EXISTENCE, UNIQUENESS AND REGULARITY OF SOLUTIONS TO THE STOCHASTIC LANDAU-LIFSHITZ-SLONCZEWSKI EQUATION

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ABSTRACT. In this paper we are concerned with the stochastic Landau-Lifshitz-Slonczewski equation (LLS) that describes magnetisation of an inifnite nanowire evolving under current driven spin torque. The current brings into the system a multiplicative gradient noise that appears as a transport term in the equation. We prove the existence, uniqueness and regularity of pathwise solutions to the equation.

CONTENTS

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1. Introduction

In this paper we are concerned with the existence, uniqueness and regularity of solutions to the stochastic Landau-Lifshitz-Slonczewski equation (LLS) equation considered on real line, see (2). To the best of our knowledge , this is the first result on a system of stochastic PDEs that combines variational structure with the transport noise in the presence of geometric constraints, more precisely, with solutions taking values in a sphere.

Let us recall briefly the physical motivation for LLS equation, see [1, 10] for more details. A deterministic version of equation (2) was introduced in [16] in order to study the magnetisation dynamics of ferromagnetic elements in presence of electric current. If the ferromagnetic element is small enough (100 nanometers) then the interaction between the electric current and the magnetisation results in the current-induced magnetisation switching and spin wave emission. It is expected that good understanding of those effects will allow us to develop new types of current-controlled magnetic memories and current controlled magnetic oscillators.

Mathematical theory of the deterministic LLS equation is at an early stage. The case, when the ferromagnetic material fills in a 3-dimensional domain is studied in an important paper [10], where the existence and uniqueness of solutions is proved and their regularity is studied. A physically important case of a nanowire is a subject of ongoing intense research in physics. Mathematical analysis of dynamics of travelling domain walls and their stability was only recently initiated in [11, 13, 14].

The necessity to include random fluctuations (such as thermal noise) into the dynamics of magnetisation has been conjectured by physicists for many years (see for example [1, 2, 12]. The existence and uniqueness of solutions for the Landau-Lifshitz equation without the Slonczewski term but including random fluctuations was intensely studied in recent years [3, 7, 8].

Let us first recall briefly the formulation of the deterministic LLS equation. We will identify an infinite nanowire made of ferromagnetic material with a real line $\mathbb{R}$ and will denote by $m(t,x) \in \mathbb{R}^3$ the magnetisation vector at a time $t \geq 0$ and at a point $x \in \mathbb{R}$. For temperatures below the Curie point the length $|m(t,x)|$ of this vector is constant in (t,x) [2], hence can be assumed equal to 1:

$$m : [0,\infty) \times \mathbb{R} \to \mathbb{S}^2.$$

The LLS equation proposed in [16] describes the dynamics of the magnetisation vector subject to the spin-velocity field (electric current):

$$v : (0,\infty) \times \mathbb{R} \to \mathbb{R}.$$

It takes the form

$$\partial_t m = -m \times \partial_{xx} m - \alpha m \times (m \times \partial_{xx} m) - v\partial_x m + \gamma m \times (v\partial_x m) , \tag{1}$$

with $\alpha > 0$ and $\gamma \in \mathbb{R}$. The term $v\partial_x m$ is known as the adiabatic term and the non-adiabatic term is given by $\gamma m \times (v\partial_x m)$. For more details on the form of this equation see [10].

We will consider a version of equation (1) with the spin-velocity field perturbed by noise:

$$\partial_t m = -m \times \partial_{xx} m - \alpha m \times (m \times \partial_{xx} m) - \partial_x m \circ (v\,dt + dW) + (m \times \gamma \partial_x m) \circ (v\,dt + dW) , \tag{2}$$

where W is an infinite-dimensional Wiener process taking values in an appropriate function space. We emphasise, that noise arises in equation (2) in a way very different way from the way it appears in stochastic Landau-Lifshitz equations studied in [3, 4, 7, 8]. While in the aforementioned papers it is a thermal noise arising inside the magnetic domain and has bounded diffusion coefficient, in (2) it is a transport noise brought into the system by the electric current, and has the gradient of the solution as a diffusion coefficient. Therefore, analysis of this equation requires more delicate arguments. We will show that for every initial condition m_0 with

$$|m_0(x)| = 1, \quad \int_{\mathbb{R}} |\partial_x m_0|^2 \, dx < \infty, \tag{3}$$

there exists a unique pathwise and strong in PDEs sense solution to (2). We will use the observation made in [10] that under the constraint $|m(t,x)| = 1$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$, we have

$$\partial_x m = -m \times (m \times \partial_x m) \,,$$

hence equation (2) can be written in the form

$$dm = -m \times (\partial_{xx} m + \alpha m \times \partial_{xx} m) \, dt + m \times (m \times \partial_x m + \gamma \partial_x m) \circ (v \, dt + dW) \,, \tag{4}$$

with $m(0) = m_0$ satisfying (3). We will assume that W is a Wiener process taking values in $H^2(\mathbb{R})$ and will prove the existence and uniqueness of pathwise solutions to this equation, see Theorem 2.4 for details. Due to the presence of gradient noise of multiplicative type, we can prove this theorem only for the Wiener process with the covariance small enough, see Theorem 2.4 for precise formulation. Let us comment on the proof of this theorem. We start with the formulation of a semidiscrete approximation scheme that allows to construction approximating solutions that satisfy the sphere constraint. The same approach was used in [10] to study the deterministic equation (1). Then we obtain a set of uniform estimates for the approximate solutions. This step requires using quadratic interpolations and is technically much more complicated than in the case of the stochastic Landau-Lifshitz equation without transport. Next, we follow compactness type argument to prove the existence of a limiting point that is a strong in PDEs sense solution to stochastic LLS equation. Then we show uniqueness of pathwise solutions and use the Yamada-Watanabe theorem in the same way as in [3].

2. Semi-discrete scheme and the main result

2.1. **Notation.**

2.1.1. *Function spaces.* Let $\rho_w(x) = (1+x^2)^{-w}$ for $w \geq 0$. Clearly,

$$\rho_w(x) \in (0,1], \quad |\rho_w'(x)| \leq w \rho_w(x), \tag{5}$$

and for $w > \frac{1}{2}$, $\int_{\mathbb{R}} \rho_w(x) \, dx < \infty$. For $p \in [1,\infty)$, define the weighted Lebesgue space $\mathbb{L}^p_w$ by

$$\mathbb{L}^p_w = \left\{ f : \mathbb{R} \to \mathbb{R}^3; \int_{\mathbb{R}} |f(x)|^p \rho_w(x) \, dx < \infty \right\}.$$

If $w = 0$, then we will write $\mathbb{L}^p$ instead of $\mathbb{L}^p_0$. We will denote by $\mathbb{H}^1_w$ the Hilbert space

$$\mathbb{H}^1_w = \left\{ f \in \mathbb{L}^2_w; Df \in \mathbb{L}^2_w \right\} .$$

Let $0 < w_1 < w_2$. Then $\rho_{w_1} > \rho_{w_2}$ and the embeddings $\mathbb{L}^2 \hookrightarrow \mathbb{L}^2_{w_1} \hookrightarrow \mathbb{L}^2_{w_2}$ are continuous with

$$|f|_{\mathbb{L}^2_{w_2}} \leq |f|_{\mathbb{L}^2_{w_1}} \leq |f|_{\mathbb{L}^2}, \quad \forall f \in \mathbb{L}^2.$$

Moreover, the embeddings

$$\mathbb{H}^1_{w_1} \hookrightarrow \mathbb{L}^2_{w_2} \quad \text{and} \quad \mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^2 \hookrightarrow \mathbb{H}^1_{w_2}$$

are compact, where $\mathring{\mathbb{H}}^2$ stands for a standard homogeneous Sobolev space of functions $f : \mathbb{R} \to \mathbb{R}^3$ with weak derivatives $Df, D^2 f \in \mathbb{L}^2$. The Laplace operator Δ considered in $\mathbb{L}^2_w$ with the domain $\mathbb{H}^2_w$ is variational and the operator $A_1 = I - \Delta$ is invertible. For $\beta > 0$, let $\mathbb{H}^\beta_w$ denote the domain of $A_1^{\beta/2}$ endowed with the norm $|\cdot|_{\mathbb{H}^\beta_w} := |A_1^{\beta/2} \cdot|_{\mathbb{L}^2_w}$. and with dual space $\mathbb{H}^{-\beta}_w$.

2.1.2. *Assumption and Notation.* Let W be an $H^2(\mathbb{R})$-valued Wiener process with the covariance operator Q. Then there exists a complete orthonormal sequence $\{f_j;\, j\geq 1\}$ of $H^2(\mathbb{R})$ made of eigenvectors of Q, that is

$$Qf_j = q_j^2 f_j, \quad q^2 := \sum_j q_j^2 < \infty,$$

and then we have

$$W(t) = \sum_{j=1}^{\infty} q_j W_j(t) f_j\,.$$

The following is a standing assumption for the rest of the paper and it will not be enunciated again.

Assumption 2.1.

$$C_\kappa^2 := \left| \sum_{j=1}^{\infty} q_j^2 \left(f_j^2 + (f_j')^2 + (f_j'')^2 \right) \right|_{\mathbb{L}^\infty} < \infty, \tag{6}$$

and v is in $\mathcal{C}([0,T];H^1(\mathbb{R}))$ with

$$C_v := \operatorname*{ess\,sup}_{t\in\mathbb{R}_+} |v(t)|_{\mathbb{L}^\infty} < \infty\,. \tag{7}$$

Define a function

$$\kappa^2(x) = \sum_{j=1}^{\infty} q_j^2 f_j^2(x), \quad x\in\mathbb{R}\,.$$

Remark 2.2. *(a) Assumption 2.1 yields*

$$|\kappa|_{\mathbb{L}^\infty} \leq C_\kappa, \quad \text{and} \quad |\kappa\kappa'|_{\mathbb{L}^\infty} \leq C_\kappa^2\,.$$

(b) Every $\mathbb{H}^2$-valued finite-dimensional Wiener process satisfies (6) provided $f_j'' \in \mathbb{L}^\infty$ for $j\geq 1$.

The following notations will be used throughout the paper. Let $G : \mathbb{L}^\infty \cap \mathring{\mathbb{H}}^1 \to \mathbb{L}^2$ be defined as

$$G(m) = m\times(m\times\partial_x m) + m\times\gamma\partial_x m\,.$$

Let $\mathcal{G}(m) := G'(m)(G(m))$, which can be expressed as

$$\begin{aligned}\mathcal{G}(m) &= (\gamma^2 - |m|^2) m\times(m\times\partial_{xx} m) - 2\gamma|m|^2 m\times\partial_{xx}m - \gamma^2\partial_x m\times(m\times\partial_x m)\\ &\quad - |m\times\partial_x m|^2 m + \gamma\langle m, \partial_x m\rangle m\times\partial_x m.\end{aligned}$$

In the rest of the paper we consider equation (4) in its Itô form:

$$dm = \left(F(m) + \frac{1}{2}S(m)\right) dt + G(m)\,dW, \quad m(0) = m_0. \tag{8}$$

Here,

$$F(m) := m\times(m\times v\partial_x m) + \gamma m\times v\partial_x m - m\times(\partial_{xx}m + \alpha m\times\partial_{xx}m)$$

and the Stratonovich correction term $S(m)$ takes the form

$$\begin{aligned}S(m) &:= \kappa^2\mathcal{G}(m) + \kappa\kappa'\left[m\times(m\times G(m)) + \gamma m\times G(m)\right]\\ &= \kappa^2\mathcal{G}(m) + \kappa\kappa'\left[(\gamma^2 - |m|^2) m\times(m\times\partial_x m) - 2\gamma|m|^2 m\times\partial_x m\right].\end{aligned}$$

2.2. **Semi-discrete scheme.**

2.2.1. *Discrete operators and discrete spaces.* Let $\mathbb{Z}_h = \{x = kh : k \in \mathbb{Z}\}$ denote a discretization of the real line of mesh size $h > 0$. For $u : \mathbb{Z}_h \to \mathbb{R}^3$, we write $u^{\pm}(x)$ for $u(x \pm h)$, and we introduce discrete gradient and discrete Laplace operators:

$$\partial^h u = \frac{1}{h}\left(u^+ - u\right) \quad \text{and} \quad \Delta^h u = \frac{1}{h}(\partial^h u - \partial^h u^-). \tag{9}$$

Let $\mathbb{L}_h^\infty$, $\mathbb{L}_h^p$, $\mathbb{H}_h^1$ and $\mathbb{E}_h := \mathbb{L}_h^\infty \cap \mathring{\mathbb{H}}_h^1$ be discrete spaces equipped with respective norms:

$$|u|_{\mathbb{L}_h^\infty} = \sup_{x\in\mathbb{Z}_h} |u(x)|, \qquad |u|^p_{\mathbb{L}_h^p} = h \sum_{x\in\mathbb{Z}_h} |u(x)|^p,$$
$$|u|^2_{\mathbb{E}_h} = |u|^2_{\mathbb{L}_h^\infty} + |\partial^h u|^2_{\mathbb{L}_h^2}, \quad |u|^2_{\mathbb{H}_h^1} = |u|^2_{\mathbb{L}_h^2} + |\partial^h u|^2_{\mathbb{L}_h^2},$$

where $p \in [1, \infty)$.

We will say that $u : [0,T] \times \Omega \times \mathbb{Z}_h \to \mathbb{R}^3$ is an $\mathbb{E}_h$-valued progressively measurable process if for every $x \in \mathbb{Z}_h$ the process $u(\cdot, x)$ is progressively measurable and for every $t \in [0,T]$,

$$|u(t)|_{\mathbb{E}_h} < \infty, \quad \mathbb{P}\text{-a.s.}$$

In particular, the process $\{|u(t)|_{\mathbb{E}_h}; t \geq 0\}$ is progressively measurable.

Let $\mathcal{E}_h$ denote the space of $\mathbb{E}_h$-valued progressively measurable processes, with norm

$$|u|_{\mathcal{E}_h} = \sup_{t\in[0,T]} \left(\mathbb{E}\left[|u(t)|^2_{\mathbb{E}_h}\right]\right)^{\frac{1}{2}}.$$

2.2.2. *Discrete equation.* For $u \in \mathbb{E}_h$, we define

$$\begin{aligned} F^h(u) &= u \times \left(u \times v\partial^h u\right) + \gamma u \times v\partial^h u - u \times \left(\Delta^h u + \alpha u \times \Delta^h u\right) \\ G^h(u) &= u \times \left(u \times \partial^h u\right) + \gamma u \times \partial^h u \\ S^h(u) &= \mathcal{G}^h_\kappa(u) + \kappa\kappa' \left[u \times (u \times G^h(u)) + \gamma u \times G^h(u)\right], \end{aligned} \tag{10}$$

where

$$\begin{aligned} \mathcal{G}^h_\kappa(u) &= \frac{1}{2}\left((\kappa^2)^- + \kappa^2\right) G^h_1(u) + \kappa^2 G^h_2(u) + (\kappa^2)^- G^h_3(u) \\ G^h_1(u) &= (\gamma^2 - |u|^2) u \times \left(u \times \Delta^h u\right) - 2\gamma|u|^2 u \times \Delta^h u \\ G^h_2(u) &= -\gamma^2 \partial^h u \times \left(u \times \partial^h u\right) - |u \times \partial^h u|^2 u \\ G^h_3(u) &= 2\gamma\langle u, \partial^h u^-\rangle u \times \partial^h u. \end{aligned}$$

Fix a terminal time $T \in (0, \infty)$, we describe the semi-discrete scheme for (8) as a stochastic differential equation in the space $\mathbb{E}_h$:

$$dm^h = \left(F^h(m^h) + \frac{1}{2} S^h(m^h)\right) dt + G^h(m^h)\, dW, \quad m^h(0) = m_0 \in \mathbb{E}_h, \tag{11}$$

In (10) and (11), $\kappa^2, v(t)$ and $W(t)$ are the restrictions of the corresponding functions to $\mathbb{Z}_h$ for every $t \in [0,T]$. The term $S^h(m^h)$ is a discretised and a modified version of the Stratonovich correction $S(m)$. It is chosen to simplify the proof in Section 3.2 without affecting the limit. For example, $G^h_3(m^h)$ with the constant 2 does not match with the term $\gamma\langle m, \partial_x m\rangle m \times \partial_x m$ in $\mathcal{G}(u)$, but if $\langle m^h, \partial^h m^{h-}\rangle$ converges to 0 in a suitable space, then the constant will be irrelevant to the final equation.

2.3. **Main result.**

Definition 2.1. *We say that a progressively measurable process m defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ where W is a Wiener process, is a solution to equation (8) if*

(a) $|m(t,x)| = 1$ *(t,x)-a.e.*
(b) for every $T \in (0,\infty)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\partial_x m|^2_{\mathbb{L}^2} + \int_0^T |\partial_{xx} m|^2_{\mathbb{L}^2}\, dt\right] < \infty,$$

(c) for every $t \in [0,T]$ the following equality holds in $\mathbb{L}^2$:

$$m(t) - m_0 = \int_0^t \left(F(m(s)) + \frac{1}{2}S(m(s))\right) ds + \int_0^t G(m(s))\, dW(s), \quad \mathbb{P}\text{-a.s.} \tag{12}$$

Note that (a)–(d) above and Assumption 2.1 yield

$$\int_0^t \left(|F(m(s))|^2_{\mathbb{L}_2} + S(m(s))|^2_{\mathbb{L}^2} + |\kappa G(m(s))|^2_{\mathbb{L}^2}\right) ds < \infty,$$

hence the Bochner integral and the Itô integral in (12) are well defined in $\mathbb{L}^2$.

