned}\int_\Sigma \frac{L_\nu^{-1}j_{\nu,n}(u+\lambda v) - L_\nu^{-1}j_{\nu,n}(u)}{\lambda}\,\mathrm{d}\mu &\ge \int_\Sigma \inf_{\lambda>0} \frac{L_\nu^{-1}j_{\nu,n}(u+\lambda v) - L_\nu^{-1}j_{\nu,n}(u)}{\lambda}\,\mathrm{d}\mu\\ &= \int_\Sigma L_\nu^{-1}j_{\nu,n}'(u)v\,\mathrm{d}\mu \ge 0,\end{aligned}$$

from where one can conclude that (2.35) holds for $j_{\nu,n}$. Since for every $h \in L^1(\Sigma,\mu)$, $j_{\nu,n}(h)$ converges to $j_\nu(h)$ a.e. on Σ and since the right hand side in (A.1) does not depend on n, it follows that (2.35) holds for j_ν. By the properties of the sequence $(j_\nu)_{\nu\ge 0}$, one easily concludes that inequality (2.35) holds for j. This completes the proof of this proposition. □

Next, we outline the proof of Proposition 2.17.

Proof of Proposition 2.17. We begin by showing that $A\phi$ is accretive in $L^1(\Sigma,\mu)$. To do so, let (u,v), $(\hat{u},\hat{v}) \in A\phi$ and (u,w), $(\hat{u},\hat{w}) \in \phi$. First, we assume that hypothesis (i) holds. Then,

$$\int_\Sigma \psi\,(v-\hat{v})\,\mathrm{d}\mu \ge 0 \tag{A.2}$$

for every $\psi \in L^\infty(\Sigma,\mu)$ satisfying $\psi(x) \in \mathrm{sign}(w(x)-\hat{w}(x))$ for a.e. $x \in \Sigma$ and since by assumption, A is single-valued, the situation $w = \hat{w}$ implies that (A.2) holds only for $\psi \equiv 0$. Consider, the function $\psi \in L^\infty(\Sigma,\mu)$ defined by

$$\psi(x) := \begin{cases} 1 & \text{if } u(x) > \hat{u}(x),\\ \mathrm{sign}_0(w(x)-\hat{w}(x)) & \text{if } u(x) = \hat{u}(x),\\ -1 & \text{if } u(x) < \hat{u}(x), \end{cases}$$

for a.e. $x \in \Sigma$. Then by construction,

$$\psi \in \ \mathrm{sign}(w(x)-\hat{w}(x)) \cap \mathrm{sign}(u(x)-\hat{u}(x)).$$

In particular, ψ satisfies (A.2) hence $A\phi$ is accretive in $L^1(\Sigma,\mu)$. If we assume that hypothesis (ii) holds, then by definition of $A\phi$ and since ϕ is a function, one has that $v \in A\phi(u)$ and $\hat{v} \in A\phi(\hat{u})$. Thus and since A is accretive in $L^1(\Sigma,\mu)$,

$$\begin{aligned} &[\phi(u)-\phi(\hat{u}), v-\hat{v}]_1\\ &\quad= \int_{\{\phi(u)\neq\phi(\hat{u})\}} \mathrm{sign}_0(\phi(u)-\phi(\hat{u}))\,(v-\hat{v})\,\mathrm{d}\mu + \int_{\{\phi(u)=\phi(\hat{u})\}} |v-\hat{v}|\,\mathrm{d}\mu \ge 0. \end{aligned}$$

Since ϕ is injective, one has that $\{\phi(u)=\phi(\hat{u})\} = \{u=\hat{u}\}$. Therefore,

$$\begin{aligned} &\int_{\{u\neq\hat{u}\}} \mathrm{sign}_0(u-\hat{u})\,(v-\hat{v})\,\mathrm{d}\mu + \int_{\{u=\hat{u}\}} |v-\hat{v}|\,\mathrm{d}\mu\\ &\quad= \int_{\{\phi(u)\neq\phi(\hat{u})\}} \mathrm{sign}_0(\phi(u)-\phi(\hat{u}))\,(v-\hat{v})\,\mathrm{d}\mu + \int_{\{\phi(u)=\phi(\hat{u})\}} |v-\hat{v}|\,\mathrm{d}\mu \ge 0, \end{aligned}$$

showing that $A\phi$ is accretive in $L^1(\Sigma,\mu)$.

Moreover, for every $\varepsilon > 0$, the sum $\varepsilon\phi_1 + A\phi$ is accretive in $L^1(\Sigma,\mu)$ under the assumption that either (i) or (ii) holds. This follows easily from the fact that the operator ϕ_1 in $L^1(\Sigma,\mu)$ of the monotone function ϕ on $\mathbb{R}$ is s-accretive in $L^1(\Sigma,\mu)$ (cf. [15]).

Similarly, one shows under the assumptions ϕ is injective and A is T-accretive in $L^1(\Sigma,\mu)$ that for every $\varepsilon \ge 0$, one has $\varepsilon\phi_1 + A\phi$ is T-accretive in $L^1(\Sigma,\mu)$ (cf. [10, Proposition 2.5]).

Next, suppose that A has a complete resolvent and ϕ is continuous satisfying $\phi(0) = 0$. Then, for every $T \in P_0$, $T\circ\phi^{-1}$ is continuous on $Rg(\phi) =]a,b[$ for some a, $b \in \mathbb{R}$, bounded and $T\circ\phi^{-1}(0) = 0$. If (ρ_n) is a standard positive mollifier sequence on $\mathbb{R}$, then $T_n := (T\circ\phi^{-1}) * \rho_n \in P_0$ and $T_n \to T\circ\phi^{-1}$ uniformly on compact subsets of $\mathbb{R}$ as $n \to \infty$. Thus, for every u, $v \in L^1(\Sigma,\mu)$, $T_n(\phi(u))v \to T(u)v$ a.e. on Σ as $n \to \infty$ and since $(T_n(\phi(u)))$ is uniformly bounded in $L^\infty(\Sigma,\mu)$, it follows that $\lim_{n\to\infty} T_n(\phi(u))v = T(u)v$ in $L^1(\Sigma,\mu)$. For every $(u,v) \in A\phi$, one has $(\phi(u),v) \in A$ hence by Proposition 2.4,

$$\int_\Sigma T(\phi(u))v\,\mathrm{d}\mu \ge 0$$

for every $T \in P_0$. Hence, for every $T \in P_0$, replacing T by T_n in the latter inequality and sending $n \to \infty$ yields

$$\int_\Sigma T(u)v \,\mathrm{d}\mu \geq 0, \tag{A.3}$$

showing that $A\phi$ has a complete resolvent. If (Σ, μ) is finite and A has a c-complete resolvent, then similar arguments and replacing Proposition 2.4 by Proposition 2.14 yields that $A\phi$ has a c-complete resolvent. Now, for every $\varepsilon > 0$, recall that $\varepsilon\phi_1$ is completely accretive in $L^1(\Sigma, \mu)$. Thus, if $\phi(0) = 0$, then ϕ_1 has a complete resolvent and so

$$\int_\Sigma T(u)\varepsilon\phi(u) \,\mathrm{d}\mu \geq 0$$

for every $T \in P_0$. For any $T \in P_0$, adding this inequality to (A.3) for $u \in D(\phi_1) \cap D(A\phi)$ and $v \in A\phi(u)$ shows that for every $\varepsilon > 0$, $\varepsilon\phi_1 + A\phi$ has a complete resolvent by Proposition 2.4. Again, the same arguments and using Proposition 2.14 yields that for every $\varepsilon > 0$, $\varepsilon\phi_1 + A\phi$ has a c-complete resolvent. □

The statements of Proposition 2.17 are used in following proof.

Proof of Proposition 2.18. Here, we have been inspired by the proof of [41, Proposition 2]. Let A_ϕ denote the operator on $L^1(\Sigma, \mu)$ given by

$$A_\phi = \left\{ (u, f) \in L^1 \times L^1(\Sigma, \mu) \;\middle|\; \begin{array}{l} \text{there are } \lambda > 0,\ g \in L^1 \cap L^\infty(\Sigma, \mu) \text{ such that} \\ \lim\limits_{\varepsilon \to 0+} J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi} g = u \text{ in } L^1(\Sigma, \mu) \text{ and } f = \dfrac{g-u}{\lambda} \end{array} \right\},$$

where for every $\lambda > 0$ and every $\varepsilon > 0$, the operator $J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi}$ denotes the resolvent of $\varepsilon\phi + A_{1\cap\infty}\phi$.

We begin by showing that under the hypotheses (i)-(iii), for every $\varepsilon > 0$ sufficiently small, $\lambda > 0$ and every $g \in L^1 \cap L^\infty(\Sigma, \mu)$, there is a unique $u_\varepsilon \in D(A_{1\cap\infty}\phi)$ satisfying

$$u_\varepsilon + \lambda(\varepsilon\phi(u_\varepsilon) + A_{1\cap\infty}\phi(u_\varepsilon)) \ni g \tag{A.4}$$

or equivalently, $u_\varepsilon = J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi} g$, and there is an $u \in L^1 \cap L^\infty(\Sigma, \mu)$ such that

$$\lim_{\varepsilon \to 0+} u_\varepsilon = u \qquad \text{in } L^1(\Sigma, \mu) \tag{A.5}$$

and

$$\lim_{\varepsilon \to 0+} \varepsilon\, \varphi(u_\varepsilon) = 0 \qquad \text{in } L^{\tilde{q}}(\Sigma, \mu), \text{ for every } 1 \leq \tilde{q} \leq \infty. \tag{A.6}$$

By Proposition 2.5, the operator $\phi_q^{-1} + \lambda A$ is m-completely accretive in $L^q(\Sigma, \mu)$. Thus, and since $(0, 0) \in \phi_q^{-1} + \lambda A$, for every $\varepsilon > 0$, there are $v_\varepsilon \in L^1 \cap L^\infty(\Sigma, \mu) \cap D(\phi_q^{-1}) \cap D(A)$ and $w_\varepsilon \in Av_\varepsilon$ satisfying

$$v_\varepsilon + \tfrac{1}{\varepsilon\lambda}(\phi^{-1}(v_\varepsilon) + \lambda w_\varepsilon) = \tfrac{1}{\varepsilon\lambda} g. \tag{A.7}$$

In fact (cf. the proof of [8, Proposition 3.8]), the solution v_ε of (A.7) is the limit

$$\lim_{\nu \to 0+} v_{\varepsilon,\nu} = v_\varepsilon \qquad \text{in } L^q(\Sigma, \mu)$$

of the sequence $(v_{\varepsilon,\nu})_{\nu>0}$ of solutions $v_{\varepsilon,\nu} \in L^1 \cap L^\infty(\Sigma,\mu) \cap D(A)$ of

$$v_{\varepsilon,\nu} + \tfrac{1}{\varepsilon\lambda}(\beta_\nu(v_{\varepsilon,\nu}) + \lambda w_{\varepsilon,\nu} = \tfrac{1}{\varepsilon\lambda} g \tag{A.8}$$

with $w_{\varepsilon,\nu} \in Av_{\varepsilon,\nu}$. Moreover, one has

$$\lim_{\nu\to 0+} \beta_\nu(v_{\varepsilon,\nu}) = \beta(v_\varepsilon) \qquad \text{weakly in } L^q(\Sigma,\mu), \tag{A.9}$$