Lemma 2.3. *For every $h > 0$, let $|m_0|_{\mathbb{E}_h} \leq K_0$. Then there exists a unique solution m^h of the semi-discrete scheme (11) in $\mathcal{E}_h$ satisfying $|m^h(t,x)| = 1$, $\mathbb{P}$-a.s. for all $t \in [0,T]$ and $x \in \mathbb{Z}_h$.*

Theorem 2.4. *There exists a solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, m)$ of (8) in the sense of Definition 2.1, such that for $p \in [1,\infty)$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\partial_x m(t)|^p_{\mathbb{L}^2} + \left(\int_0^T |\partial_{xx} m(t)|^2_{\mathbb{L}^2}\, dt\right)^p\right] < \infty,$$

and for every $T > 0$ and $\alpha \in \left(0, \frac{1}{2}\right)$,

$$m - m_0 \in C^\alpha\left([0,T]; \mathbb{L}^2\right), \quad \mathbb{P}\text{-a.s.}$$

Moreover, there exists a convergent subsequence $\{m_h\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that m_h has the same law as a quadratic interpolation of m^h for every $h > 0$, and m is the $\mathbb{P}$-a.s. limit of $\{m_h\}$ in $\mathcal{C}([0,T]; \mathbb{H}^{-1}_w)$ for some $w \geq 1$.

Theorem 2.5. *The solution m of (8) is pathwise unique and therefore unique in law.*

3. Discretization

From the definition of discrete operators and the discrete $\mathbb{L}^p_h$ norm, we deduce following results.

Remark 3.1. *For $u : \mathbb{Z}_h \to \mathbb{R}^3$,*

(a) $\partial^h(\partial^h u) = \frac{1}{h}(\partial^h u^+ - \partial^h u) = \Delta^h u^+$,

(b) for any $p \in [1,\infty]$, $|u|_{\mathbb{L}_h^p} = |u^+|_{\mathbb{L}_h^p} = |u^-|_{\mathbb{L}_h^p}$, which implies $|\partial^h u|_{\mathbb{L}_h^p} = |\partial^h u^+|_{\mathbb{L}_h^p} = |\partial^h u^-|_{\mathbb{L}_h^p}$, and hence,

$$|\partial^h u|^2_{\mathbb{L}_h^p} \le \frac{4}{h^2}|u|^2_{\mathbb{L}_h^p}, \quad |\Delta^h u|^2_{\mathbb{L}_h^p} \le \frac{4}{h^2}|\partial^h u|^2_{\mathbb{L}_h^p},$$

(c) Lemma A.1 indicates $|u|_{\mathbb{L}_h^\infty} \le C|u|_{\mathbb{H}_h^1}$ for any $u \in \mathbb{H}_h^1$.

3.1. **Existence of a unique solution of the semi-discrete scheme.**

Lemma 3.2. *For every $h>0$, if $f,g:\mathbb{E}_h \to \mathbb{E}_h$ are locally Lipschitz and satisfy $f(0)=g(0)=0$, then $f\times g$, $\langle f,g\rangle$ and $\partial^h f$ are also locally Lipschitz on $\mathbb{E}_h$.*

The result in Lemma 3.2 is clear and we omit the proof here. Then we check that the coefficients in (11) are locally Lipschitz on $\mathbb{E}_h$.

Lemma 3.3. *For every $h>0$, $F^h, G^h, S^h : \mathbb{E}_h \to \mathbb{E}_h$ are locally Lipschitz on $\mathbb{E}_h$.*

Proof. Let $u,w \in \mathbb{E}_h$. It follows from Remark 3.1(b) that

$$\begin{aligned}|\partial^h u - \partial^h w|^2_{\mathbb{E}_h} &= |\partial^h(u-w)|^2_{\mathbb{L}_h^\infty} + |\partial^h\partial^h(u-w)|^2_{\mathbb{L}_h^2}\\ &\le \frac{4}{h^2}\left(|u-w|^2_{\mathbb{L}_h^\infty} + |\partial^h(u-w)|^2_{\mathbb{L}_h^2}\right) = \frac{4}{h^2}|u-w|^2_{\mathbb{E}_h},\end{aligned}$$

and

$$\begin{aligned}|\Delta^h u - \Delta^h w|^2_{\mathbb{E}_h} &= |\Delta^h(u-w)|^2_{\mathbb{L}_h^\infty} + |\partial^h\Delta^h(u-w)|^2_{\mathbb{L}_h^2}\\ &\le \frac{4}{h^2}\left(|\partial^h(u-w)|^2_{\mathbb{L}_h^\infty} + |\Delta^h(u-w)|^2_{\mathbb{L}_h^2}\right)\\ &\le \frac{16}{h^4}\left(|u-w|^2_{\mathbb{L}_h^\infty} + |\partial^h(u-w)|^2_{\mathbb{L}_h^2}\right) = \frac{16}{h^4}|u-w|^2_{\mathbb{E}_h}.\end{aligned}$$

By Lemma 3.2 and (7), F^h, G^h and S^h are locally Lipschitz. □

Proof of Lemma 2.3. For each $n \in \mathbb{N}$ and $r^h = F^h, S^h$ and G^h, define

$$r_n^h(u) = \begin{cases} r^h(u) & \text{if } |u|_{\mathbb{E}_h} \le n\\ r^h\left(\frac{nu}{|u|_{\mathbb{E}_h}}\right) & \text{if } |u|_{\mathbb{E}_h} > n.\end{cases}$$

Then F_n^h, S_n^h and G_n^h are Lipschitz on $\mathbb{E}_h$.

Fix $n \in \mathbb{N}$. Let $A_n : \mathcal{E}_h \to \mathcal{E}_h$ be given by

$$\begin{aligned}A_n(u)(t) &= m_0 + \int_0^t \left(F_n^h(u(s)) + \frac{1}{2}S_n^h(u(s))\right)ds + \int_0^t G_n^h(u(s))\,dW(s)\\ &= m_0 + I_n(t) + J_n(t).\end{aligned} \tag{13}$$

We first verify that $A_n(u) \in \mathcal{E}_h$ for $u \in \mathcal{E}_h$. Note that F_n^h and S_n^h are bounded on $\mathbb{E}_h$, with

$$\mathbb{E}\left[|I_n(t)|^2_{\mathbb{E}_h}\right] \le T\mathbb{E}\left[\int_0^t \left|F_n^h(u(s)) + \frac{1}{2}S_n^h(u(s))\right|^2_{\mathbb{E}_h} ds\right]$$

$$\leq C_1(h,n)T\mathbb{E}\left[\int_0^t |u(s)|^2_{\mathbb{E}_h} ds\right]$$
$$\leq C_1(h,n)T^2|u|^2_{\mathcal{E}_h},$$

for some constant C_1 that depends on h and n. For $J_n(t)$, we have

$$\sum_j q_j^2|f_jG^h(u(s))|^2_{\mathbb{L}^2_h} = \sum_j q_j^2\left|f_ju(s)\times\left(u(s)\times\partial^h u(s)\right)+\gamma f_ju(s)\times\partial^h u(s)\right|^2_{\mathbb{L}^2_h}$$
$$\leq 2|\kappa^2|_{\mathbb{L}^\infty_h}\left(|u(s)|^4_{\mathbb{L}^\infty_h}+\gamma^2|u(s)|^2_{\mathbb{L}^\infty_h}\right)|\partial^h u(s)|^2_{\mathbb{L}^2_h}.$$

where the last inequality holds by Tonelli's theorem. This together with Remark 3.1(b) implies

$$\sum_j q_j^2|\partial^h(f_jG^h(u(s)))|^2_{\mathbb{L}^2_h} \leq \frac{4}{h^2}\sum_j q_j^2|f_jG^h(u(s))|^2_{\mathbb{L}^2_h}$$
$$\leq \frac{8}{h^2}|\kappa^2|_{\mathbb{L}^\infty_h}\left(|u(s)|^4_{\mathbb{L}^\infty_h}+\gamma^2|u(s)|^2_{\mathbb{L}^\infty_h}\right)\ |\partial^h u(s)|^2_{\mathbb{L}^2_h}.$$

Then by the definition of G^h_n, the assumption (6) and Fubini's theorem,

$$\mathbb{E}\left[\int_0^t\sum_j q_j^2\left|f_jG^h_n(u(s))\right|^2_{\mathbb{H}^1_h}\,ds\right] \leq 2C_\kappa^2(n^4+\gamma^2n^2)\left(1+\frac{4}{h^2}\right)T\sup_{s\in[0,t]}\mathbb{E}\left[|u(s)|^2_{\mathbb{E}_h}\right]$$
$$= C_2(h,n,\kappa)T|u|^2_{\mathcal{E}_h},$$

for $C_2(h,n,\kappa)=2C_\kappa^2(n^4+\gamma^2n^2)(1+\frac{4}{h^2})T$. Thus, J_n is a $\mathbb{H}^1_h$-valued continuous martingale. By Lemma A.1 (or Remark 3.1(c)), there exists a constant $C>0$ such that

$$|J_n(t)|^2_{\mathbb{E}_h} = |J_n(t)|^2_{\mathbb{L}^\infty_h}+|\partial^h J_n(t)|^2_{\mathbb{L}^2_h} \leq (C^2+1)|J_n(t)|^2_{\mathbb{H}^1_h}. \tag{14}$$

From [5, Corollary 4.29],

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|J_n(t)|^2_{\mathbb{H}^1_h}\right] &= \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t G^h_n(u(s))\,dW(s)\right|^2_{\mathbb{H}^1_h}\right]\\ &\leq \mathbb{E}\left[\int_0^T\sum_j q_j^2\left|f_jG^h_n(u(s))\right|^2_{\mathbb{H}^1_h}\,ds\right]\\ &\leq C_2(h,n,\kappa)T|u|^2_{\mathcal{E}_h}.\end{aligned} \tag{15}$$

It follows from (14) and (15) that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|J_n(t)|^2_{\mathbb{E}_h}\right] \leq \left(C^2+1\right)C_2(h,n,\kappa)T|u|^2_{\mathcal{E}_h}. \tag{16}$$

Thus, $A_n(u)\in\mathcal{E}_h$ for $u\in\mathcal{E}_h$.

For $u,v\in\mathcal{E}_h$, there exists a constant $C>0$ such that

$$|A_n(v)-A_n(u)|^2_{\mathcal{E}_h} \leq C\sup_{t\in[0,T]}\mathbb{E}\left[\left|\int_0^t F^h_n(v(s))-F^h_n(u(s))\,ds\right|^2_{\mathbb{E}_h}\right]$$
$$+C\sup_{t\in[0,T]}\mathbb{E}\left[\left|\int_0^t\frac{1}{2}\left(S^h_n(v(s))-S^h_n(u(s))\right)\,ds\right|^2_{\mathbb{E}_h}\right]$$

$$+C\sup_{t\in[0,T]}\mathbb{E}\left[\left|\int_0^t\left(G_n^h(\nu(s))-G_n^h(u(s))\right)dW(s)\right|_{\mathbb{E}_h}^2\right].$$

Similarly, $M_n(t):=\int_0^t\left(G_n^h(\nu(s))-G_n^h(u(s))\right)dW(s)$ is a $\mathbb{H}_h^1$-valued continuous martingale, and replacing J_n by M_n in (14) and (15), we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}|M_n(t)|_{\mathbb{E}_h}^2\right]\le\mathbb{E}\left[\int_0^T\sum_j q_j^2\left|f_j\left(G_n^h(\nu(s))-G_n^h(u(s))\right)\right|_{\mathbb{H}_h^1}^2 ds\right]. \tag{17}$$

By construction, if $|\nu(s)|_{\mathbb{E}_h},|u(s)|_{\mathbb{E}_h}\le n$, then

$$\begin{aligned}|G_n^h(\nu(s))-G_n^h(u(s))|_{\mathbb{L}_h^2}&\le|\nu(s)-u(s)|_{\mathbb{L}_h^\infty}\left(|\nu(s)|_{\mathbb{L}_h^\infty}+|u(s)|_{\mathbb{L}_h^\infty}+|\gamma|\right)|\partial^h\nu(s)|_{\mathbb{L}_h^2}\\&\quad+\left(|u(s)|_{\mathbb{L}_h^\infty}^2+|\gamma||u(s)|_{\mathbb{L}_h^\infty}\right)|\partial^h\nu(s)-\partial^h u(s)|_{\mathbb{L}_h^2}\\&\le(3n^2+2|\gamma|n)\;|\nu(s)-u(s)|_{\mathbb{E}_h}.\end{aligned}$$

If $|\nu(s)|_{\mathbb{E}_h},|u(s)|_{\mathbb{E}_h}>n$, then let $n_s^\nu=n|\nu(s)|_{\mathbb{E}_h}^{-1}$ and $n_s^u=n|u(s)|_{\mathbb{E}_h}^{-1}$, we have

$$|n_s^\nu-n_s^u|\;|u|_{\mathbb{E}_h}\le|\nu(s)-u(s)|_{\mathbb{E}_h},$$

and

$$\begin{aligned}&|G_n^h(\nu(s))-G_n^h(u(s))|_{\mathbb{L}_h^2}\\&=|G^h(n_s^\nu\nu(s))-G^h(n_s^u u(s))|_{\mathbb{L}_h^2}\\&\le\left(n_s^\nu|\nu(s)-u(s)|_{\mathbb{L}_h^\infty}+|n_s^\nu-n_s^u||u(s)|_{\mathbb{L}_h^\infty}\right)\left(n_s^\nu|\nu(s)|_{\mathbb{L}_h^\infty}+n_s^u|u(s)|_{\mathbb{L}_h^\infty}+|\gamma|\right)n_s^\nu|\partial^h\nu(s)|_{\mathbb{L}_h^2}\\&\quad+\left(n_s^\nu|\partial^h\nu(s)-\partial^h u(s)|_{\mathbb{L}_h^2}+|n_s^\nu-n_s^u||\partial^h u(s)|_{\mathbb{L}_h^2}\right)\left((n_s^u)^2|u(s)|_{\mathbb{L}_h^\infty}^2+|\gamma|n_s^u|u(s)|_{\mathbb{L}_h^\infty}\right)\\&\le2(3n^2+2|\gamma|n)\;|\nu(s)-u(s)|_{\mathbb{E}_h}.\end{aligned}$$

If $|\nu(s)|_{\mathbb{E}_h}>n$ and $|u(s)|_{\mathbb{E}_h}\le n$, then

$$|n_s^\nu-1|\le n^{-1}|n-|\nu(s)|_{\mathbb{E}_h}|\le n^{-1}||u(s)|_{\mathbb{E}_h}-|\nu(s)|_{\mathbb{E}_h}|\le n^{-1}|u(s)-\nu(s)|_{\mathbb{E}_h},$$

implying that $|n_s^\nu-1|\;|u|_{\mathbb{E}_h}\le|u(s)-\nu(s)|_{\mathbb{E}_h}$ and

$$|G_n^h(\nu(s))-G_n^h(u(s))|_{\mathbb{L}_h^2}\le2(3n^2+2|\gamma|n)\;|\nu(s)-u(s)|_{\mathbb{E}_h}.$$

Similar result follow for $\partial^h(G_n^h(\nu)-G_n^h(u))$ using Remark 3.1(b). Then, by (17), Lemma 3.3 and Hölder's inequality, there exist constants $L_1(h,n,T)$ and $L_2(h,n,T)$ such that

$$\begin{aligned}|A_n(\nu)-A_n(u)|_{\mathcal{E}_h}^2&\le L_1(h,n,T)\sup_{t\in[0,T]}\mathbb{E}\left[\int_0^t|\nu(s)-u(s)|_{\mathbb{E}_h}^2\,ds\right]\\&\le L_2(h,n,T)|\nu-u|_{\mathcal{E}_h}^2.\end{aligned}$$

Consider the discrete equation

$$dm_n^h(t)=\left(F_n^h(m_n^h(t))+\frac{1}{2}S_n^h(m_n^h(t))\right)dt+G_n^h(m_n^h(t))\;dW(t), \tag{18}$$

with $m_n^h(0)=m_0\in\mathbb{E}_h$ on intervals $[(k-1)\tilde{T},k\tilde{T}]$ for $k\ge1$, where $\tilde{T}$ satisfies $L_2(h,n,\tilde{T})<1$. By the Banach fixed point theorem, there exists a unique solution $m_n^h\in\mathcal{E}_h$ of (18) on $[0,T]$.

Define the stopping times

$$\tau_n := \inf\{t \geq 0 : |m_n^h(t)|_{\mathbb{E}_h} > n\}, \quad \tau_n' := \inf\{t \geq 0 : |m_{n+1}^h(t)|_{\mathbb{E}_h} > n\}.$$

Let $\tau = \tau_n \wedge \tau_n'$. Then $A_n(m_{n+1}^h) = A_{n+1}(m_{n+1}^h)$ on $[0, \tau)$, and by (17),

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \left|m_{n+1}^h(t\wedge\tau) - m_n^h(t\wedge\tau)\right|_{\mathbb{E}_h}^2\right] &= \mathbb{E}\left[\sup_{t\in[0,T]} \left|A_n(m_{n+1}^h)(t\wedge\tau) - A_n(m_n^h)(t\wedge\tau)\right|_{\mathbb{E}_h}^2\right] \\ &\leq L_1(h,n,T)\mathbb{E}\left[\int_0^T \left|m_{n+1}^h(s) - m_n^h(s)\right|_{\mathbb{E}_h}^2 \, ds\right],\end{aligned}$$

which implies

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|m_{n+1}^h(t\wedge\tau) - m_n^h(t\wedge\tau)\right|_{\mathbb{E}_h}^2\right] = 0,$$

by Grönwall's lemma. Thus, $m_{n+1}^h(\cdot\wedge\tau) = m_n^h(\cdot\wedge\tau)$ and $\tau = \tau_n$, $\mathbb{P}$-a.s. and the discrete equation (11) admits a local solution $m^h(t) = m_n^h(t)$ for $t \in [0, \tau_n]$.

Recall that $m_0 \in \mathbb{E}_h$ and $|m_0(x)| = 1$ for all $x \in \mathbb{Z}_h$. Applying Itô's lemma to $\frac{1}{2}|m^h(t,x)|^2$,

$$\begin{aligned}\frac{1}{2}d|m^h(t,x)|^2 &= \left\langle F^h(m^h(t))(x) + \frac{1}{2}S^h(m^h(t))(x), m^h(t,x)\right\rangle dt + \frac{1}{2}\kappa^2(x)|G^h(m^h(t))(x)|^2 dt \\ &\quad + \left\langle G^h(m^h(t))(x), m^h(t,x)\right\rangle dW(t).\end{aligned}$$

By $\langle a, a\times b\rangle = 0$, for any $t \in [0, \tau_n]$ and $x \in \mathbb{Z}_h$,

$$\begin{aligned}&\left\langle F^h(m^h), m^h\right\rangle(t,x) = 0, \\ &\left\langle S^h(m^h), m^h\right\rangle(t,x) = \left\langle \kappa^2 G_2^h(m^h), m^h\right\rangle(t,x) = -\kappa^2(x)|G^h(m^h)|^2(t,x), \\ &\left\langle G^h(m^h), m^h\right\rangle(t,x) = 0.\end{aligned}$$

Therefore, $|m^h(t,x)| = |m_n^h(t,x)| = |m_0(x)| = 1$ for any $t \in [0, \tau_n]$ and $x \in \mathbb{Z}_h$.

For any fixed h and n, the unique solution m_n^h of (18) satisfies

$$\begin{aligned}\mathbb{E}\left[|m_n^h(t\wedge\tau_n)|_{\mathbb{E}_h}^2\right] &= \mathbb{E}\left[|m_n^h(t\wedge\tau_n)|_{\mathbb{L}_h^\infty}^2 + |\partial^h m_n^h(t\wedge\tau_n)|_{\mathbb{L}_h^2}^2\right] \\ &= 1 + \mathbb{E}\left[|\partial^h m_n^h(t\wedge\tau_n)|_{\mathbb{L}_h^2}^2\right].\end{aligned}$$

We apply Itô's lemma to $\frac{1}{2}|\partial^h m_n^h(t\wedge\tau_n)|_{\mathbb{L}_h^2}^2$,

$$\begin{aligned}&\frac{1}{2}|\partial^h m_n^h(t\wedge\tau_n)|_{\mathbb{L}_h^2}^2 - \frac{1}{2}|\partial^h m_0|_{\mathbb{L}_h^2}^2 \\ &= \int_0^{t\wedge\tau_n} \left\langle -\Delta^h m_n^h(s), F^h(m_n^h(s)) + \frac{1}{2}S^h(m_n^h(s))\right\rangle_{\mathbb{L}_h^2} ds \\ &\quad + \int_0^{t\wedge\tau_n} \frac{1}{2}\sum_j q_j^2 \left|\partial^h\left(f_j G^h(m_n^h(s))\right)\right|_{\mathbb{L}_h^2}^2 ds + \int_0^{t\wedge\tau_n} \left\langle -\Delta^h m_n^h(s), G^h(m_n^h(s))\, dW(s)\right\rangle_{\mathbb{L}_h^2}.\end{aligned}$$

Since $|m_n^h(t,x)| = 1$ for $(t,x) \in [0,\tau_n] \times \mathbb{Z}_h$, and

$$|\Delta^h m_n^h(t)|^2_{\mathbb{L}^2_h} + |\partial^h m_n^h(t)|^4_{\mathbb{L}^2_h} \leq \frac{8}{h^2} |\partial^h m_n^h(t)|^2_{\mathbb{L}^2_h},$$

there exist constants β_1 and β_2 that depend on h (not n) such that

$$\left\langle -\Delta^h m_n^h(s), F^h(m_n^h(s)) + \frac{1}{2} S^h(m_n^h(s)) \right\rangle_{\mathbb{L}^2_h} + \frac{1}{2} \sum_j q_j^2 \left| \partial^h \left(f_j G^h(m_n^h(s)) \right) \right|^2_{\mathbb{L}^2_h} \leq \beta_1(h) |\partial^h m_n^h(s)|^2_{\mathbb{L}^2_h},$$

and

$$\begin{aligned} \left\langle \Delta^h m_n^h(s), G^h(m_n^h(s)) \right\rangle^2_{\mathbb{L}^2_h} &\leq \frac{1}{4} \left(|m_n^h(s) \times \Delta^h m_n^h(s)|^2_{\mathbb{L}^2_h} + |\partial^h m_n^h(s)|^2_{\mathbb{L}^2_h} \right)^2 \\ &\leq \beta_2(h) |\partial^h m_n^h(s)|^2_{\mathbb{L}^2_h}. \end{aligned}$$

Then, by the boundedness of κ^2, the stochastic integral $\int_0^{t\wedge\tau_n} \langle \Delta^h m_n^h(s), G^h(m_n^h(s))\, dW(s) \rangle_{\mathbb{L}^2_h}$ is a square integrable continuous martingale for $m_n^h \in \mathcal{E}_h$, for every $h > 0$. Now, we have

$$\begin{aligned} |\partial^h m_n^h(t\wedge\tau_n)|^2_{\mathbb{L}^2_h} \leq |\partial^h m_0|^2_{\mathbb{L}^2_h} &+ 2 \int_0^{t\wedge\tau_n} \beta_1(h) |\partial^h m_n^h(s)|^2_{\mathbb{L}^2_h}\, ds \\ &- 2 \int_0^{t\wedge\tau_n} \left\langle \Delta^h m_n^h(s), G^h(m_n^h)(s)\, dW(s) \right\rangle_{\mathbb{L}^2_h}, \end{aligned}$$

where the stochastic integral part vanishes after taking expectation. We obtain

$$\mathbb{E}\left[|m_n^h(t\wedge\tau_n)|^2_{\mathbb{E}_h} \right] \leq \mathbb{E}\left[|m_0|^2_{\mathbb{E}_h} + 2\beta_1(h) \int_0^t |m_n^h(s\wedge\tau_n)|^2_{\mathbb{E}_h}\, ds \right].$$

By Grönwall's lemma,

$$\mathbb{E}\left[|m_n^h(t\wedge\tau_n)|^2_{\mathbb{E}_h} \right] \leq \mathbb{E}\left[|m_0|^2_{\mathbb{E}_h} \right] \exp\left(\int_0^t 2\beta_1(h) ds \right) \leq K(h,t). \tag{19}$$

By the definition of τ_n, the left-hand-side of (19) is greater than $n^2 \mathbb{P}(\tau_n \in [0,t])$, thus

$$\lim_{n\to\infty} \mathbb{P}(\tau_n \in [0,t]) \leq \lim_{n\to\infty} K(h,t) n^{-2} = 0.$$

In other words, $\tau_n \to \infty$, $\mathbb{P}$-a.s, as $n \to \infty$. Thus, the process $m^h(t) = \lim_{n\to\infty} m_n^h(t\wedge\tau_n)$ is the unique solution of the semi-discrete scheme (11). □

3.2. **Uniform estimates for the solution m^h of the discrete SDE.** For every $h > 0$, let

$$M^h(t) := \int_0^t \langle \Delta^h m^h(s), G^h(m^h(s))\, dW(s) \rangle_{\mathbb{L}^2_h}.$$

In the following lemma, we deduce an upper bound of the stochastic integral $M^h(t)$ which is used in the proof of Lemma 3.5 to obtain uniform estimates for m^h.

Lemma 3.4. *For any $p \in (0,\infty)$, there exists a constant b_p independent of h such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| M^h(t) \right|^p \right] \leq \frac{1}{2} b_p (1+|\gamma|)^p C_\kappa^p\, \mathbb{E}\left[\sup_{t\in[0,T]} |\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h} + \left(\int_0^T |m^h(t) \times \Delta^h m^h(t)|^2_{\mathbb{L}^2_h}\, dt \right)^p \right].$$

Proof. We observe that for every $j \geq 1$,

$$\begin{aligned}
&\langle \Delta^h m^h(t), q_j f_j G^h(m^h(t))\rangle_{\mathbb{L}^2_h} \\
&= h\sum_x q_j f_j \langle m^h \times \Delta^h m^h, m^h \times \partial^h m^h + \gamma \partial^h m^h\rangle(t,x) \\
&\leq (1+|\gamma|)h\sum_x \left|q_j f_j \partial^h m^h\right| \left|m^h \times \Delta^h m^h\right|(t,x) \\
&\leq (1+|\gamma|)\left(h\sum_x q_j^2 f_j^2 |\partial^h m^h|^2(t,x)\right)^{\frac{1}{2}} \left(h\sum_x |m^h \times \Delta^h m^h|^2(t,x)\right)^{\frac{1}{2}},
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_j \langle \Delta^h m^h(t), q_j f_j G^h(m^h(t))\rangle^2_{\mathbb{L}^2_h} &\leq (1+|\gamma|)^2\ |m^h(t)\times \Delta^h m^h(t)|^2_{\mathbb{L}^2_h} \left(h\sum_x\sum_j q_j^2 f_j^2 |\partial^h m^h|^2(t,x)\right) \\
&\leq (1+|\gamma|)^2\ C_\kappa^2\ |m^h(t)\times \Delta^h m^h(t)|^2_{\mathbb{L}^2_h}\ |\partial^h m^h(t)|^2_{\mathbb{L}^2_h}.
\end{aligned}$$

Then as in the proof of Lemma 2.3, for every fixed h,

$$\sum_j \langle \Delta^h m^h(t), q_j f_j G^h(m^h(t))\rangle^2_{\mathbb{L}^2_h} < \infty, \quad \mathbb{P}\text{-a.s.}$$

implying that $M^h(t)$ is a continuous martingale. By the Burkholder-Davis-Gundy inequality, for $p \in (0,\infty)$, there exists a constant b_p such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |M^h(t)|^p\right] &\leq b_p \mathbb{E}\left[\left(\int_0^T \sum_j \left\langle \Delta^h m^h(t), q_j f_j G^h(m^h(t))\right\rangle^2_{\mathbb{L}^2_h} dt\right)^{\frac{p}{2}}\right] \\
&\leq b_p(1+|\gamma|)^p C_\kappa^p\ \mathbb{E}\left[\left(\int_0^T\ |\partial^h m^h(t)|^2_{\mathbb{L}^2_h}\ |m^h(t)\times\Delta^h m^h(t)|^2_{\mathbb{L}^2_h}\ dt\right)^{\frac{p}{2}}\right].
\end{aligned}$$

Taking the supremum over t for $|\partial^h m^h(t)|^2_{\mathbb{L}^2_h}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |M^h(t)|^p\right] \\
&\leq b_p(1+|\gamma|)^p C_\kappa^p\ \mathbb{E}\left[\sup_{t\in[0,T]} |\partial^h m^h(t)|^p_{\mathbb{L}^2_h}\ \left(\int_0^T |m^h(t)\times\Delta^h m^h(t)|^2_{\mathbb{L}^2_h}\ dt\right)^{\frac{p}{2}}\right] \\
&\leq \frac{1}{2} b_p(1+|\gamma|)^p C_\kappa^p\ \mathbb{E}\left[\sup_{t\in[0,T]} |\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h} + \left(\int_0^T |m^h(t)\times\Delta^h m^h(t)|^2_{\mathbb{L}^2_h}\ dt\right)^{p}\right].
\end{aligned}$$

□

Lemma 3.5. *For any $p \in [1,\infty)$, assume that $\{(q_j, f_j)\}_{j\geq 1}$ satisfies*

$$\begin{aligned}
N_{1,p} &:= 1 - 4^{p-1} b_p (1+|\gamma|)^p C_\kappa^p > 0, \\
N_{2,p} &:= 2^p\left(\alpha - (1+2\gamma^2)C_\kappa^2 - \delta\right)^p - 4^{p-1} b_p (1+|\gamma|)^p C_\kappa^p > 0,
\end{aligned} \tag{20}$$

for some small $\delta > 0$, where b_p is the constant in Lemma 3.4. Let $|m_0|_{\mathbb{E}_h} \le K_0$. Then, there exist constants $K_{1,p}$ and $K_{2,p}$ that are independent of h, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\right] \le K_{1,p}, \tag{21}$$

$$\mathbb{E}\left[\left(\int_0^T |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\,ds\right)^p\right] \le K_{2,p}, \tag{22}$$

for all $h > 0$.

Proof. As in Lemma 2.3, let $\phi(u) = \frac{1}{2}|\partial^h u|^2_{\mathbb{L}^2_h}$ for $u \in \mathbb{E}_h$. Then, for $v, w \in \mathring{\mathbb{H}}^1_h$,

$$\phi'(u)v = \langle \partial^h u, \partial^h v\rangle_{\mathbb{L}^2_h} = -\langle \Delta^h u, v\rangle_{\mathbb{L}^2_h}, \quad \phi''(u)(v,w) = \langle \partial^h v, \partial^h w\rangle_{\mathbb{L}^2_h}.$$

By Itô's lemma,

$$\begin{aligned}\frac{1}{2}|\partial^h m^h(t)|^2_{\mathbb{L}^2_h} - \frac{1}{2}|\partial^h m^h(0)|^2_{\mathbb{L}^2_h} &= \phi(m^h(t)) - \phi(m^h(0)) \\ &= -\int_0^t \left\langle \Delta^h m^h(s), F^h(m^h)(s)\right\rangle_{\mathbb{L}^2_h} ds \\ &\quad - \int_0^t \left\langle \Delta^h m^h(s), \frac{1}{2}S^h(m^h)(s)\right\rangle_{\mathbb{L}^2_h} ds \\ &\quad + \frac{1}{2}\int_0^t \sum_j |\partial^h(q_j f_j G^h(m^h)(s))|^2_{\mathbb{L}^2_h}\,ds \\ &\quad - \int_0^t \langle \Delta^h m^h(s), G^h(m^h)(s)\,dW(s)\rangle_{\mathbb{L}^2_h} \\ &:= \int_0^t (T_1(s)+T_2(s)+T_3(s))\,ds - M^h(t),\end{aligned} \tag{23}$$

where $M^h(t)$ is already estimated in Lemma 3.4.