where β_ν denotes the Yosida operator of $\beta := \phi^{-1}$. We note that for every $\nu > 0$, $v_{\varepsilon,\nu} \in D(A_{1\cap\infty})$ owing to the Lipschitz continuity of β_ν and since $\beta_\nu(0) = 0$. First, multiplying equation (A.8) with $\beta_\nu(v_{\varepsilon,\nu})$ with respect to the 1-bracket $[\cdot,\cdot]_1$, then using that β_ν is accretive in $L^1(\Sigma,\mu)$ and that β_ν satisfies (2.15) for $q = 1$, we see that

$$\begin{aligned}
\|\beta_\nu(v_{\varepsilon,\nu})\|_1 &\le [\beta_\nu(v_{\varepsilon,\nu}), v_{\varepsilon,\nu}]_1 + [\beta_\nu(v_{\varepsilon,\nu}), \beta_\nu(v_{\varepsilon,\nu})]_1 + [\beta_\nu(v_{\varepsilon,\nu}), w_{\varepsilon,\nu}]_1\\
&= \tfrac{1}{\varepsilon\lambda}[\beta_\nu(v_{\varepsilon,\nu}), g]_1\\
&\le \tfrac{1}{\varepsilon\lambda}\,\|g\|_1
\end{aligned}$$

for all $\nu > 0$. By this estimate together with (A.9) and Hölder's inequality yields that there is a constant $C > 0$ such that

$$\|\beta_\nu(v_{\varepsilon,\nu})\|_p \le C \qquad \text{for all } \nu > 0 \text{ and } 1 < p < q$$

hence, the weak limit $\beta(v_\varepsilon)$ satisfies

$$\|\beta(v_\varepsilon)\|_p \le C \qquad \text{for all } 1 < p < q.$$

Sending $p \to 1+$ in the latter inequality and using Fatou's lemma, we obtain that $\beta(v_\varepsilon) = \phi^{-1}(v_\varepsilon) \in L^1(\Sigma,\mu)$ and so by continuity of ϕ^{-1} on $\mathbb{R}$, $\phi^{-1}(v_\varepsilon) \in L^1 \cap L^\infty(\Sigma,\mu)$. Thus, equation (A.7) yields $v_\varepsilon \in D(A_{1\cap\infty})$ with $w_\varepsilon \in L^1 \cap L^\infty(\Sigma,\mu)$, hence, for every $\varepsilon > 0$, there is $v_\varepsilon \in L^1 \cap L^\infty(\Sigma,\mu) \cap D(\phi_q^{-1}) \cap D(A_{1\cap\infty})$ such that $\phi^{-1}(v_\varepsilon) \in L^1 \cap L^\infty(\Sigma,\mu)$ and

$$v_\varepsilon + \tfrac{1}{\varepsilon\lambda}(\phi^{-1}(v_\varepsilon) + \lambda A_{1\cap\infty} v_\varepsilon) \ni \tfrac{1}{\varepsilon\lambda} g.$$

Taking $u_\varepsilon = \phi^{-1}(v_\varepsilon)$, one has $\phi(u_\varepsilon) = v_\varepsilon$. Thus and by the last inclusion, we have shown that for every $\varepsilon > 0$, there is an $u_\varepsilon \in L^1 \cap L^\infty(\Sigma,\mu)$ such that $\phi(u_\varepsilon) \in D(A_{1\cap\infty})$ and (A.4) holds, or, equivalently, $u_\varepsilon = J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi} g$. We still need to show that the (A.5) and (A.6) hold.

We begin, by assuming that hypothesis (i) holds. Then, by Proposition 2.18, for every $\varepsilon > 0$, $\varepsilon\phi + A_{1\cap\infty}\phi$ is T-accretive in $L^1(\Sigma,\mu)$. Thus, for every $\tilde g \in L^1 \cap L^\infty(\Sigma,\mu)$ satisfying $\tilde g \ge 0$ and $\varepsilon > 0$, one has that $\tilde u_\varepsilon := J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi} \tilde g$ satisfies $\tilde u_\varepsilon \ge 0$, $\|\tilde u_\varepsilon\|_{\tilde q} \le \|\tilde g\|_{\tilde q}$ for $1 \le \tilde q \le \infty$ hence, by the assumptions on $A\phi$ and ϕ,

$$\begin{aligned}
\|\tilde u_\varepsilon + \lambda\varepsilon\phi(\tilde u_\varepsilon)\|_1 &= \|\tilde u_\varepsilon\|_1 + \lambda\varepsilon\|\phi(\tilde u_\varepsilon)\|_1\\
&\le \|\tilde g - \lambda\varepsilon\phi(\tilde u_\varepsilon)\|_1 + \lambda\varepsilon\|\phi(\tilde u_\varepsilon)\|_1 = \|\tilde g\|_1
\end{aligned} \tag{A.10}$$

for every sufficiently small $\varepsilon > 0$. Moreover, if $\tilde w_\varepsilon \in A_{1\cap\infty}\phi(\tilde u_\varepsilon)$ satisfies $\tilde u_\varepsilon + \lambda(\eta\phi(\tilde u_\varepsilon) + \tilde w_\varepsilon)) = \tilde g$, then for every $\varepsilon > \eta > 0$,

$$\tilde u_\varepsilon + \lambda(\eta\phi(\tilde u_\varepsilon) + \tilde w_\varepsilon)) = \tilde g - \lambda(\varepsilon - \eta)\tilde g \le \tilde g = \tilde u_\eta + \lambda(\eta\phi(\tilde u_\eta) + \tilde w_\eta))$$

and so, since the resolvent $J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi}$ of $\varepsilon\phi + A_{1\cap\infty}\phi$ is order-preserving, one has that $\tilde u_\varepsilon \le \tilde u_\eta$ for every $\varepsilon > \eta > 0$. Since $\tilde u_\varepsilon \ge 0$ and $\sup_{\varepsilon>0}\|\tilde u_\varepsilon\|_1 \le \|\tilde g\|_1$, Beppo-Levi's monotone convergence theorem implies that there is $u_+ \in L^1 \cap L^\infty(\Sigma,\mu)$