<u>An estimate on T_1:</u>

$$\begin{aligned}T_1(s) &= -\langle \Delta^h m^h(s), F^h(m^h)(s)\rangle_{\mathbb{L}^2_h} \\ &= \alpha\left\langle \Delta^h m^h(s), m^h(s)\times\left(m^h(s)\times\Delta^h m^h(s)\right)\right\rangle_{\mathbb{L}^2_h} \\ &\quad - \left\langle \Delta^h m^h(s), m^h(s)\times\left(m^h(s)\times v^h(s)\partial^h m^h(s) + \gamma v^h(s)\partial^h m^h(s)\right)\right\rangle_{\mathbb{L}^2_h} \\ &= -\alpha|m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h} \\ &\quad + \left\langle m^h(s)\times\Delta^h m^h(s), m^h(s)\times v^h(s)\partial^h m^h(s) + \gamma v^h(s)\partial^h m^h(s)\right\rangle_{\mathbb{L}^2_h}.\end{aligned} \tag{24}$$

The second term on the right hand side of (24) is estimated using (7) and the fact that $|m^h| = 1$ $\mathbb{P}$-a.s., as follows

$$\begin{aligned}&\left\langle m^h(s)\times\Delta^h m^h(s), m^h(s)\times v^h(s)\partial^h m^h(s) + \gamma v^h(s)\partial^h m^h(s)\right\rangle_{\mathbb{L}^2_h} \\ &\le \varepsilon^2|m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h} + \frac{1}{2\varepsilon^2}C_v^2(1+\gamma^2)|\partial^h m^h(s)|^2_{\mathbb{L}^2_h},\end{aligned} \tag{25}$$

for arbitrary $\varepsilon > 0$. An estimate of T_1 is obtained from (24) and (25)

$$T_1 \le (\varepsilon^2 - \alpha)|m^h(t) \times \Delta^h m^h(t)|^2_{\mathbb{L}^2_h} + \frac{1}{2\varepsilon^2} C_v^2 (1+\gamma^2)|\partial^h m^h(t)|^2_{\mathbb{L}^2_h}. \tag{26}$$

An estimate on T_2:

$$\begin{aligned}
T_2(s) &= -\frac{1}{2}\langle \Delta^h m^h(s), S^h(m^h(s))\rangle_{\mathbb{L}^2_h} \\
&= \frac{1}{2}\left\langle m^h \times \Delta^h m^h, \kappa\kappa' \left(m^h \times G^h(m^h) + \gamma G^h(m^h) \right) \right\rangle_{\mathbb{L}^2_h} \\
&\quad - \frac{1}{2}\left\langle \Delta^h m^h, \mathcal{G}^h_\kappa(m^h) \right\rangle_{\mathbb{L}^2_h} \\
&= T_{21} + T_{22}.
\end{aligned}$$

Using $|m^h| = 1$, $\mathbb{P}$-a.s., we estimate T_{21}:

$$\begin{aligned}
T_{21} &= \frac{1}{2}\left\langle m^h \times \Delta^h m^h, \kappa\kappa' \left[(\gamma^2 - 1) m^h \times \partial^h m^h + 2\gamma m^h \times (m^h \times \partial^h m^h) \right] \right\rangle_{\mathbb{L}^2_h} \\
&\le \frac{1}{2}\varepsilon^2 |m^h(s) \times \Delta^h m^h(s)|^2_{\mathbb{L}^2_h} + \frac{1}{4\varepsilon^2}\left((\gamma^2 - 1)^2 + 4\gamma^2 \right) |\kappa\kappa'|^2_{\mathbb{L}^\infty_h} |\partial^h m^h(s)|^2_{\mathbb{L}^2_h}.
\end{aligned} \tag{27}$$

where $|\kappa\kappa'|^2_{\mathbb{L}^\infty_h} \le C_\kappa^4$ by (6).

To estimate T_{22}, we first note that for any $u : \mathbb{Z}_h \to \mathbb{S}^2$,

$$\begin{aligned}
\mathcal{G}^h_\kappa(u) &= \frac{1}{2}\left((\kappa^2)^- + \kappa^2 \right) \left[(\gamma^2 - 1) u \times \left(u \times \Delta^h u \right) - 2\gamma u \times \Delta^h u \right] \\
&\quad - \gamma^2 \kappa^2 \partial^h u \times (u \times \partial^h u) \\
&\quad - \kappa^2 |u \times \partial^h u|^2 u + 2\gamma (\kappa^2)^- \langle u, (\partial^h u)^- \rangle u \times \partial^h u.
\end{aligned}$$

By (97), we deduce

$$\begin{aligned}
T_{22}(s) &= -\frac{1}{2} h \sum_x \left\langle \Delta^h m^h, \mathcal{G}^h_\kappa(m^h) \right\rangle (s,x) \\
&= \frac{1}{4}(\gamma^2 - 1) h \sum_x \left((\kappa^2)^- + \kappa^2 \right) |m^h \times \Delta^h m^h|^2 (s,x) \\
&\quad + \frac{1}{2}\gamma^2 h \sum_x \kappa^2 \left\langle \Delta m^h, \partial^h m^h \times (m^h \times \partial^h m^h) \right\rangle (s,x) \\
&\quad - \frac{1}{4} h \sum_x \kappa^2 |m^h \times \partial^h m^h|^2 \left(|\partial^h m^h|^2 + |(\partial^h m^h)^-|^2 \right) (s,x) \\
&\quad - \gamma h \sum_x (\kappa^2)^- \left\langle \Delta^h m^h, m^h \times \partial^h m^h \right\rangle \left\langle m^h, (\partial^h m^h)^- \right\rangle (s,x) \\
&= T_{22a}(s) + T_{22b}(s) + T_{22c}(s) + T_{22d}(s).
\end{aligned} \tag{28}$$

It is clear that

$$\begin{aligned}
T_{22a}(s) &= \frac{1}{4}(\gamma^2 - 1) h \sum_x \left((\kappa^2)^- + \kappa^2 \right) |m^h \times \Delta^h m^h|^2 (s,x) \\
&\le \frac{1}{2}\gamma^2 |\kappa|^2_{\mathbb{L}^\infty_h} |m^h \times \Delta^h m^h|^2_{\mathbb{L}^2_h}(s) - \frac{1}{4} h \sum_x \left((\kappa^2)^- + \kappa^2 \right) |m^h \times \Delta^h m^h|^2 (s,x),
\end{aligned} \tag{29}$$

where the second term on the right-hand side will cancel with parts of T_3.

To estimate T_{22b}, we observe that for any $u : \mathbb{Z}_h \to \mathbb{S}^2$, using (97),

$$\begin{aligned}\left\langle \Delta^h u, \partial^h u \times (u \times \partial^h u) \right\rangle &= |\partial^h u|^2 \left\langle u, \Delta^h u \right\rangle - \left\langle u, \partial^h u \right\rangle \left\langle \partial^h u, \Delta^h u \right\rangle \\ &= -\frac{1}{2}|\partial^h u|^2 \left(|\partial^h u|^2 + |\partial^h u^-|^2 \right) + \frac{1}{2}|\partial^h u|^2 \left(|\partial^h u|^2 - \left\langle \partial^h u, \partial^h u^- \right\rangle \right) \\ &= -\frac{1}{2}|\partial^h u|^2 |\partial^h u^-|^2 - \frac{1}{2}|\partial^h u|^2 \left\langle \partial^h u, \partial^h u^- \right\rangle, \end{aligned} \tag{30}$$

where

$$\left\langle \partial^h u, \partial^h u^- \right\rangle = \frac{1}{2} \left(|\partial^h u|^2 + |\partial^h u^-|^2 - h^2 |\Delta^h u|^2 \right). \tag{31}$$

If $|\partial^h u(x)| \le |\partial^h u^-(x)|$ at some $x \in \mathbb{Z}_h$, then

$$-\left\langle \partial^h u, \partial^h u^- \right\rangle (x) \le |\partial^h u^-(x)|^2,$$

and by (30),

$$\begin{aligned}\left\langle \Delta^h u, \partial^h u \times (u \times \partial^h u) \right\rangle (x) &\le -\frac{1}{2}|\partial^h u(x)|^2 |\partial^h u^-(x)|^2 + \frac{1}{2}|\partial^h u(x)|^2 |\partial^h u^-(x)|^2 \\ &= 0.\end{aligned}$$

If $|\partial^h u(x)| \ge |\partial^h u^-(x)|$, then we can show that the term given by (30) is bounded by $|u \times \Delta^h u|^2(x)$. Explicitly, by (31),

$$\begin{aligned}&|u \times \Delta^h u|^2(x) - \left\langle \Delta^h u, \partial^h u \times (u \times \partial^h u) \right\rangle (x) \\ &= |\Delta^h u|^2 - \left\langle u, \Delta^h u \right\rangle^2 + \frac{1}{2}|\partial^h u|^2 |\partial^h u^-|^2 + \frac{1}{2}|\partial^h u|^2 \left\langle \partial^h u, \partial^h u^- \right\rangle \\ &= |\Delta^h u|^2 - \frac{1}{4}\left(|\partial^h u|^2 + |\partial^h u^-|^2 \right)^2 + \frac{1}{2}|\partial^h u|^2 |\partial^h u^-|^2 + \frac{1}{4}|\partial^h u|^2 \left(|\partial^h u|^2 + |\partial^h u^-|^2 - h^2 |\Delta^h u|^2 \right) \\ &= \left(1 - \frac{1}{4}h^2 |\partial^h u|^2 \right) |\Delta^h u|^2 - \frac{1}{4}|\partial^h u^-|^4 + \frac{1}{4}|\partial^h u|^2 |\partial^h u^-|^2 \\ &\ge 0,\end{aligned}$$

where the last inequality holds by $h^2 |\partial^h u|^2 \le 4$ and $|\partial^h u(x)| \ge |\partial^h u^-(x)|$. Combining the two cases and replacing u by $m^h(s)$, we have

$$\begin{aligned}T_{22b}(s) &= \frac{1}{2}\gamma^2 h \sum_x \kappa^2 \left\langle \Delta^h m^h, \partial^h m^h \times (m^h \times \partial^h m^h) \right\rangle (s,x) \\ &\le \frac{1}{2}\gamma^2 h \sum_x \kappa^2 |m^h \times \Delta^h m^h|^2(s,x) \; \mathbb{1}_{\{|\partial^h m^h(s,x)| \ge |\partial^h m^{h-}(s,x)|\}} \\ &\le \frac{1}{2}\gamma^2 C_\kappa^2 \; |m^h \times \Delta^h m^h|^2_{\mathbb{L}^2_h}(s).\end{aligned} \tag{32}$$

We will see later in the proof that T_{22c} and T_{22d} also cancel with parts of T_3.

<u>An estimate on T_3:</u>

$$T_3(s) = \frac{1}{2}\sum_j \left| \partial^h (q_j f_j G^h(m^h)) \right|^2_{\mathbb{L}^2_h}(s)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_j q_j^2 \left|\partial^h (f_j G^h(m^h))\right|^2_{\mathbb{L}^2_h}(s) \\
&= \frac{1}{2} h \sum_x \sum_j q_j^2 \left|\partial^h \left(f_j m^h \times (m^h \times \partial^h m^h)\right)\right|^2 (s,x) + \frac{1}{2}\gamma^2 h \sum_x \sum_j q_j^2 \left|\partial^h (f_j m^h \times \partial^h m^h)\right|^2 (s,x) \\
&\quad + \gamma h \sum_x \sum_j q_j^2 \left\langle \partial^h \left(f_j m^h \times (m^h \times \partial^h m^h)\right), \partial^h \left(f_j m^h \times \partial^h m^h\right)\right\rangle (s,x) \\
&= T_{31}(s) + T_{32}(s) + T_{33}(s).
\end{aligned}
$$

We first estimate $T_{31}(s)$. For $u : \mathbb{Z}_h \to \mathbb{S}^2$, we have for every $j \geq 1$,

$$
\begin{aligned}
&\partial^h \left(f_j u \times (u \times \partial^h u)\right)(x) \\
&= \frac{1}{2}\left(\partial^h f_j\, u^+ \times (u^+ \times \partial^h u^+) + f_j \partial^h \left(u \times (u \times \partial^h u)\right)\right)(x) \\
&\quad + \frac{1}{2}\left(\partial^h f_j\, u \times (u \times \partial^h u) + f_j^+ \partial^h \left(u \times (u \times \partial^h u)\right)\right)(x) \\
&= \frac{1}{2}\partial^h f_j \left(u^+ \times (u^+ \times \partial^h u^+) + u \times (u \times \partial^h u)\right)(x) \\
&\quad + \frac{1}{2}\left[f_j \partial^h u \times (u \times \partial^h u) + f_j^+ u \times \partial^h (u \times \partial^h u)\right](x) \\
&\quad + \frac{1}{2}\left[f_j^+ \partial^h u \times (u^+ \times \partial^h u^+) + f_j u^+ \times \partial^h (u \times \partial^h u)\right](x) \\
&= \frac{1}{2}A_0(x) + \frac{1}{2}\left[A_1(x) + B_1(x)\right] + \frac{1}{2}\left[A_2(x) + B_2(x)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
A_0(x) &= \partial^h f_j \left(u^+ \times (u^+ \times \partial^h u^+) + u \times (u \times \partial^h u)\right)(x), \\
A_1(x) &= f_j \partial^h u \times (u \times \partial^h u), \\
A_2(x) &= f_j^+ \partial^h u \times (u^+ \times \partial^h u^+), \\
B_1(x) &= f_j^+ u \times \partial^h (u \times \partial^h u), \\
B_2(x) &= f_j u^+ \times \partial^h (u \times \partial^h u).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{2}\left|\partial^h \left(f_j u \times (u \times \partial^h u)\right)\right|^2 (x) \\
&= \frac{1}{2}\left(\frac{1}{4}|A_0|^2(x) + \frac{1}{4}|A_1 + B_1 + A_2 + B_2|^2(x) + \frac{1}{2}\langle A_0, A_1 + A_2 + B_1 + B_2\rangle (x)\right) \\
&\leq \frac{1}{2}\left(\frac{1}{4}|A_0|^2(x) + \frac{1}{2}|A_1 + B_1|^2(x) + \frac{1}{2}|A_2 + B_2|^2(x) + \frac{1}{2}\langle A_0, A_1 + A_2 + B_1 + B_2\rangle (x)\right) \\
&= \frac{1}{8}|A_0|^2(x) + \frac{1}{4}\left(|A_1|^2 + |B_1|^2 + |A_2|^2 + |B_2|^2\right)(x) \\
&\quad + \frac{1}{2}\langle A_1, B_1\rangle (x) + \frac{1}{2}\langle A_2, B_2\rangle (x) + \frac{1}{4}\langle A_0, A_1 + A_2 + B_1 + B_2\rangle (x).
\end{aligned}
$$

For the square $\frac{1}{8}|A_0|^2(x)$:

$$
\begin{aligned}
\frac{1}{8}h\sum_x\sum_j q_j^2|A_0|^2(x) &= \frac{1}{8}h\sum_x\sum_j q_j^2|\partial^h f_j|^2\ \left|u^+\times(u^+\times\partial^h u^+)+u\times(u\times\partial^h u)\right|^2(x)\\
&\leq \frac{1}{8}C_\kappa^2\, h\sum_x\left|u^+\times(u^+\times\partial^h u^+)+u\times(u\times\partial^h u)\right|^2(x)\\
&\leq \frac{1}{2}C_\kappa^2\ |\partial^h u|^2_{\mathbb{L}^2_h},
\end{aligned}
\tag{33}
$$

where the second inequality holds by applying the Mean Value Theorem to f_j on the interval $[x,x+h]$ for every $j\geq 1$, such that there exists some $\xi_h\in(x,x+h)$ satisfying

$$
\left|\frac{f_j(x+h)-f_j(x)}{h}\right| = |f_j'(\xi_h)|,
$$

and $|\sum_j q_j^2(f_j')^2|_{\mathbb{L}^\infty}\leq C_\kappa^2$ by assumption (6).

For the squares $\frac{1}{4}|A_1|^2(x)$ and $\frac{1}{4}|A_2|^2(x)$:

$$
\begin{aligned}
&\frac{1}{4}h\sum_x\sum_j q_j^2\left(|A_1|^2(x)+|A_2|^2(x)\right)\\
&=\frac{1}{4}\sum_x\sum_j q_j^2\left(\left|f_j\partial^h u\times(u\times\partial^h u)\right|^2+\left|f_j^+\partial^h u\times(u^+\times\partial^h u^+)\right|^2\right)(x)\\
&\leq \frac{1}{4}h\sum_x\sum_j q_j^2 f_j^2\ |\partial^h u|^2\ |u\times\partial^h u|^2(x)+\frac{1}{4}h\sum_x\sum_j q_j^2(f_j^+)^2\ |\partial^h u|^2\ |u^+\times\partial^h u^+|^2(x)\\
&=\frac{1}{4}h\sum_x\left(\sum_j q_j^2 f_j^2\right)|u\times\partial^h u|^2\left(|\partial^h u|^2+|\partial^h u^-|^2\right)(x)\\
&=\frac{1}{4}h\sum_x\kappa^2|u\times\partial^h u|^2\left(|\partial^h u|^2+|\partial^h u^-|^2\right)(x),
\end{aligned}
\tag{34}
$$

where the right-hand side cancels with $T_{22c}(s)$ in (28) when u is replaced with $m^h(s)$.

For the squares $\frac{1}{4}|B_1|^2(x)$ and $\frac{1}{4}|B_2|^2(x)$, we first observe that

$$
\partial^h\left(u\times\partial^h u\right)(x) = u^+\times\Delta^h u^+.
\tag{35}
$$

Then,

$$
\begin{aligned}
&\frac{1}{4}\left(|B_1|^2(x)+|B_2|^2(x)\right)\\
&=\frac{1}{4}(f_j^+)^2\left|u\times\left(u^+\times\Delta^h u^+\right)\right|^2(x)+\frac{1}{4}f_j^2\left|u^+\times\left(u^+\times\Delta^h u^+\right)\right|^2(x)\\
&\leq\frac{1}{4}\left((f_j^+)^2+f_j^2\right)\left|u^+\times\Delta^h u^+\right|^2(x).
\end{aligned}
$$

This implies that

$$\begin{aligned}\frac{1}{4}h\sum_x\sum_j q_j^2\left(|B_1|^2(x)+|B_2|^2(x)\right)&=\frac{1}{4}h\sum_x\sum_j q_j^2\left((f_j^-)^2+f_j^2\right)\left|u\times\Delta^h u\right|^2(x)\\&=\frac{1}{4}h\sum_x\left((\kappa^2)^-+\kappa^2\right)\left|u\times\Delta^h u\right|^2(x),\end{aligned}\tag{36}$$

where the right-hand side cancels with a part of the estimate for T_{22a} in (29) when $u=m^h(s)$ as aforementioned.