such that $\tilde{u}_\varepsilon \uparrow u_+$ in $L^1(\Sigma,\mu)$ as $\varepsilon \downarrow 0+$. Similarly, one shows that for every $\tilde{g} \in L^1 \cap L^\infty(\Sigma,\mu)$ satisfying $\tilde{g} \le 0$, one has $\tilde{u}_\varepsilon \le 0$, $\tilde{u}_\varepsilon \ge \tilde{u}_\eta$ for every $\varepsilon > \eta > 0$ and there is $u_- \in L^1 \cap L^\infty(\Sigma,\mu)$ such that $\tilde{u}_\varepsilon \downarrow u_-$ in $L^1(\Sigma,\mu)$ as $\varepsilon \downarrow 0+$. Now, we apply this to a general function $g \in L^1 \cap L^\infty(\Sigma,\mu)$. Let $g^+ = g \vee 0$ be the positive part of g and $g^- = (-g) \vee 0$ be the negative part of g. Since by assumption, $J_\lambda^{A_{1\cap\infty}\phi}$ is order-preserving, u_ε, $u_{\varepsilon^+} := J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi}(g^+)$ and $u_{\varepsilon^-} := J_\lambda^{\varepsilon\phi + A_{1\cap\infty}\phi}(-g^-)$ satisfy

$$u_{\varepsilon^-} \le u_\varepsilon \le u_{\varepsilon^+} \qquad \text{for every } \varepsilon > 0, \tag{A.11}$$

and there are u_+, $u_- \in L^1 \cap L^\infty(\Sigma,\mu)$ satisfying $u_+ \ge 0$, $u_- \le 0$, $u_\varepsilon \uparrow u_+$ in $L^1(\Sigma,\mu)$ as $\varepsilon \downarrow 0$ and $u_\varepsilon \downarrow u_-$ in $L^1(\Sigma,\mu)$ as $\varepsilon \uparrow 0+$. In particular,

$$u_- \le u_\varepsilon \le u_+ \qquad \text{for every } \varepsilon > 0.$$

Thus, to see that (A.5) holds for some function $u \in L^1 \cap L^\infty(\Sigma,\mu)$, it is enough to show that for every sequence $(\varepsilon_n)_{n\ge1} \subseteq (0,1)$ with $\varepsilon_n > \varepsilon_{n+1}$ and every $\delta > 0$, one has

$$\lim_{n,m\to\infty} \mu\left(\{|u_{\varepsilon_n} - u_{\varepsilon_m}| > \delta\}\right) = 0, \tag{A.12}$$

that is, $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence μ-measure. First, we note that by the boundedness of $(u_{\varepsilon_n})_{n\ge1}$ and by the continuity and infectivity of ϕ, for every given $\delta > 0$, there is an $N > 0$ such that

$$\{|u_{\varepsilon_n} - u_{\varepsilon_m}| > \delta\} \subseteq \{|\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m})| > N\} \qquad \text{for every } n, m \ge 1.$$

By (A.10) and (A.11), every $\phi(u_{\varepsilon_n}) \in L^1(\Sigma,\mu)$. Furthermore, by the continuity of ϕ and since $\|u_{\varepsilon_n}\|_\infty \le \|g\|_\infty$, we obtain that every $\phi(u_{\varepsilon_n}) \in L^\infty(\Sigma,\mu)$. Thus, $\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m}) \in L^1 \cap L^\infty(\Sigma,\mu)$ and so, $\psi(\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m})) \in L^q(\Sigma,\mu)$ for $\psi(r) := 1$ if $r > N$, $\psi(r) = 0$ if $|r| \le N$ and $\psi(r) = -1$ if $r < -N$. Thus, multiplying inclusion

$$(u_{\varepsilon_n} - u_{\varepsilon_m}) + \lambda(A_{1\cap\infty}\phi(u_{\varepsilon_n}) - A_{1\cap\infty}\phi(u_{\varepsilon_m})) \ni \varepsilon_m\phi(u_{\varepsilon_m}) - \varepsilon_n\phi(u_{\varepsilon_n}),$$

by $(\psi(\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m})))_q$, and using that $A_{1\cap\infty}$ is accretive in $L^q(\Sigma,\mu)$ together with Hölder's inequality yields

$$\begin{aligned}
&\int_{\{|\phi(u_{\varepsilon_n})-\phi(u_{\varepsilon_m})|>N\}} |u_{\varepsilon_n} - u_{\varepsilon_m}|\,\mathrm{d}\mu \\
&\qquad \le \lambda \int_{\{|\phi(u_{\varepsilon_n})-\phi(u_{\varepsilon_m})|>N\}} |\varepsilon_m\phi(u_{\varepsilon_m}) - \varepsilon_n\phi(u_{\varepsilon_n})|\,\mathrm{d}\mu \\
&\qquad \le \lambda\, \|\varepsilon_m\phi(u_{\varepsilon_m}) - \varepsilon_n\phi(u_{\varepsilon_n})\|_2\, \mu\left(\{|\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m})| > N\}\right)^{1/2}.
\end{aligned} \tag{A.13}$$

By continuity of ϕ and boundedness of $(u_{\varepsilon_n})_{n\ge1}$, there is an $M > 0$ such that

$$\{|\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m})| > N\} \subseteq \{|u_{\varepsilon_n} - u_{\varepsilon_m}| > M\}$$

and so, (A.13) gives

$$M\,\mu\left(\{|\phi(u_{\varepsilon_n}) - \phi(u_{\varepsilon_m})| > N\}\right)^{1/2} \le \lambda\, \|\varepsilon_m\phi(u_{\varepsilon_m}) - \varepsilon_n\phi(u_{\varepsilon_n})\|_2 \tag{A.14}$$

for every n, $m \ge 1$. By (A.10) and (A.11), one has $(\varepsilon\phi(u_\varepsilon))_{\varepsilon>0}$ is bounded in $L^1(\Sigma,\mu)$. By continuity of ϕ and since $\|u_\varepsilon\|_\infty \le \|g\|_\infty$, one has $\lim_{\varepsilon\to0+} \varepsilon\phi(u_\varepsilon) = 0$ in $L^\infty(\Sigma,\mu)$. Thus, under the hypothesis (i), for every $g \in L^1 \cap L^\infty(\Sigma,\mu)$, (A.6) holds. In particular, the right hand side in (A.14) tends to zero as n, $m \to \infty$ showing that (A.12) holds and hence (A.5) holds for some function

$u \in L^1\cap L^\infty(\Sigma,\mu)$ provided hypothesis (i) holds. Next, suppose that hypotheses (ii) and (iii) hold. Since $u_\varepsilon = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g$ can be rewritten as $u_\varepsilon = J_\lambda^{A_{1\cap\infty}\phi}[g - \lambda\varepsilon\phi(u_\varepsilon)]$, the accretivity of $A_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$ yields

$$\|u_\varepsilon - u_\eta\|_1 \le \lambda\, \|\varepsilon\phi(u_\varepsilon) - \eta\phi(u_\eta)\|_1 \tag{A.15}$$

for every ε, $\eta > 0$. If hypothesis (ii) holds, then for every $g \in L^1\cap L^\infty(\Sigma,\mu)$, there is another $K_1 > 0$ such that

$$|\phi(r)| \le K_1\,|r| \qquad \text{for every } |r| \le \|g\|_\infty. \tag{A.16}$$

Moreover, since $\varepsilon\phi + A_{1\cap\infty}\phi$ has a complete resolvent, $u_\varepsilon = J_\lambda^{\varepsilon\phi+A_1\phi}g$ satisfies

$$\|u_\varepsilon\|_\infty \le \|g\|_\infty \tag{A.17}$$

for every $\varepsilon > 0$ and so, (A.16) yields

$$|\phi(u_\varepsilon)| \le K_1\,|u_\varepsilon| \qquad \text{for a.e. } x\in\Sigma \text{ and all } \varepsilon > 0. \tag{A.18}$$

Thus and since $\|u_\varepsilon\|_{\tilde q} \le \|g\|_{\tilde q}$ for every $1 \le q \le \infty$, it follows that

$$\|\varepsilon\phi(u_\varepsilon)\|_{\tilde q} \le K_1\,\varepsilon\,\|u_\varepsilon\|_{\tilde q} \le K_1\,\varepsilon\,\|g\|_{\tilde q},$$

for every for every $1 \le q \le \infty$, from where we can conclude that the sequence $(u_\varepsilon)_{\varepsilon>0}$ has limit (A.6) under hypothesis (ii). In particular, by (A.15), $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $L^1(\Sigma,\mu)$. Therefore and by (A.17), (A.5) holds for some function $u \in L^1\cap L^\infty(\Sigma,\mu)$ also under hypothesis (ii). Moreover, if we assume that hypothesis (iii) holds, then the continuous embedding of $L^\infty(\Sigma,\mu)$ into $L^1(\Sigma,\mu)$ and the boundedness of ϕ on $[-\|g\|_\infty, \|g\|_\infty]$ imply that (A.6) holds. Thus and by (A.17), (A.5) holds for some function $u \in L^1\cap L^\infty(\Sigma,\mu)$ also under hypothesis (iii).

With these preliminaries, we can begin proving the statements of this proposition. First, we show that A_ϕ is an extension of $A_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$. To do so, let $(\hat u,\hat v) \in A_{1\cap\infty}\phi$ and for $\lambda > 0$, set $g = \hat u + \lambda\hat v$. Then, $f := \frac{g-\hat u}{\lambda} = \hat v$ and for every $\varepsilon > 0$ sufficiently small, there is a unique $u_\varepsilon = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g \in D(A_{1\cap\infty}\phi)$ and there is a function $u \in L^1\cap L^\infty(\Sigma,\mu)$ satisfying (A.5). Since $\hat u = J_\lambda^{A_{1\cap\infty}\phi}g$ can be rewritten as $\hat u = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}[g + \lambda\varepsilon\phi(\hat u)]$ and since the operator $\varepsilon\phi + A_{1\cap\infty}\phi$ is accretive in $L^1(\Sigma,\mu)$,

$$\|u_\varepsilon - \hat u\|_1 = \|J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g - J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}[g+\lambda\varepsilon\phi(\hat u)]\|_1 \le \lambda\,\varepsilon\,\|\phi(\hat u)\|_1.$$

Since by assumption, $\hat u \in D(A_{1\cap\infty}\phi)$, one has $\phi(u) \in L^1(\Sigma,\mu)$. Thus, sending $n\to\infty$ in the last inequality yields $u = \hat u$ and so, $u \in D(A_\phi)$ with $f = v \in A_\phi u$.

Next, we show that A_ϕ is contained in the closure $\overline{A_{1\cap\infty}\phi}$ of $A_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$. Let $(u,f) \in A_\phi$. Then, by definition of A_ϕ, there are $\lambda > 0$ and $g \in L^1\cap L^\infty(\Sigma,\mu)$ such that $f = \frac{g-u}{\lambda}$ and for every $\varepsilon > 0$ sufficiently small, there is $u_\varepsilon = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g \in L^1\cap L^\infty(\Sigma,\mu)$ and $u \in L^1\cap L^\infty(\Sigma,\mu)$ satisfying (A.5). By definition of the resolvent $J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}$ of $\varepsilon\phi + A_{1\cap\infty}\phi$, one has

$$(u_\varepsilon, \frac{g-u_\varepsilon}{\lambda} - \varepsilon\phi(u_\varepsilon)) \in A_{1\cap\infty}\phi$$

and so, by (A.6) for $\tilde q = 1$, we can conclude that $(u, \frac{g-u}{\lambda}) = (u,f) \in \overline{A_{1\cap\infty}\phi}$.