For the cross terms $\frac{1}{2}\langle A_1,B_1\rangle(x)$ and $\frac{1}{2}\langle A_2,B_2\rangle(x)$:

$$\begin{aligned}\langle A_1,B_1\rangle(x)&=f_jf_j^+\left\langle\partial^h u\times(u\times\partial^h u),u\times\partial^h(u\times\partial^h u)\right\rangle(x)\\&=f_jf_j^+\left\langle|\partial^h u|^2u-\langle\partial^h u,u\rangle\partial^h u,u\times\partial^h(u\times\partial^h u)\right\rangle(x)\\&=f_jf_j^+\langle u,\partial^h u\rangle\left\langle u\times\partial^h u,\partial^h(u\times\partial^h u)\right\rangle(x),\end{aligned}$$

and similarly,

$$\begin{aligned}\langle A_2,B_2\rangle(x)&=f_jf_j^+\left\langle\partial^h u\times(u^+\times\partial^h u^+),u^+\times\partial^h(u\times\partial^h u)\right\rangle(x)\\&=f_jf_j^+\langle u^+,\partial^h u\rangle\left\langle u^+\times\partial^h u^+,\partial^h(u\times\partial^h u)\right\rangle(x).\end{aligned}$$

Then,

$$\begin{aligned}&\langle A_1,B_1\rangle+\langle A_2,B_2\rangle(x)\\&=f_jf_j^+\left\langle\langle u,\partial^h u\rangle u\times\partial^h u+\langle u^+,\partial^h u\rangle u^+\times\partial^h u^+,\ \partial^h(u\times\partial^h u)\right\rangle(x).\end{aligned}\tag{37}$$

By (35), the left term in the inner product (37) can be simplified as

$$\begin{aligned}&\langle u,\partial^h u\rangle u\times\partial^h u+\langle u^+,\partial^h u\rangle u^+\times\partial^h u^+\\&=\left(\langle u,\partial^h u\rangle+\langle u^+,\partial^h u\rangle\right)u\times\partial^h u+\langle u^+,\partial^h u\rangle h\left(u^+\times\Delta^h u^+\right)\\&=\langle u^+,u^+-u\rangle\left(u^+\times\Delta^h u^+\right),\end{aligned}$$

where the second equality holds by observing $\langle u+u^+,\partial^h u\rangle(x)=0$ due to $|u(x)|=1$ for all x. Recall that the right term in the inner product (37) is $\partial^h(u\times\partial^h u)=u^+\times\Delta^h u^+$ by (35). Then,

$$\begin{aligned}\frac{1}{2}\left(\langle A_1,B_1\rangle+\langle A_2,B_2\rangle(x)\right)&=\frac{1}{2}f_jf_j^+\langle u^+,u^+-u\rangle\left|u^+\times\Delta^h u^+\right|^2(x)\\&\le\frac{1}{2}\left(f_j^2+(f_j^+)^2\right)|u^+\times\Delta^h u^+|^2(x)\end{aligned}$$

Taking the sum over $x\in\mathbb{Z}_h$,

$$\begin{aligned}\frac{1}{2}h\sum_x\sum_j q_j^2\left(\langle A_1,B_1\rangle+\langle A_2,B_2\rangle\right)(x)&\le\frac{1}{2}h\sum_x\left(\kappa^2+(\kappa^2)^+\right)|u^+\times\Delta^h u^+|^2(x)\\&\le C_\kappa^2\,|u\times\Delta^h u|^2_{\mathbb{L}^2_h}.\end{aligned}\tag{38}$$

For the cross term $\frac{1}{4}\langle A_0,\, A_1+A_2+B_1+B_2\rangle(x)$:

$$A_0(x) = \frac{f_j^+ - f_j}{h}\left(u^+\times(u^+\times\partial^h u^+) + u\times(u\times\partial^h u)\right)(x),$$

and

$$\begin{aligned}(A_1+A_2+B_1+B_2)(x) &= (f_j+f_j^+)\partial^h\left(u\times(u\times\partial^h u)\right)(x)\\ &= \frac{f_j+f_j^+}{h}\left(u^+\times(u^+\times\partial^h u^+) - u\times(u\times\partial^h u)\right)(x),\end{aligned}$$

which imply

$$\begin{aligned}&\frac{1}{4}\langle A_0,\, A_1+A_2+B_1+B_2\rangle(x)\\ &= \frac{1}{4h^2}\left((f_j^+)^2 - f_j^2\right)\left(|u^+\times(u^+\times\partial^h u^+)|^2 - |u\times(u\times\partial^h u)|^2\right)(x).\end{aligned}$$

Then, using again the Mean Value Theorem for $\Delta^h(f_j)^2$,

$$\begin{aligned}&\frac{1}{4}h\sum_x\sum_j q_j^2\langle A_0,\, A_1+A_2+B_1+B_2\rangle(x)\\ &= \frac{1}{4}h\sum_x\sum_j q_j^2\frac{1}{h^2}\left(f_j^2-(f_j^-)^2-(f_j^+)^2+f_j^2\right)|u\times(u\times\partial^h u)|^2(x)\\ &= -\frac{1}{4}h\sum_x\sum_j q_j^2\Delta^h\left(f_j^2\right)|u\times(u\times\partial^h u)|^2(x)\\ &\le \frac{1}{2}C_\kappa^2\,|\partial^h u|^2_{\mathbb{L}_h^2}.\end{aligned} \tag{39}$$

Therefore, by (33), (34), (36), (38) and (39),

$$\begin{aligned}T_{31}(s) &= \frac{1}{2}h\sum_x\left|\partial^h\left(f_j m^h\times(m^h\times\partial^h m^h)\right)\right|^2(s,x)\\ &\le C_\kappa^2|\partial^h m^h|^2_{\mathbb{L}_h^2}(s) + C_\kappa^2|m^h\times\Delta^h m^h|^2_{\mathbb{L}_h^2}(s)\\ &\quad + \frac{1}{4}h\sum_x\kappa^2|m^h\times\partial^h m^h|^2\left(|\partial^h m^h|^2 + |(\partial^h m^h)^-|^2\right)(s,x)\\ &\quad + \frac{1}{4}h\sum_x\left(\kappa^2+(\kappa^2)^-\right)|m^h\times\Delta^h m^h|^2(s,x).\end{aligned} \tag{40}$$

Next, we estimate $T_{32}(s)$. Using (35),

$$
\begin{aligned}
T_{32}(s) &= \frac{1}{2}\gamma^2 h \sum_x \sum_j q_j^2 |\partial^h (f_j m^h \times \partial^h m^h)|^2 (s,x) \\
&= \frac{1}{2}\gamma^2 h \sum_x \sum_j q_j^2 |(\partial^h f_j) m^h \times \partial^h m^h + f_j^+ (m^h \times \Delta^h m^h)^+|^2 (s,x) \\
&\le \gamma^2 h \sum_x \sum_j q_j^2 |\partial^h f_j|^2 \, |\partial^h m^h|^2 (s,x) + \gamma^2 h \sum_x \left(\sum_j q_j^2 f_j^2 \right) |m^h \times \Delta^h m^h|^2 (s,x) \\
&\le \gamma^2 C_\kappa^2 \left(|\partial^h m^h|^2_{\mathbb{L}^2_h}(s) + |m^h \times \Delta^h m^h|^2_{\mathbb{L}^2_h} \right).
\end{aligned}
\tag{41}
$$

Finally, we estimate $T_{33}(s)$. We note that for $u = u(x)$ with $|u(x)| = 1$ for all x and for all $j \ge 1$,

$$
\begin{aligned}
&\left\langle \partial^h \left(f_j u \times (u \times \partial^h u) \right), \partial^h \left(f_j u \times \partial^h u \right) \right\rangle \\
&= \left\langle (\partial^h f_j) u^+ \times (u^+ \times \partial^h u^+) + f_j \partial^h (u \times (u \times \partial^h u)), (\partial^h f_j) u^+ \times \partial^h u^+ + f_j (u \times \Delta^h u)^+ \right\rangle \\
&= (\partial^h f_j) f_j \left\langle u^+ \times (u^+ \times \partial^h u^+), (u \times \Delta^h u)^+ \right\rangle \\
&\quad + \left\langle f_j \partial^h u \times (u^+ \times \partial^h u^+) + f_j u \times (u \times \Delta^h u)^+, (\partial^h f_j) u^+ \times \partial^h u^+ + f_j (u \times \Delta^h u)^+ \right\rangle \\
&= (\partial^h f_j) f_j \left\langle (u^+ - u) \times (u^+ \times \partial^h u^+), (u \times \Delta^h u)^+ \right\rangle \\
&\quad + f_j^2 \left\langle \partial^h u \times (u^+ \times \partial^h u^+), u^+ \times \Delta^h u^+ \right\rangle.
\end{aligned}
$$

Since $\sum_j q_j^2 |f_j| |\partial^h f_j|(x) \le C_\kappa^2$ for all $x \in \mathbb{Z}_h$, we have

$$
\begin{aligned}
T_{33}(s) &= \gamma h \sum_x \sum_j q_j^2 \left\langle \partial^h \left(f_j m^h \times m^h \times \partial^h m^h \right), \partial^h \left(f_j m^h \times \partial^h m^h \right) \right\rangle (s,x) \\
&\le |\gamma| h \sum_x \sum_j q_j^2 |\partial^h f_j^-| \, |f_j^-| \left(\frac{1}{\varepsilon^2} |\partial^h m^h|^2 + \varepsilon^2 |m^h \times \Delta^h m^h|^2 \right) (s,x) \\
&\quad + \gamma h \sum_x \sum_j q_j^2 (f_j^-)^2 \left\langle m^h, (\partial^h m^h)^- \right\rangle \left\langle m^h \times \partial^h m^h, \Delta^h m^h \right\rangle (s,x) \\
&= |\gamma| C_\kappa^2 \left(\frac{1}{\varepsilon^2} |\partial^h m^h|^2_{\mathbb{L}^2_h}(s) + \varepsilon^2 |m^h \times \Delta^h m^h|^2_{\mathbb{L}^2_h}(s) \right) - T_{22d}(s),
\end{aligned}
\tag{42}
$$

where $T_{22d}(s)$ is given in (28).

An estimate on $T_1 + T_2 + T_3$:

$$
T_1 + T_2 + T_3 = T_1 + T_{21} + T_{22a} + T_{22b} + T_{22c} + T_{22d} + T_{31} + T_{32} + T_{33},
$$

where by (29) and (40),

$$
T_{22a}(s) + T_{22c}(s) + T_{31}(s) \le \left(\frac{1}{2}\gamma^2 + 1 \right) C_\kappa^2 |m^h \times \Delta^h m^h|^2_{\mathbb{L}^2_h}(s) + C_\kappa^2 |\partial^h m^h|^2_{\mathbb{L}^2_h}(s),
$$

and by (42),

$$
T_{22d}(s) + T_{33}(s) \le |\gamma| C_\kappa^2 \left(\frac{1}{\varepsilon^2} |\partial^h m^h|^2_{\mathbb{L}^2_h}(s) + \varepsilon^2 |m^h \times \Delta^h m^h|^2_{\mathbb{L}^2_h}(s) \right).
$$

Then, by (26), (27), (32) and (41),

$$T_1(s)+T_2(s)+T_3(s) \le \frac{1}{2}C_{1,\varepsilon}\, |\partial^h m^h(s)|^2_{\mathbb{L}^2_h} + \left(\frac{1}{2}C_{2,\varepsilon}-\alpha\right)\, |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}, \tag{43}$$

where

$$\begin{aligned} C_{1,\varepsilon} &:= \frac{1}{2\varepsilon^2}\left[\left(\gamma^2+1\right)^2 C_\kappa^4 + 4|\gamma| C_\kappa^2 + 2C_\nu^2(1+\gamma^2)\right] + 2\left(1+\gamma^2\right)C_\kappa^2, \\ C_{2,\varepsilon} &:= \left(4\gamma^2+2\right)C_\kappa^2 + \varepsilon^2\left(3+2|\gamma|C_\kappa^2\right). \end{aligned} \tag{44}$$

Uniform estimate of $\partial^h m^h$:

Using (23) and (43), we have

$$\begin{aligned} &|\partial^h m^h(t)|^2_{\mathbb{L}^2_h} + (2\alpha - C_{2,\varepsilon})\int_0^t |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\, ds \\ &\le |\partial^h m_0|^2_{\mathbb{L}^2_h} + C_{1,\varepsilon}\int_0^t |\partial^h m^h(s)|^2_{\mathbb{L}^2_h}\, ds + 2\sup_{t\in[0,T]}|M^h(t)|. \end{aligned} \tag{45}$$

Taking a sufficiently small ε such that $\frac{1}{2}\varepsilon^2\left(3+2|\gamma|C_\kappa^2\right) < \delta$, we have from (20):

$$2\alpha - C_{2,\varepsilon} > 0. \tag{46}$$

Then, for $p\ge 1$ and $q=\frac{p}{p-1}$,

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h} + (2\alpha-C_{2,\varepsilon})^p\left(\int_0^T |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\, ds\right)^p\right] \\ &\le \mathbb{E}\left[\left(\sup_{t\in[0,T]}|\partial^h m^h(t)|^2_{\mathbb{L}^2_h} + (2\alpha-C_{2,\varepsilon})\int_0^T |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\, ds\right)^p\right] \\ &\le \mathbb{E}\left[\left(|\partial^h m_0|^2_{\mathbb{L}^2_h} + C_{1,\varepsilon}\int_0^T |\partial^h m^h(s)|^2_{\mathbb{L}^2_h}\, ds + 2\sup_{t\in[0,T]}|M^h(t)|\right)^p\right] \\ &\le (2^{p-1})^2\mathbb{E}\left[|\partial^h m_0|^{2p}_{\mathbb{L}^2_h} + C^p_{1,\varepsilon}T^{\frac{p}{q}}\int_0^T \sup_{t\in[0,s]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\, ds\right] + 2^{p-1}\mathbb{E}\left[2^p\sup_{t\in[0,T]}|M^h(t)|^p\right] \\ &\le 4^{p-1}\mathbb{E}\left[|\partial^h m_0|^{2p}_{\mathbb{L}^2_h} + C^p_{1,\varepsilon}T^{\frac{p}{q}}\int_0^T \sup_{t\in[0,s]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\, ds\right] \\ &\quad + 4^{p-1}b_p(1+|\gamma|)^p C^p_\kappa\, \mathbb{E}\left[\sup_{t\in[0,T]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h} + \left(\int_0^T |m^h(t)\times\Delta^h m^h(t)|^2_{\mathbb{L}^2_h}\, dt\right)^p\right], \end{aligned}$$

where the second inequality holds by (45) and the last inequality holds by Lemma 3.4. Then, by the definitions of $N_{1,p}$ and $N_{2,p}$,

$$\begin{aligned} &N_{1,p}\mathbb{E}\left[\sup_{t\in[0,T]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\right] + N_{2,p}\mathbb{E}\left[\left(\int_0^T |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\, ds\right)^p\right] \\ &\le 4^{p-1}\mathbb{E}\left[|\partial^h m_0|^{2p}_{\mathbb{L}^2_h} + C^p_{1,\varepsilon}T^{\frac{p}{q}}\int_0^T \sup_{t\in[0,s]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\, ds\right]. \end{aligned} \tag{47}$$

Hence, by (20) and (47),

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\right] \le N_{1,p}^{-1}4^{p-1}\left(\mathbb{E}\left[|\partial^h m_0|^{2p}_{\mathbb{L}^2_h}\right]+C^p_{1,\varepsilon}T^{\frac{p}{q}}\mathbb{E}\left[\int_0^T \sup_{t\in[0,s]}|\partial^h m^h(t)|^{2p}_{\mathbb{L}^2_h}\,ds\right]\right). \tag{48}$$

By Fubini's theorem and Grönwall's inequality,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\partial^{+h} m^h(t)|^{2p}_{\mathbb{L}^2_h}\right] \le N_{1,p}^{-1}4^{p-1}\mathbb{E}\left[|\partial^h m_0|^{2p}_{\mathbb{L}^2_h}\right]\exp\left(\int_0^T N_{1,p}^{-1}4^{p-1}C^p_{1,\varepsilon}T^{\frac{p}{q}}\,dt\right) = K_{1,p}, \tag{49}$$

where $K_{1,p}$ depends on $p, C_\nu, C_\kappa, \varepsilon, T$ and K_0, but not on h, proving (21).

Finally, by (20), (47) and (49),

$$\mathbb{E}\left[\left(\int_0^T |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\,ds\right)^p\right] \le N_{2,p}^{-1}4^{p-1}\mathbb{E}\left[|\partial^h m_0|^{2p}_{\mathbb{L}^2_h}+C^p_{1,\varepsilon}T^{\frac{p}{q}+1}K_{1,p}\right] = K_{2,p},$$

where $K_{2,p}$ depends on $p, C_\nu, C_\kappa, \varepsilon, T$ and K_0, but not on h, proving (22). □

Remark 3.6. *Fix $p\in[1,\infty)$, if*

$$C_\kappa \le \frac{\alpha-\delta}{1+2\gamma^2+2^{1-\frac{2}{p}}b_p^{\frac{1}{p}}(1+|\gamma|)} \wedge \frac{1}{4b_p^{\frac{1}{p}}(1+|\gamma|)} \wedge 1-\delta,$$

then the assumption (20) of Lemma 3.5 is satisfied.