The operator A_ϕ is accretive in $L^1(\Sigma,\mu)$ since by construction of A_ϕ, the operator A_ϕ is contained in the *limit inferior* $\liminf_{\varepsilon\to 0+}(\varepsilon\phi + A_1\phi)$ (see, for instance, [15, Definition (2.17) and Proposition (2.18)] or [8, Proposition 4.4]) of the family $(\varepsilon\phi + A_{1\cap\infty}\phi)_{\varepsilon>0}$ of accretive operators $\varepsilon\phi + A_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$ (see Proposition 2.17). Moreover, A_ϕ is m-accretive in $L^1(\Sigma,\mu)$. To see that A_ϕ satisfies the range condition (2.14) for $X = L^1(\Sigma,\mu)$, note that under the hypotheses (i)-(iii), A_ϕ is closed in $L^1(\Sigma,\mu)$. Hence, it is sufficient to show that the set

$$L^1\cap L^\infty(\Sigma,\mu) \subseteq Rg(I+\lambda A_\phi). \tag{A.19}$$

To this end, let $g \in L^1\cap L^\infty(\Sigma,\mu)$ and $\lambda > 0$. Then, by following the arguments in the first part of this proof, we see that for every $\varepsilon > 0$, there is $u_\varepsilon = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g \in D(A_{1\cap\infty}\phi)$. Since $u_\varepsilon = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g$ is equivalent to $u_\varepsilon = J_\lambda^{A_{1\cap\infty}\phi}[g-\lambda\varepsilon\phi(u_\varepsilon)]$, we have that

$$\|u_\varepsilon - u_\eta\|_1 \le \lambda\|\varepsilon\phi(u_\varepsilon) - \eta\phi(u_\eta)\|_1$$

for every $\varepsilon,\eta > 0$. Thus and since under the hypotheses (i)-(iii), (A.6) holds for $\tilde{q} = 1$, we can conclude from the previous inequality that $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $L^1(\Sigma,\mu)$ as $\varepsilon \to 0+$. Therefore, there is an $u \in L^1(\Sigma,\mu)$ such that (A.5) holds and so, by definition of A_ϕ, $(u,f) \in A_\phi$ with $f := \frac{g-u}{\lambda} \in L^1(\Sigma,\mu)$ and $g = u + \lambda f \in (I+\lambda A_\phi)u$. Thus, the range condition (A.19) holds.

Summarising, we have shown that A_ϕ in contained in the closure $\overline{A_{1\cap\infty}\phi}$ of $A_{1\cap\infty}\phi$ in $L^1(\Sigma,\mu)$ and A_ϕ is m-accretive in $L^1(\Sigma,\mu)$. Thus and since $\overline{A_{1\cap\infty}\phi}$ is accretive in $L^1(\Sigma,\mu)$, statement (2.21) implies that $\overline{A_{1\cap\infty}\phi} = A_\phi$.

Next, we show that the range condition (2.44) holds under the hypotheses (ii) and (iii). For this, let $g \in L^1\cap L^\infty(\Sigma,\mu)$. Then, for every $\varepsilon > 0$ sufficiently small,

$$u_\varepsilon = J_\lambda^{\varepsilon\phi+A_{1\cap\infty}\phi}g \in L^1\cap L^\infty(\Sigma,\mu) \quad\text{and}\quad (\phi(u_\varepsilon), \frac{g-u_\varepsilon}{\lambda} - \varepsilon\phi(u_\varepsilon)) \in A.$$

Thus, if we can show that

$$\lim_{\varepsilon\to 0+}\phi(u_\varepsilon) = \phi(u) \qquad \text{in } L^q(\Sigma,\mu) \tag{A.20}$$

and

$$\lim_{\varepsilon\to 0+}\frac{g-u_\varepsilon}{\lambda} - \varepsilon\phi(u_\varepsilon) = \frac{g-u}{\lambda} \qquad \text{in } L^q(\Sigma,\mu) \tag{A.21}$$

then by the assumption, A is m-accretive in the uniformly convex Banach space $L^q(\Sigma,\mu)$ (cf. [8, Proposition 3.4]), we have that

$$\left(\phi(u), \frac{g-u}{\lambda}\right) \in A.$$

To see that (A.20) holds, recall that by (A.17) and (A.5), one has

$$\lim_{\varepsilon\to 0+} u_\varepsilon = u \qquad \text{in } L^q(\Sigma,\mu). \tag{A.22}$$

If hypothesis (ii) holds, then combining (A.22) with the continuity of φ and by (A.18), it follows that (A.20) holds and

$$\|\varepsilon\phi(u_\varepsilon)\|_q \le \varepsilon K_1\,\|u_\varepsilon\|_q \le \varepsilon K_1\|g\|_q,$$

from where we can conclude that (A.21) holds. If hypothesis (iii) holds, then by (A.22), the continuity of ϕ, and by eventually passing to a subsequence, we see that $\lim_{\varepsilon\to 0+}\phi(u_\varepsilon(x)) = \phi(u(x))$ a.e. on Σ. Thus, by (A.17) and the embedding of $L^\infty(\Sigma,\mu)$ into $L^q(\Sigma,\mu)$, we see that (A.20) and (A.21) hold. Moreover, using that $\|u_\varepsilon\|_p \le \|g\|_p$ for all $\varepsilon > 0$ and $1 \le p \le \infty$, we can conclude that $u \in L^1 \cap L^\infty(\Sigma,\mu)$ and by the hypotheses (ii) and (iii), that $\phi(u) \in L^1 \cap L^\infty(\Sigma,\mu)$. Thus,

$$\Big(\phi(u), \frac{g-u}{\lambda}\Big) \in A_{1\cap\infty},$$

proving the range condition (2.44). This completes the proof of this proposition. □

Appendix B. The link between mean spaces and L^p

The first part of the following theorem has been proved in [65, Théorème 1.1 of Chapter IV] by using so-called *discrete mean spaces* (cf. [65, Chapter II]). Here, we improve this result by showing that both spaces are isometrically isomorphic. This result serves us in the proof of Theorem 4.10 and Theorem 4.13 to determine the convergence of the constants in inequality (4.28) as $m \to \infty$.