Lemma 3.7. *For any $p\in[1,\infty)$, under the conditions of Lemma 3.5, there exists a constant $K_{3,p}$ independent of h such that*

$$\mathbb{E}\left[\left(\int_0^T |\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}\,ds\right)^p\right] \le K_{3,p}. \tag{50}$$

Proof. Since $|m^h(s,x)|=1$ for all $(s,x)\in[0,T]\times\mathbb{Z}_h$, we have

$$|\Delta^h m^h(s,x)|^2 = |m^h(s,x)\times\Delta^h m^h(s,x)|^2+\langle m^h(s,x),\Delta^h m^h(s,x)\rangle^2.$$

Then,

$$\begin{aligned}|\Delta^h m^h(s)|^2_{\mathbb{L}^2_h} &= h\sum_{x\in\mathbb{Z}_h}\left(|m^h(s,x)\times\Delta^h m^h(s,x)|^2+\langle m^h(s,x),\Delta^h m^h(s,x)\rangle^2\right)\\ &= |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}+h\sum_{x\in\mathbb{Z}_h}\left(\frac{1}{2}(|\partial^h m^h(s,x)|^2+|(\partial^h m^h)^-(s,x)|^2)\right)^2\\ &\le |m^h(s)\times\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}+|\partial^h m^h(s)|^4_{\mathbb{L}^4_h},\end{aligned}$$

where

$$|\partial^h m^h(s)|^4_{\mathbb{L}^4_h} \le |\partial^h m^h(s)|^2_{\mathbb{L}^\infty_h}\,|\partial^h m^h(s)|^2_{\mathbb{L}^2_h}.$$

Applying Lemma A.1 on $\partial^h m^h(s)$,

$$|\partial^h m^h(s)|_{\mathbb{L}^\infty_h} \le C|\partial^h m^h(s)|^{\frac{1}{2}}_{\mathbb{L}^2_h}|\partial^h(\partial^h m^h(s))|^{\frac{1}{2}}_{\mathbb{L}^2_h}, \tag{51}$$

where $|\partial^h(\partial^h m^h(s))|_{\mathbb{L}^2_h} = |\Delta^h m^h(s)|_{\mathbb{L}^2_h}$. Thus,

$$\begin{aligned} |\partial^h m^h(s)|^4_{\mathbb{L}^4_h} &\leq C^2 |\partial^h m^h(s)|^3_{\mathbb{L}^2_h} \, |\Delta^h m^h(s)|_{\mathbb{L}^2_h} \\ &\leq \frac{1}{2} C^4 |\partial^h m^h(s)|^6_{\mathbb{L}^2_h} + \frac{1}{2} |\Delta^h m^h(s)|^2_{\mathbb{L}^2_h}. \end{aligned} \tag{52}$$

We have

$$|\Delta^h m^h(s)|^2_{\mathbb{L}^2_h} \leq 2|m^h(s) \times \Delta^h m^h(s)|^2_{\mathbb{L}^2_h} + C^4 |\partial^h m^h(s)|^6_{\mathbb{L}^2_h}.$$

Then, by Lemma 3.5,

$$\mathbb{E}\left[\left(\int_0^T |\Delta^h m^h(t)|^2_{\mathbb{L}^2_h} \, dt\right)^p\right] \leq 2^{p(p-1)} K_{2,p} + 2^{p-1} C^{4p} K_{1,3p} T^p = K_{3,p},$$

proving (50). □

4. Quadratic interpolation for the solution m^h of (11)

4.1. **Interpolations.** For any fixed $h > 0$, let $x_k = kh \in \mathbb{Z}_h$, for $k \in \mathbb{Z}$. We introduce interpolations of discrete functions defined on $\mathbb{Z}_h$ to functions defined on $\mathbb{R}$.

Given $u : \mathbb{Z}_h \to \mathbb{R}^3$, let $\overline{u} : \mathbb{R} \to \mathbb{R}^3$ denote a quadratic interpolation of u, given by

$$\overline{u}(x) = \frac{1}{2}\left(u(x_k) + u(x_{k-1})\right) + \partial^h u^-(x_k)(x - x_k) + \frac{1}{2}\Delta^h u(x_k)(x - x_k)^2, \tag{53}$$

for $x \in [x_k, x_{k+1})$, $k \in \mathbb{Z}$, where $\overline{u}$ is continuously differentiable with

$$\begin{aligned} D\overline{u}(x) &= \partial^h u^-(x_k) + \Delta^h u(x_k)(x - x_k), \qquad & x \in [x_k, x_{k+1}), \\ D^2\overline{u}(x) &= \Delta^h u(x_k), & x \in (x_k, x_{k+1}). \end{aligned} \tag{54}$$

Let $\widehat{u} : \mathbb{R} \to \mathbb{R}^3$ denote the piecewise constant interpolation of u, given by

$$\widehat{u}(x) = u(x_k), \quad x \in [x_k, x_{k+1}), \tag{55}$$

for any $k \in \mathbb{Z}$. In terms of $\widehat{u}$, we can express $\overline{u}$ as

$$\begin{aligned} \overline{u}(x) &= \frac{1}{2}\left(\widehat{u}(x) + \widehat{u^-}(x)\right) + \partial^h \widehat{u^-}(x)(x - x_k) + \frac{1}{2}\Delta^h \widehat{u}(x)(x - x_k)^2 \\ &= \widehat{u}(x) + \left(D\overline{u}(x) - D^2\overline{u}(x)(x - x_k)\right)\left(x - x_k - \frac{h}{2}\right) + \frac{1}{2}D^2\overline{u}(x)(x - x_k)^2, \end{aligned} \tag{56}$$

for $x \in [x_k, x_{k+1})$, $k \in \mathbb{Z}$.

We collect estimates of $\overline{u}$ and $\widehat{u}$ in terms of u in the following remark.

Remark 4.1. *Let $u : \mathbb{Z}_h \to \mathbb{R}^3$. Then*

$$\begin{aligned} &|\widehat{u}|_{\mathbb{L}^\infty} = |u|_{\mathbb{L}^\infty_h}, \\ &|\widehat{u}|_{\mathbb{L}^2_w} \leq |\widehat{u}|_{\mathbb{L}^2} = |u|_{\mathbb{L}^2_h}, \quad w > 0, \end{aligned}$$

and

$$
\begin{aligned}
&|\overline{u}|_{\mathbb{L}^\infty} \le 5|u|_{\mathbb{L}^\infty_h},\\
&|D\overline{u}|_{\mathbb{L}^2} \le 3|\partial^h u|_{\mathbb{L}^2_h},\\
&|D^2\overline{u}|_{\mathbb{L}^2} = |\Delta^h u|_{\mathbb{L}^2_h},\\
&|D\overline{u}|^4_{\mathbb{L}^4} \le \frac{1}{2}C^4|D\overline{u}|^6_{\mathbb{L}^2} + \frac{1}{2}|D^2\overline{u}|^2_{\mathbb{L}^2}.
\end{aligned} \tag{57}
$$

4.2. An equation and estimates for $\overline{m}^h$.

4.2.1. *Equation for* $\overline{m}^h$. Since m^h is the solution of the semi-discrete scheme (11), the piecewise constant interpolation $\widehat{m}^h$ satisfies

$$
\widehat{m}^h(t) = m_0 + \int_0^t \left(F^h_{\widehat{\nu}}(\widehat{m}^h(s)) + \frac{1}{2}S^h_{\widehat{\kappa}}(\widehat{m}^h(s))\right) ds + \int_0^t G^h(\widehat{m}^h(s))\, d\widehat{W}^h(s), \tag{58}
$$

where $F^h_{\widehat{\nu}}$ and $S^h_{\widehat{\kappa}}$ are defined as in (10) but with $\widehat{\nu}$, $\widehat{\kappa}$ and $\widehat{\kappa\kappa'}$ in place of ν, κ and $\kappa\kappa'$ respectively, and

$$
\widehat{W}^h(t) := \sum_{j=1}^{\infty} q_j W_j(t) \widehat{f}_j, \quad t \in [0,T].
$$

In particular, for every fixed $h > 0$, $m^h \in \mathcal{C}([0,T];\mathbb{E}_h)$ and $\widehat{m}^h \in \mathcal{C}([0,T];\mathbb{L}^2_w \cap \mathring{\mathbb{H}}^1)$ for $w \ge 1$.

In order to obtain an equation for $\overline{m}^h$, by using (56) we note that

$$
\begin{aligned}
&\widehat{m}^h = \overline{m}^h - R_0\overline{m}^h, \quad \partial^h\widehat{m}^h = D\overline{m}^h - R_1\overline{m}^h,\\
&\widehat{m}^h \times \partial^h\widehat{m}^h = \overline{m}^h \times D\overline{m}^h - P_1\overline{m}^h, \quad \widehat{m}^h \times \left(\widehat{m}^h \times \partial^h\widehat{m}^h\right) = \overline{m}^h \times \left(\overline{m}^h \times D\overline{m}^h\right) - P_2\overline{m}^h,\\
&|\widehat{m}^h \times \partial^h\widehat{m}^h|^2\widehat{m}^h = \left|\overline{m}^h \times D\overline{m}^h\right|^2 \overline{m}^h - P_3\overline{m}^h,\\
&\partial^h\widehat{m}^h \times \left(\widehat{m}^h \times \partial^h\widehat{m}^h\right) = D\overline{m}^h \times \left(\overline{m}^h \times D\overline{m}^h\right) - P_4\overline{m}^h,\\
&\langle \widehat{m}^h, \partial^h\widehat{m}^{h-}\rangle\ \widehat{m}^h \times \partial^h\widehat{m}^h = \langle \overline{m}^h, D\overline{m}^h\rangle\ \overline{m}^h \times D\overline{m}^h - P_5\overline{m}^h,\\
&\widehat{m}^h \times \Delta^h\widehat{m}^h = \overline{m}^h \times D^2\overline{m}^h - Q_1\overline{m}^h, \quad \widehat{m}^h \times \left(\widehat{m}^h \times \Delta^h\widehat{m}^h\right) = \overline{m}^h \times \left(\overline{m}^h \times D^2\overline{m}^h\right) - Q_2\overline{m}^h,
\end{aligned}
$$

where for $u : \mathbb{R} \to \mathbb{R}^3$ with well-defined weak derivatives,

$$
\begin{aligned}
R_0u(x) &:= \left(Du(x) - D^2u(x)(x - x_k)\right)\left(x - x_k - \frac{h}{2}\right) + \frac{1}{2}D^2u(x)(x - x_k)^2,\\
R_1u(x) &:= D^2u(x)(x - x_k - h),\\
P_1u(x) &:= R_0u(x) \times Du(x) + (u - R_0u)(x) \times R_1u(x),
\end{aligned} \tag{59}
$$

and

$$
\begin{aligned}
P_2u(x) &:= [u\times P_1u + R_0u\times(u\times Du - P_1u)](x),\\
P_3u(x) &:= \left[\langle 2u\times Du - P_1u, P_1u\rangle u + |u\times Du - P_1u|^2 R_0u\right](x),\\
P_4u(x) &:= [R_1u\times(u\times Du) + (Du - R_1u)\times P_1u](x),\\
P_5u(x) &:= \left(\langle R_0u(x), Du(x)\rangle + \langle (u-R_0u)(x), D^2u(x)(x-x_k)\rangle\right)u(x)\times Du(x)\\
&\quad + \langle (u-R_0u)(x), Du(x) - D^2u(x)(x-x_k)\rangle P_1u(x),\\
Q_1u(x) &:= R_0u(x)\times D^2u(x),\\
Q_2u(x) &:= \left[u\times Q_1u + R_0u\times\left((u-R_0u)\times D^2u\right)\right](x),
\end{aligned}
\tag{60}
$$

for $x\in[x_k,x_{k+1})$, $k\in\mathbb{Z}$.

Moreover, for $u:[0,T]\times\mathbb{R}\to\mathbb{R}^3$, define

$$
\begin{aligned}
F_{\widehat{\nu}}(u) &:= -u\times\left(D^2u + \alpha u\times D^2u\right) + \widehat{\nu}(u\times(u\times Du) + \gamma u\times Du),\\
S_{\widehat{\kappa}}(u) &:= \frac{1}{2}\left((\widehat{\kappa}^-)^2 + \widehat{\kappa}^2\right)\left((\gamma^2-1)u\times\left(u\times D^2u\right) - 2\gamma u\times D^2u\right)\\
&\quad - \widehat{\kappa}^2\left(\gamma^2 Du\times(u\times Du) + |u\times Du|^2u\right) + 2\gamma(\widehat{\kappa}^2)^-\langle u, Du\rangle u\times Du\\
&\quad + \widehat{\kappa\kappa'}\left[(\gamma^2-1)u\times(u\times Du) - 2\gamma u\times Du\right].
\end{aligned}
\tag{61}
$$

By (58), we arrive at the equation of $\overline{m}^h$. For $t\in[0,T]$,

$$
\begin{aligned}
\overline{m}^h(t) &= m_0 + \int_0^t F_{\widehat{\nu}}(\overline{m}^h(s))\,ds + \frac{1}{2}\int_0^t S_{\widehat{\kappa}}(\overline{m}^h(s))\,ds + \int_0^t G(\overline{m}^h(s))\,d\widehat{W}^h(s)\\
&\quad + R_0\overline{m}^h(t) + \int_0^t\left(-\widehat{\nu}(\gamma P_1 + P_2)\overline{m}^h + Q_1\overline{m}^h + \alpha Q_2\overline{m}^h\right)(s)\,ds\\
&\quad + \frac{1}{4}\int_0^t\left((\widehat{\kappa}^2)^- + \widehat{\kappa}^2\right)\left[2\gamma Q_1\overline{m}^h - (\gamma^2-1)Q_2\overline{m}^h\right](s)\,ds\\
&\quad + \frac{1}{2}\int_0^t\left(\widehat{\kappa}^2(P_3\overline{m}^h + \gamma^2P_4\overline{m}^h) - 2\gamma(\widehat{\kappa}^2)^-P_5\overline{m}^h\right)(s)\,ds\\
&\quad + \frac{1}{2}\int_0^t\widehat{\kappa\kappa'}\left[2\gamma P_1\overline{m}^h - (\gamma^2-1)P_2\overline{m}^h\right](s)\,ds\\
&\quad - \int_0^t(\gamma P_1 + P_2)\overline{m}^h(s)\,d\widehat{W}^h(s).
\end{aligned}
\tag{62}
$$

4.2.2. *Estimates for $\overline{m}^h$.* For $p\in[1,\infty)$ and $w\geq 1$, we deduce from Lemmata 3.5, 3.7 and Remark 4.1:

$$
\sup_{t\in[0,T]}|\overline{m}^h(t)|^p_{\mathbb{L}^\infty} \leq 5^p, \quad \mathbb{P}\text{-a.s.} \tag{63}
$$

and

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|D\overline{m}^h(t)|^{2p}_{\mathbb{L}^2} + \left(\int_0^T\left(|\overline{m}^h(t)|^p_{\mathbb{L}^p_w} + |D\overline{m}^h(t)|^4_{\mathbb{L}^4} + |D^2\overline{m}^h(t)|^2_{\mathbb{L}^2}\right)dt\right)^p\right] \leq C(p,T,w), \tag{64}
$$

for some constant $C(p,T,w)$.

Results of convergence of $R_0\overline{m}^h$, $R_1\overline{m}^h$, $P_1\overline{m}^h, \dots, P_5\overline{m}^h$ and $Q_1\overline{m}^h$, $Q_2\overline{m}^h$ are proved in the following lemma.

Lemma 4.2. *For $f = R_1$, P_1 or P_2,*

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}|R_0\overline{m}^h(t)|^2_{\mathbb{L}^2}+\int_0^T|f\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right]=0.$$

Moreover, for $f = P_3$, P_4, P_5, Q_1 or Q_2, for any measurable process $\varphi \in L^4(\Omega; L^4(0,T;\mathbb{L}^4))$,

$$\lim_{h\to 0}\mathbb{E}\left[\int_0^T\left\langle f\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\right]=0.$$

Proof. By construction, $|x-x_k|<h$ for $x\in[x_k,x_{k+1})$, $k\in\mathbb{Z}$, and $\sup_{t\in[0,T]}|\overline{m}^h(t)|_{\mathbb{L}^\infty}\le 5$, $\mathbb{P}$-a.s. Thus, for $p\in[1,\infty)$, there is a constant C_{R_0} independent of h such that

$$\sup_{t\in[0,T]}|R_0\overline{m}^h(t)|^p_{\mathbb{L}^\infty}\le C^p_{R_0},\quad \mathbb{P}\text{-a.s.}$$

Using (54), we can often re-write $R_0,\dots,Q_2$ in terms of $\widehat{m}^h$ to simplify the estimates.

An estimate on $R_0\overline{m}^h$:

$$R_0\overline{m}^h(t,x)=\partial^h\widehat{m}^h(t,x-h)\left(x-x_k-\frac{h}{2}\right)+\frac{1}{2}\Delta^h\widehat{m}(t,x)(x-x_k)^2,\quad x\in[x_k,x_{k+1}],$$

which implies

$$\mathbb{E}\left[\sup_{t\in[0,T]}|R_0\overline{m}^h(t)|^2_{\mathbb{L}^2}\right]\le\frac{h^2}{2}\mathbb{E}\left[\sup_{t\in[0,T]}\left(|\partial^h\widehat{m}^h(t)|^2_{\mathbb{L}^2}+|\partial^h\widehat{m}^h(t)-\partial^h\widehat{m}^{h-}(t)|^2_{\mathbb{L}^2}\right)\right].\tag{65}$$

The expectation on the right-hand side of (65) is bounded by Lemma 3.5, thus the left-hand side converges to 0 as $h\to 0$. As a result,

$$\mathbb{E}\left[\int_0^T|R_0\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\le\mathbb{E}\left[C^2_{R_0}T\sup_{t\in[0,T]}|R_0\overline{m}^h(t)|^2_{\mathbb{L}^2}\right]\overset{h\to 0}{\to}0.\tag{66}$$

An estimate on $R_1\overline{m}^h$:

$$\mathbb{E}\left[\int_0^T|R_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right]\le\frac{h^2}{2}\mathbb{E}\left[\int_0^T|D^2\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dx\,dt\right],$$

implying $R_1\overline{m}^h\to 0$ in $L^2(\Omega;L^2(0,T;\mathbb{L}^2))$ by (64).

An estimate on $P_1\overline{m}^h$:

$$\begin{aligned}\mathbb{E}\left[\int_0^T|P_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right]&=\mathbb{E}\left[\int_0^T|R_0\overline{m}^h(t)\times D\overline{m}^h(t)+\widehat{m}^h(t)\times R_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right]\\&\le 2\left(\mathbb{E}\left[\int_0^T|R_0\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12}\left(\mathbb{E}\left[\int_0^T|D\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12}\\&\quad+2\mathbb{E}\left[\int_0^T|R_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right],\end{aligned}$$

where $D\overline{m}^h \in L^4(\Omega; L^4(0,T;\mathbb{L}^4))$ for all $h>0$ by (64). Then, by the L^4-convergence of $R_0\overline{m}^h$ in (66) and the L^2-convergence of $R_1\overline{m}^h$, we have $P_1\overline{m}^h \to 0$ in $L^2(\Omega; L^2(0,T;\mathbb{L}^2))$.

An estimate on $P_2\overline{m}^h$:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |P_2\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right] &= \mathbb{E}\left[\int_0^T |\widehat{m}^h(t)\times P_1\overline{m}^h(t) + R_0\overline{m}^h(t)\times(\overline{m}^h(t)\times D\overline{m}^h(t))|^2_{\mathbb{L}^2}\,dt\right] \\
&\le 2\mathbb{E}\left[\int_0^T |\widehat{m}^h(t)\times P_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right] \\
&\quad + 2\left(\mathbb{E}\left[\int_0^T |R_0\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12}\left(\mathbb{E}\left[\int_0^T |\overline{m}^h(t)\times D\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12},
\end{aligned}$$

which implies that $P_2\overline{m}^h \to 0$ in $L^2(\Omega;L^2(0,T;\mathbb{L}^2))$ by (63) and (64) together with the convergences of $P_1\overline{m}^h$ and $R_0\overline{m}^h$.

An estimate on $P_3\overline{m}^h$:

$$\begin{aligned}
P_3\overline{m}^h &= |\widehat{m}^h\times\partial^h\widehat{m}^h|^2 R_0\overline{m}^h + \left\langle \overline{m}^h\times D\overline{m}^h + \widehat{m}^h\times\partial^h\widehat{m}^h, P_1\overline{m}^h\right\rangle \overline{m}^h \\
&=: P_{31}\overline{m}^h + P_{32}\overline{m}^h.
\end{aligned}$$

Then for $\varphi \in L^4(\Omega;L^4(0,T;\mathbb{L}^4))$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\langle P_{31}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\,dt\right] &\le \left(\mathbb{E}\left[\int_0^T |\widehat{m}^h(t)\times\partial^h\widehat{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12} \\
&\quad\times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac14}\left(\mathbb{E}\left[\int_0^T |R_0\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac14},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\langle P_{32}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}dt\right] &\le 5\left(\mathbb{E}\left[\int_0^T \left|\overline{m}^h(t)\times D\overline{m}^h(t) + \widehat{m}^h(t)\times\partial^h\widehat{m}^h(t)\right|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac14} \\
&\quad\times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac14}\left(\mathbb{E}\left[\int_0^T |P_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right]\right)^{\frac12}.
\end{aligned}$$

By Lemmata 3.5 and 3.7, (63), (64) and the property of φ, the expectations on the right-hand side of the two inequalities above are finite. Then by the convergences of $R_0\overline{m}^h$ and $P_1\overline{m}^h$, we obtain the weak convergence of $P_3\overline{m}^h$ as desired.

An estimate on $P_4\overline{m}^h$:

$$\begin{aligned}
P_4\overline{m}^h &= R_1\overline{m}^h\times(\overline{m}^h\times D\overline{m}^h) + \partial^h\widehat{m}^h\times P_1\overline{m}^h \\
&=: P_{41}\overline{m}^h + P_{42}\overline{m}^h.
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\langle P_{41}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\,dt\right] &\le 5\left(\mathbb{E}\left[\int_0^T |R_1\overline{m}^h(t)|^2_{\mathbb{L}^2}\,dt\right]\right)^{\frac12}\left(\mathbb{E}\left[\int_0^T |D\overline{m}^h(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac14} \\
&\quad\times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac14},
\end{aligned}$$

and

$$\mathbb{E}\left[\int_0^T \left\langle P_{42}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\,dt\right] \le \left(\mathbb{E}\left[\int_0^T |P_1\overline{m}^h(t)|_{\mathbb{L}^2}^2\,dt\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T |\partial^h\widehat{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}} \times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}.$$

Using (64), (52) and the convergences of $R_1\overline{m}^h$ and $P_1\overline{m}^h$, the right-hand side of each of the two inequalities above converges to 0 as $h\to 0$.

<u>An estimate on $P_5\overline{m}^h$:</u>

$$\begin{aligned}P_5\overline{m}^h(x) &= \langle R_0\overline{m}^h(x), D\overline{m}^h(x)\rangle\overline{m}^h(x)\times D\overline{m}^h(x) + (x-x_k)\langle\widehat{m}^h(x), D^2\overline{m}^h(x)\rangle\overline{m}^h(x)\times D\overline{m}^h(x)\\ &\quad + \langle\widehat{m}^h(x),\partial^h\widehat{m}^{h-}(x)\rangle P_1u(x)\\ &=: P_{51}\overline{m}^h(x)+P_{52}\overline{m}^h(x)+P_{53}\overline{m}^h(x),\quad x\in[x_k,x_{k+1}).\end{aligned}$$

By Lemma 2.3 and (63), we have

$$\mathbb{E}\left[\int_0^T \left\langle P_{51}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\,dt\right] \le 5\left(\mathbb{E}\left[\int_0^T |R_0\overline{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}\left(\mathbb{E}\left[\int_0^T |D\overline{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{2}} \times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}},$$

$$\mathbb{E}\left[\int_0^T \left\langle P_{52}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\,dt\right] \le 5h\left(\mathbb{E}\left[\int_0^T |D^2\overline{m}^h(t)|_{\mathbb{L}^2}^2\,dt\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T |D\overline{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}} \times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}},$$

and

$$\mathbb{E}\left[\int_0^T \left\langle P_{53}\overline{m}^h(t),\varphi(t)\right\rangle_{\mathbb{L}^2}\,dt\right] \le \left(\mathbb{E}\left[\int_0^T |P_1\overline{m}^h(t)|_{\mathbb{L}^2}^2\,dt\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T |\partial^h\widehat{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}} \times\left(\mathbb{E}\left[\int_0^T |\varphi(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}.$$

Similarly, by Lemma 3.5, (64) and the convergences of $R_0\overline{m}^h$ and $P_1\overline{m}^h$, the right-hand side of each of the inequalities above converges to 0 as $h\to 0$.

<u>An estimate on $Q_1\overline{m}^h$:</u>

$$\begin{aligned}Q_1\overline{m}^h(t,x) &= R_0\overline{m}^h(t,x)\times D^2\overline{m}^h(t,x)\\ &= \partial^h\widehat{m}^{h-}(t,x)\times\Delta^h\widehat{m}^h(t,x)\left(x-x_k-\frac{h}{2}\right),\quad x\in[x_k,x_{k+1}).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \langle Q_1\overline{m}^h(t),\varphi(t)\rangle_{\mathbb{L}^2}\,dt\right]\\ &\le \frac{h}{2}\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} |\partial^h\widehat{m}^{h-}(t,x)||\Delta^h\widehat{m}^h(t,x)||\varphi(t,x)|\,dx\,dt\right]\end{aligned}$$

$$\leq \frac{h}{2}\left(\mathbb{E}\left[\int_0^T |\partial^h \widehat{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac14}\left(\mathbb{E}\left[\int_0^T |\Delta^h \widehat{m}^h(t)|_{\mathbb{L}^2}^2\,dt\right]\right)^{\frac12}\left(\mathbb{E}\left[\int_0^T |\varphi(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac14},$$

where the three expectation terms on the right-hand side are finite, proving that the right-hand side converges to 0 as $h \to 0$.

An estimate on $Q_2\overline{m}^h$:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \langle Q_2\overline{m}^h(t), \varphi(t)\rangle_{\mathbb{L}^2}\,dt\right]\\
&= \mathbb{E}\left[\int_0^T \langle Q_1\overline{m}^h(t), \varphi(t)\times\overline{m}^h(t)\rangle_{\mathbb{L}^2}\,dt + \int_0^T \langle R_0\overline{m}^h(t), (\widehat{m}^h(t)\times\Delta^h\widehat{m}^h(t))\times\varphi(t)\rangle_{\mathbb{L}^2}\,dt\right]\\
&\leq \mathbb{E}\left[\int_0^T \langle Q_1\overline{m}^h(t), \varphi(t)\times\overline{m}^h(t)\rangle_{\mathbb{L}^2}\,dt\right]\\
&\quad + \left(\mathbb{E}\left[\int_0^T |R_0\overline{m}^h(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac14}\left(\int_0^T |\widehat{m}^h(t)\times\Delta^h\widehat{m}^h(t)|_{\mathbb{L}^2}^2\,dt\right)^{\frac12}\left(\int_0^T |\varphi(t)|_{\mathbb{L}^4}^4\,dt\right)^{\frac14},
\end{aligned}$$

where on the right-hand side, the first term converges to 0 as $h \to 0$ by the argument for $Q_1\overline{m}^h$ with $\varphi\times\overline{m}^h \in L^4(\Omega; L^4(0,T;\mathbb{L}^4))$, and the second term converges to 0 by (66). □

We also obtain uniform bounds for $\overline{m}^h$ in weighted spaces.

Lemma 4.3. *For any $w \geq 1$, the quadratic interpolation $\overline{m}^h$ satisfies*

(i) $\sup_h \mathbb{E}[|\overline{m}^h|_{B_w}^2] < \infty$ *for* $B_w := L^2(0,T;\mathbb{L}^2_w \cap \mathring{\mathbb{H}}^2)\cap\mathcal{C}([0,T];\mathbb{L}^2_w\cap\mathring{\mathbb{H}}^1)$,

(ii) $\sup_h \mathbb{E}[|\overline{m}^h|^2_{W^{\alpha,p}(0,T;\mathbb{L}^2_w)}] < \infty$, *for* $p\in[2,\infty)$ *and* $\alpha\in(0,\frac12)$ *such that* $\alpha - \frac1p < \frac12$,

(iii) $|\overline{m}^h| \to 1$ *in* $L^2(\Omega; L^2(0,T;\mathbb{L}^2))$.

Proof. Part (i). For every fixed $h>0$, $\widehat{m}^h$ is in $\mathcal{C}([0,T];\mathbb{L}^2_w\cap\mathring{\mathbb{H}}^1)$ and so does $\overline{m}^h$. Then part (i) follows directly from the estimates in (63) and (64).

Part (ii). Recall (58), we have from the definition (53) that

$$\overline{m}^h(t,x) = I_0(x) + I_1(t,x) + I_2(t,x) + I_3(t,x) + I_4(t,x),$$

where for $x\in[x_k,x_{k+1})$, $k\in\mathbb{Z}$,

$$\begin{aligned}
I_0(x) &= \frac12(m_0(x_k)+m_0(x_{k-1})) + (x-x_k)\partial^h m_0(x_{k-1}) + \frac12(x-x_k)^2\Delta^h m_0(x_k),\\
I_1(t,x) &= \frac12\int_0^t\left(F^h(\widehat{m}^h(s,x_k)) + \frac12 S^h(\widehat{m}^h(s,x_k))\right)ds\\
&\quad + \frac12\int_0^t\left(F^h(\widehat{m}^h(s,x_{k-1})) + \frac12 S^h(\widehat{m}^h(s,x_{k-1}))\right)ds,\\
I_2(t,x) &= (x-x_k)\int_0^t \partial^h\left(F^h(\widehat{m}^h) + \frac12 S^h(\widehat{m}^h)\right)(s,x_{k-1})\,ds,\\
I_3(t,x) &= \frac12(x-x_k)^2\int_0^t \Delta^h\left(F^h(\widehat{m}^h) + \frac12 S^h(\widehat{m}^h)\right)(s,x_k)\,ds,
\end{aligned}$$

$$\begin{aligned} I_4(t,x) &= \frac{1}{2}\int_0^t \left(G^h(\widehat{m}^h(s,x_k)) + G^h(\widehat{m}^h(s,x_{k-1})) \right) d\widehat{W}^h(s) \\ &+ (x-x_k)\int_0^t \partial^h G^h(\widehat{m}^h(s,x_{k-1}))\, d\widehat{W}^h(s) \\ &+ \frac{1}{2}(x-x_k)^2 \int_0^t \Delta^h G^h(\widehat{m}^h(s,x_k))\, d\widehat{W}^h(s). \end{aligned}$$

By the $\mathbb{L}^\infty$-estimate of $\overline{m}^h$ in (63),

$$\mathbb{E}\left[\int_0^T |\overline{m}^h(t)|^2_{\mathbb{L}^2_w}\, dt\right] \le 5^2\pi T.$$

For I_1, by Lemma 3.5, there exists a constant a_1 that may depend on $C_v, C_\kappa, \alpha, \gamma, T, K_{1,1}, K_{1,3}$ and $K_{2,1}$ such that

$$\mathbb{E}\left[\int_0^T \left| F^h(\widehat{m}^h(t)) + \frac{1}{2}S^h(\widehat{m}^h(t)) \right|^2_{\mathbb{L}^2} dt\right] \le a_1,$$

For I_2 and I_3, since $|x-x_k| \le h$ and

$$\begin{aligned} h|\partial^h u(x_{k-1})| &\le |u(x_k)| + |u(x_{k-1})|, \\ h^2|\Delta^h u(x_k)| &\le |u(x_{k+1})| + |u(x_{k-1})| + 2|u(x_k)|, \end{aligned}$$

there also exist constants a_2, a_3 such that

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left| h\partial^h \left(F^h(\widehat{m}^h) + \frac{1}{2}S^h(\widehat{m}^h) \right)(t) \right|^2_{\mathbb{L}^2} dt\right] &\le a_2, \\ \mathbb{E}\left[\int_0^T \left| h^2\Delta^h \left(F^h(\widehat{m}^h) + \frac{1}{2}S^h(\widehat{m}^h) \right)(t) \right|^2_{\mathbb{L}^2} dt\right] &\le a_3. \end{aligned}$$

Similarly, for the stochastic integrals in I_4, we only need to verify that $\int_0^T G^h(\widehat{m}^h(s))\, d\widehat{W}^h(s)$ is bounded in $L^p(\Omega; W^{\alpha,p}(0,T;\mathbb{L}^2))$. By [6, Lemma 2.1], there exist a constant C depending on α, p, γ, T and a constant a_4 depending on C, C_κ and $K_{1,p}$ such that for $p \in [2,\infty)$ and $\alpha \in (0,\frac{1}{2})$,

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^T G^h(\widehat{m}^h(s))\, d\widehat{W}^h(s)\right|^p_{W^{\alpha,p}(0,T;\mathbb{L}^2)}\right] &= \mathbb{E}\left[\left|\int_0^T G^h(m^h(s))\, dW(s)\right|^p_{W^{\alpha,p}(0,T;\mathbb{L}^2_h)}\right] \\ &\le C\,\mathbb{E}\left[\int_0^T \left(\sum_j q_j^2 \left|f_j G^h(m^h(s))\right|^2_{\mathbb{L}^2_h}\right)^{\frac{p}{2}} ds\right] \\ &\le C|\kappa|^p_{\mathbb{L}^\infty}\mathbb{E}\left[\int_0^T \left|G^h(\widehat{m}^h(s))\right|^p_{\mathbb{L}^2} ds\right] \le a_4. \end{aligned}$$

Since $\mathbb{L}^2 \hookrightarrow \mathbb{L}^2_w$ for $w \ge 1$, the estimates above hold for the $\mathbb{L}^2_w$-norm. By Lemma A.4, the embedding $W^{1,2}(0,T;\mathbb{L}^2_w) \hookrightarrow W^{\alpha,p}(0,T;\mathbb{L}^2_w)$ is continuous for $\alpha - \frac{1}{p} < \frac{1}{2}$. Thus,

$$\sup_h \mathbb{E}\left[|\overline{m}^h|^2_{W^{\alpha,p}(0,T;\mathbb{L}^2_w)}\right] < \infty.$$

Part (iii). Since $|m^h(t,x)| = 1$, $\mathbb{P}$-a.s. for all $(t,x) \in [0,T] \times \mathbb{Z}_h$, we observe that

$$|\overline{m}^h(t,x)| \leq 1 + \left| \partial^h m^h(t,x_{k-1})(x-x_k) + \frac{1}{2}\Delta^h m^h(t,x_k)(x-x_k)^2 \right|,$$

for $(t,x) \in [0,T] \times [x_k, x_{k+1})$, $k \in \mathbb{Z}$. This implies

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left| |\overline{m}^h(t,x)| - 1 \right|^2 \, dx\, dt\right] \\
&\leq \mathbb{E}\left[\int_0^T \sum_k \int_{x_k}^{x_{k+1}} \left| \partial^h m^h(t,x_{k-1})(x-x_k) + \frac{1}{2}\Delta^h m^h(t,x_k)(x-x_k)^2 \right|^2 \, dx\, dt\right] \\
&\leq \frac{2}{3} h^2 K_{1,1} T + \frac{1}{10} h^4 K_{3,1},
\end{aligned}$$

where the last inequality holds by Lemmata 3.5 and 3.7, and we obtain the convergence after taking $h \to 0$. □

5. Existence of solution

In this section, we first show that the sequence $\{(\overline{m}^h, W)\}_h$ is tight and then by using the Skorohod theorem we obtain its almost sure convergence, up to a change of probability space. Finally, we prove that the limit is a solution of the stochastic LLS equation (4) in the sense of Definition 2.1.