Theorem B.1. *Let (Σ,μ) be a σ-finite measure space, (X_0, X_1) be an interpolation couple, $1 \le p_0, p_1 \le \infty$ and $0 < \theta < 1$. Then for $1 \le p \le \infty$ given by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{B.1}$$

one has that

$$(L^{p_0}(\Sigma, X_0;\mu), L^{p_1}(\Sigma, X_1;\mu))_{\theta,p_0,p_1} = L^p(\Sigma, (X_0, X_1)_{\theta,p_0,p_1};\mu) \tag{B.2}$$

with equal norms.

Proof of Theorem B.1. We only outline the proof for $1 \le p_0 < \infty$ and $1 \le p_1 < \infty$ since the other cases are shown similarly.

First, let u be an element of $(L^{p_0}(\Sigma, X_0;\mu), L^{p_1}(\Sigma, X_1;\mu))_{\theta,p_0,p_1}$. By definition, there are measurable functions $v_i : (0,\infty) \to L^{p_i}(\Sigma,\mu)$ for $i = 0,1$ such that $t^{-\theta}v_0 \in L^{p_0}_*(L^{p_0}(\Sigma, X_0;\mu))$, $t^{1-\theta}v_1 \in L^{p_1}(\Sigma, X_1;\mu)$ and

$$u(x) = v_0(t,x) + v_1(t,x)$$

for a.e. $(t,x) \in (0,\infty)\times\Sigma$. Since (Σ,μ) and $(\mathbb{R}_+, \frac{dt}{t})$ are both σ-finite measure spaces, Fubini's theorem implies that

$$t^{-\theta}v_0(\cdot,x) \in L^{p_0}_*(X_0) \quad\text{ and }\quad t^{1-\theta}v_1(\cdot,x) \in L^{p_1}(\Sigma,\mu)$$

for a.e. $x \in \Sigma$. Thus by definition of the mean space and by (4.7), one has $a(x) \in (X_0, X_1)_{\theta,p_0,p_1}$ for a.e. $x \in \Sigma$ and

$$\|u(x)\|_{(X_0,X_1)_{\theta,p_0,p_1}} \le \|t^{-\theta}v_0(\cdot,x)\|^{1-\theta}_{L^{p_0}_*(X_0)}\, \|t^{1-\theta}v_1(\cdot,x)\|^{\theta}_{L^{p_1}_*(X_1)}.$$

Integrating the latter inequality over Σ, taking pth root, applying Hölder's inequality (where one uses (B.1)) and Fubini's theorem, we see that

$$\begin{aligned}
&\|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}\\
&\quad\le \Big[\int_\Sigma \|t^{-\theta}v_0(\cdot,x)\|^{p_0}_{L^{p_0}_*(X_0)}\,\mathrm{d}\mu\Big]^{\frac{1-\theta}{p_0}} \Big[\int_\Sigma \|t^{1-\theta}v_1(\cdot,x)\|^{p_1}_{L^{p_1}_*(X_1)}\,\mathrm{d}\mu\Big]^{\frac{\theta}{p_1}}\\
&\quad= \|t^{-\theta}v_0\|^{1-\theta}_{L^{p_0}_*(L^{p_0}(\Sigma,X_0;\mu))}\, \|t^{1-\theta}v_1\|^{\theta}_{L^{p_1}_*(L^{p_1}(\Sigma,X_1;\mu))}.
\end{aligned}$$

Taking in this inequality the infimum over all representation pairs (v_0, v_1) of u and applying (4.7) yields

$$\|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)} \le \|u\|_{(L^{p_0}(\Sigma,X_0;\mu),L^{p_1}(\Sigma,X_1;\mu))_{\theta,p_0,p_1}}.$$

Now, let $u \in L^p(\Sigma, (X_0, X_1)_{\theta,p_0,p_1}; \mu)$ be a step function given by

$$u(x) = \sum_{\nu=1}^{m} a_\nu \, \mathbb{1}_{B_\nu}(x)$$

for finitely many different values $a_\nu \in (X_0, X_1)_{\theta,p_0,p_1}$ attained on pairwise disjoint measurable subsets B_ν of Σ. Let $\varepsilon > 0$. By the definition of $(X_0, X_1)_{\theta,p_0,p_1}$ and the infimum, for every $\nu = 1, \dots, m$, there are measurable functions $v_{i\nu} : (0, \infty) \to X_i$ for $i = 0, 1$ satisfying

$$a_\nu = v_{0\nu}(t) + v_{1\nu}(t) \tag{B.3}$$

for a.e. $t \in (0, \infty)$ and

$$\max\left\{ \|t^{-\theta} v_{0\nu}\|_{L_*^{p_0}(X_0)}, \|t^{1-\theta} v_{1\nu}\|_{L_*^{p_1}(X_1)} \right\} \le \|a_\nu\|_{(X_0,X_1)_{\theta,p_0,p_1}} (1+\varepsilon). \tag{B.4}$$

Set $\lambda = (p_0 - p)/\theta\, p_0$ and for every $\nu = 1, \dots, m$ and $i = 0, 1$ define

$$w_{i\nu}(t) = v_{i\nu}(\|a_\nu\|^{\lambda}_{(X_0,X_1)_{\theta,p_0,p_1}} t).$$

Then applying the substitution $s = \|a_\nu\|^{\lambda}_{(X_0,X_1)_{\theta,p_0,p_1}} t$ together with (B.4) yields

$$\begin{aligned} \|t^{-\theta} w_{0\nu}\|^{p_0}_{L_*^{p_0}(X_0)} &= \|a_\nu\|^{-\lambda\,\theta\, p_0}_{(X_0,X_1)_{\theta,p_0,p_1}} \int_0^\infty \|s^{-\theta} v_{0\nu}(s)\|^{p_0}_{X_0} \tfrac{\mathrm{d}s}{s} \\ &\le (1+\varepsilon)^{p_0} \|a_\nu\|^{p_0 - \lambda\,\theta\, p_0}_{(X_0,X_1)_{\theta,p_0,p_1}} \\ &= (1+\varepsilon)^{p_0} \, \|a_\nu\|^{p}_{(X_0,X_1)_{\theta,p_0,p_1}} \end{aligned} \tag{B.5}$$

for every $\nu = 1, \dots, m$. By the relation (B.1), one sees that the same λ satisfies $p_1 + \lambda(1-\theta)p_1 = p$. Thus the same arguments gives

$$\|t^{1-\theta} w_{1\nu}\|^{p_1}_{L_*^{p_1}(X_1)} \le (1+\varepsilon)^{p_1} \, \|a_\nu\|^{p}_{(X_0,X_1)_{\theta,p_0,p_1}}. \tag{B.6}$$

For $i = 0, 1$ and every $t \in (0, \infty)$, we define the step functions

$$w_i(t, x) = \sum_{\nu=1}^{m} w_{i\nu}(t) \, \mathbb{1}_{B_\nu}(x)$$

for a.e. $x \in \Sigma$. Then by (B.5) and (B.6) as well as by Fubini's theorem,

$$\begin{aligned} \int_0^\infty \|t^{-\theta} w_0(t, \cdot)\|^{p_0}_{L^{p_0}(\Sigma,X_0;\mu)} \tfrac{\mathrm{d}t}{t} &= \int_\Sigma \|t^{-\theta} w_0(\cdot, x)\|^{p_0}_{L_*^{p_0}(X_0)} \, \mathrm{d}\mu \\ &\le (1+\varepsilon)^{p_0} \sum_{i=1}^{m} \|a_\nu\|^{p}_{(X_0,X_1)_{\theta,p_0,p_1}} \mu(B_\nu) \\ &= (1+\varepsilon)^{p_0} \, \|u\|^{p}_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)} \end{aligned} \tag{B.7}$$

and similarly,

$$\int_0^\infty \|t^{1-\theta} w_1(t, \cdot)\|^{p_1}_{L^{p_1}(\Sigma,X_1;\mu)} \tfrac{\mathrm{d}t}{t} \le (1+\varepsilon)^{p_1} \, \|u\|^{p}_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}. \tag{B.8}$$

Therefore, for $i = 0, 1$, the functions $w_i : (0, \infty) \to L^{p_i}(\Sigma, X_i; \mu)$ are well defined step functions and so strongly measurable. In addition, by (B.3),

$$\begin{aligned} w_0(t,x) + w_1(t,x) &= \sum_{\nu=1}^{m} \left(v_{0\nu}(\|a_\nu\|^{\lambda}_{(X_0,X_1)_{\theta,p_0,p_1}}\, t) + v_{1\nu}(\|a_\nu\|^{\lambda}_{(X_0,X_1)_{\theta,p_0,p_1}}\, t)\right) \mathbb{1}_{B_\nu}(x) \\ &= \sum_{\nu=1}^{m} a_\nu \, \mathbb{1}_{B_\nu}(x) \\ &= u(x) \end{aligned}$$

for a.e. $x \in \Sigma$. Thus $u \in (L^{p_0}(\Sigma, X_0; \mu), L^{p_1}(\Sigma, X_1; \mu))_{\theta,p_0,p_1}$ and by (4.7), (B.7), (B.8), and (B.1),

$$\begin{aligned} \|u\|_{(L^{p_0}(\Sigma,X_0;\mu),L^{p_1}(\Sigma,X_1;\mu))_{\theta,p_0,p_1}} &\le \|t^{-\theta} w_0\|^{1-\theta}_{L^{p_0}_*(L^{p_0}(\Sigma,X_0;\mu))} \, \|t^{1-\theta} w_1\|^{\theta}_{L^{p_1}_*(L^{p_1}(\Sigma,X_1;\mu))} \\ &\le (1+\varepsilon)\, \|u\|^{\frac{(1-\theta)p}{p_0}+\frac{\theta p}{p_1}}_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)} \\ &= (1+\varepsilon)\, \|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}. \end{aligned}$$

Sending $\varepsilon \to 0+$ shows that inequality

$$\|u\|_{(L^{p_0}(\Sigma,X_0;\mu),L^{p_1}(\Sigma,X_1;\mu))_{\theta,p_0,p_1}} \le \|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}.$$

holds for step functions. Since the set of step functions is dense in the space $L^p(\Sigma, (X_0, X_1)_{\theta,p_0,p_1}; \mu)$ the claim of this theorem holds. $\square$

As an immediate consequence of Theorem B.1, we obtain the following corollary improving the statement in [65, Corollaire 1.1 of Chapter IV].

Corollary B.2. *Let (Σ, μ) be a σ-finite measure space, $1 \le p_0, p_1 \le \infty$ and $0 < \theta < 1$. Then for $1 \le p \le \infty$ satisfying the relation (B.1), one has that*

$$(L^{p_0}(\Sigma, \mu), L^{p_1}(\Sigma, \mu))_{\theta,p_0,p_1} = L^p(\Sigma, \mu)$$

with equal norms.

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