5.1. **Tightness and construction of new probability space and processes.**

Fix w_1, w_2 such that $w_2 > w_1 \geq 1$. Define

$$\begin{aligned}
E_0 &:= L^2(0,T;\mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^2) \cap W^{\alpha,4}(0,T;\mathbb{L}^2_{w_2}), \\
E &:= L^2(0,T;\mathbb{H}^1_{w_2}) \cap \mathcal{C}([0,T];\mathbb{H}^{-1}_{w_1}).
\end{aligned}$$

Recall $\mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^2 \overset{\text{compact}}{\hookrightarrow} \mathbb{H}^1_{w_2} \hookrightarrow \mathbb{L}^2_{w_2}$. By (98) and Lemma A.2,

$$E_0 \hookrightarrow L^2(0,T;\mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^2) \cap W^{\beta,2}(0,T;\mathbb{L}^2_{w_2}) \overset{\text{compact}}{\hookrightarrow} L^2(0,T;\mathbb{H}^1_{w_2}),$$

where $\overline{m}^h \in E_0$, $\mathbb{P}$-a.s. by Lemma 4.3. Also, since the embeddings $\mathbb{H}^1_{w_1} \hookrightarrow \mathbb{L}^2_{w_2} \hookrightarrow \mathbb{H}^{-1}_{w_1}$ are compact and $4\alpha > 1$, it holds by Lemma A.3 that

$$W^{\alpha,4}(0,T;\mathbb{L}^2_{w_2}) \overset{\text{compact}}{\hookrightarrow} \mathcal{C}([0,T];\mathbb{H}^{-1}_{w_1}).$$

In summary, E_0 is compactly embedded in E. For any $r > 0$,

$$\mathbb{P}\left(|\overline{m}^h|_{E_0} > r\right) \leq \frac{1}{r^2}\mathbb{E}\left[|\overline{m}^h|^2_{E_0}\right],$$

where $\{|\overline{m}^h|_{E_0} \leq r\}$ is compact in E, and the right-hand-side converges to 0 as r tends to infinity. Therefore, the set of laws $\{\mathcal{L}(\overline{m}^h)\}$ on the Banach space E is tight, which implies the following convergence result.

Lemma 5.1. *There exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and there exists a sequence (m^*_h, W^*_h) of $E \times \mathcal{C}([0,T];H^2(\mathbb{R}))$-valued random variables defined on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, such that the laws of $(\overline{m}^h, W)$ and (m^*_h, W^*_h) on $E \times \mathcal{C}([0,T];H^2(\mathbb{R}))$ are equal for every h, and there exists an $E \times \mathcal{C}([0,T];H^2(\mathbb{R}))$-valued random variable (m^*, W^*) defined on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ such that*

$$m^*_h \to m^* \text{ in } E, \quad \mathbb{P}^*\text{-a.s.} \tag{67}$$

and

$$W_h^* \to W^* \text{ in } \mathcal{C}([0,T];H^2(\mathbb{R})), \quad \mathbb{P}^*\text{-a.s.} \tag{68}$$

Proof. Since $E \times \mathcal{C}([0,T];H^2(\mathbb{R}))$ is a separable metric space, the result holds by the Skorohod theorem. □

Since the laws of $(\overline{m}^h, W)$ and (m_h^*, W_h^*) on $E \times \mathcal{C}([0,T];H^2(\mathbb{R}))$ are equal, due to the following remark we obtain the same estimates for m_h^*.

Remark 5.2. *By Kuratowski's theorem, the Borel sets of*

$$B := B_{w_1} = L^2(0,T;\mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^2) \cap \mathcal{C}([0,T];\mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^1)$$

are Borel sets of $E = L^2(0,T;\mathbb{H}^1_{w_2}) \cap \mathcal{C}([0,T];\mathbb{H}^{-1}_{w_1})$ *for* $w_1 < w_2$*, where*

$$\mathbb{P}\left(\overline{m}^h \in B\right) = 1.$$

We can assume that m_h^* *takes values in* B *and the laws on* B *of* $\overline{m}^h$ *and* m_h^* *are equal.*

By Remark 5.2, the sequence $(m_h^*)_h$ satisfies the same estimates as $(\overline{m}^h)_h$ on B. By (64), for any $p \in [1,\infty)$,

$$\sup_h \mathbb{E}^* \left[\sup_{t\in[0,T]} |m_h^*(t)|^{2p}_{\mathbb{L}^2_{w_1}} \right] < \infty, \tag{69}$$

$$\sup_h \mathbb{E}^* \left[\sup_{t\in[0,T]} |Dm_h^*(t)|^{2p}_{\mathbb{L}^2} \right] < \infty, \tag{70}$$

$$\sup_h \mathbb{E}^* \left[\left(\int_0^T |D^2 m_h^*(t)|^2_{\mathbb{L}^2}\, dt \right)^p \right] < \infty. \tag{71}$$

Since $|\rho'_w| \le w\rho$ for $w > 0$, by Gagliardo-Nirenberg inequality,

$$\begin{aligned}
|m_h^*(t)\rho_{w_1}^{\frac{1}{2}}|_{\mathbb{L}^\infty} &\le C\, |D(m_h^*(t)\rho_{\frac{w_1}{2}})|^{\frac{1}{2}}_{\mathbb{L}^2}\, |m_h^*(t)\rho_{\frac{w_1}{2}}|^{\frac{1}{2}}_{\mathbb{L}^2} \\
&\le C\left(|Dm_h^*(t)\rho_{\frac{w_1}{2}}|_{\mathbb{L}^2} + |m_h^*(t)\rho'_{\frac{w_1}{2}}|_{\mathbb{L}^2} \right)^{\frac{1}{2}}\, |m_h^*(t)|^{\frac{1}{2}}_{\mathbb{L}^2_{w_1}} \\
&\le C\left(|Dm_h^*(t)|_{\mathbb{L}^2} + \frac{w_1}{2}|m_h^*(t)|_{\mathbb{L}^2_{w_1}} \right)^{\frac{1}{2}}\, |m_h^*(t)|^{\frac{1}{2}}_{\mathbb{L}^2_{w_1}} \\
&\le \frac{C}{2}\left(|Dm_h^*(t)|_{\mathbb{L}^2} + \frac{w_1}{2}|m_h^*(t)|_{\mathbb{L}^2_{w_1}} + |m_h^*(t)|_{\mathbb{L}^2_{w_1}} \right),
\end{aligned}$$

which implies

$$\sup_h \mathbb{E}^* \left[\sup_{t\in[0,T]} |m_h^*(t)\rho_{w_1}^{\frac{1}{2}}|^{2p}_{\mathbb{L}^\infty} \right] < \infty, \quad p \in [1,\infty). \tag{72}$$

Thus, by (69) – (72), for $p \in [1,\infty)$,

$$\sup_h \mathbb{E}^* \left[\left(\int_0^T |m_h^*(t) \times Dm_h^*(t)|^2_{\mathbb{L}^2_{w_1}} \, dt \right)^p \right] < \infty,$$
$$\sup_h \mathbb{E}^* \left[\left(\int_0^T |m_h^*(t) \times D^2 m_h^*(t)|^2_{\mathbb{L}^2_{w_1}} \, dt \right)^p \right] < \infty, \tag{73}$$
$$\sup_h \mathbb{E}^* \left[\left(\int_0^T |m_h^*(t) \times (m_h^*(t) \times D^2 m_h^*(t))|^2_{\mathbb{L}^2_{2w_1}} \, dt \right)^p \right] < \infty,$$

As in (52),

$$|Dm_h^*(t)|^4_{\mathbb{L}^4} \leq \frac{C^2}{2} |D^2 m_h^*(t)|^2_{\mathbb{L}^2} + \frac{1}{2} |Dm_h^*(t)|^6_{\mathbb{L}^2},$$

which implies

$$\sup_h \mathbb{E}^* \left[\left(\int_0^T |Dm_h^*(t)|^4_{\mathbb{L}^4} \, dt \right)^p \right] < \infty, \quad p \in [1,\infty). \tag{74}$$

5.2. Identification of the limit (m^*, W^*) and pathwise uniqueness.

5.2.1. *Convergence of functions of m_h^*.* For $p \in [1,\infty)$, by the pointwise convergence of m_h^* in (67) and the uniform integrability of m_h^* and Dm_h^* in (69) – (70), we have

$$m_h^* \to m^* \quad \text{in } L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2_{w_2})), \tag{75}$$
$$Dm_h^* \to Dm^* \quad \text{in } L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2_{w_2})). \tag{76}$$

By (70), Dm_h^* also converges weakly to a measurable process X in $L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2))$, which implies that $X = Dm^* \in L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2))$ by the uniqueness of the limit of weak convergence in $L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2_{w_2}))$. By (76) and integration-by-parts,

$$D^2 m_h^* \rightharpoonup D^2 m^* \quad \text{in } L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2_{w_2})). \tag{77}$$

Similarly, by (71), $D^2 m_h^*$ converges weakly to a measurable process Y in $L^{2p}(\Omega^*; \mathbb{L}^2(0,T; \mathbb{L}^2))$, thus $Y = D^2 m^* \in L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2))$ and

$$D^2 m_h^* \rightharpoonup D^2 m^* \quad \text{in } L^{2p}(\Omega^*; L^2(0,T; \mathbb{L}^2)). \tag{78}$$

Lemma 5.3. *We have*

(i) $|m^*(t,x)| = 1$, *(t,x)-a.e. $\mathbb{P}^*$-a.s.*

(ii) $m_h^* \rho_{w_2}^{\frac{1}{2}} \to m^* \rho_{w_2}^{\frac{1}{2}}$ *in* $L^p(\Omega^*; L^p(0,T; \mathbb{L}^p))$, *for* $p \in [2,\infty)$,

(iii) $Dm^* \in L^4(\Omega^*; L^4(0,T; \mathbb{L}^4)) \cap L^p(\Omega^*; L^\infty(0,T; \mathbb{L}^2))$, *for* $p \in [2,\infty)$.

Proof. Part (i). Recall Lemma 4.3(iii), a similar argument holds for $\mathbb{L}^2_{w_2}$ (in place of $\mathbb{L}^2$). Then,

$$\mathbb{E}^* \left[\int_0^T \int_{\mathbb{R}} ||m^*(t,x)| - 1|^2 \rho_{w_2}(x) \, dx \, dt \right]$$
$$\leq 2\mathbb{E}^* \left[\int_0^T \int_{\mathbb{R}} ||m_h^*(t,x)| - 1|^2 \rho_{w_2}(x) \, dx \, dt \right] + 2\mathbb{E}^* \left[\int_0^T |m_h^*(t) - m^*(t)|^2_{\mathbb{L}^2_{w_2}} \, dt \right],$$

where the first expectation on the right-hand side converges to 0 since the laws of $\overline{m}^h$ and m_h^* are the same on $L^2(0,T;\mathbb{L}^2_{w_2})$, and the second expectation converges to 0 by (75). Thus,

$$\mathbb{E}^*\left[\int_0^T\int_{\mathbb{R}}\left||m^*(t,x)|-1\right|^2\rho_{w_2}(x)\,dx\,dt\right]=0,$$

which implies $|m^*(t,x)|=1$ a.e. on $[0,T]\times\mathbb{R}$, $\mathbb{P}^*$-a.s. This also means

$$m^*\in L^p(\Omega^*;L^p(0,T;\mathbb{L}^p_w)),\quad \forall p\in[1,\infty),\ w\geq 1.$$

Part (ii). For $w_2>w_1\geq 1$ and $p\in[2,\infty)$,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T|(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|^p_{\mathbb{L}^p}\,dt\right]\\
&\leq\mathbb{E}^*\left[\sup_{t\in[0,T]}\left|(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}\right|^{p-1}_{\mathbb{L}^\infty}\int_0^T\int_{\mathbb{R}}|(m_h^*(t,x)-m^*(t,x))\rho_{w_2}^{\frac12}(x)|\,dx\,dt\right]\\
&\leq C\left(\mathbb{E}^*\left[\sup_{t\in[0,T]}\left(|m_h^*(t)\rho_{w_1}^{\frac12}|^{p-1}_{\mathbb{L}^\infty}+|m^*(t)\rho_{w_1}^{\frac12}|^{p-1}_{\mathbb{L}^\infty}\right)^2\right]\right)^{\frac12}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T\int_{\mathbb{R}}|m_h^*(t,x)-m^*(t,x)|^2\rho_{w_2}(x)\,dx\,dt\right]\right)^{\frac12},
\end{aligned}$$

for some constant C that depends on p and T. Then, by the $\mathbb{L}^\infty$-estimate (72), Lemma 5.3(i), (5) and the strong convergence (75),

$$\lim_{h\to 0}\mathbb{E}^*\left[\int_0^T|(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|^p_{\mathbb{L}^p}\,dt\right]=0,\quad p\in[2,\infty).$$

Part (iii). From part (i), we have $\frac12 D|m^*(t,x)|^2=\langle m^*,Dm^*\rangle(t,x)=0$ and thus

$$\left\langle m^*(t,x),D^2m^*(t,x)\right\rangle=-|Dm^*(t,x)|^2,$$

for (t,x)-a.e. $\mathbb{P}^*$-a.s. Since $D^2m^*\in L^2(\Omega^*;L^2(0,T;\mathbb{L}^2))$, we deduce

$$\begin{aligned}
\mathbb{E}^*\left[\int_0^T|Dm^*(t)|^4_{\mathbb{L}^4}\,dt\right]&=\mathbb{E}^*\left[\int_0^T\int_{\mathbb{R}}\left\langle m^*(t,x),D^2m^*(t,x)\right\rangle^2\,dx\,dt\right]\\
&\leq\mathbb{E}^*\left[\int_0^T|D^2m^*(t)|^2_{\mathbb{L}^2}\,dt\right]<\infty.
\end{aligned}$$

As in [3], we extend the definition of the $\mathbb{L}^2_{w_2}\cap\mathring{\mathbb{H}}^1$-norm to $\mathbb{H}^{-1}_{w_1}$ such that $|u|_{\mathbb{L}^2_{w_2}\cap\mathring{\mathbb{H}}^1}=\infty$ if the function u is in $\mathbb{H}^{-1}_{w_1}$ but not $\mathbb{L}^2_{w_2}\cap\mathring{\mathbb{H}}^1$, where the extended map

$$u\mapsto\sup_{t\in[0,T]}|u(t)|_{\mathbb{L}^2_{w_2}\cap\mathring{\mathbb{H}}^1},\quad u\in\mathcal{C}([0,T];\mathbb{H}^{-1}_{w_1}),$$

is lower semicontinuous. Then, by the pointwise convergence (67), Fatou's lemma and (70),

$$\mathbb{E}^*\left[\sup_{t\in[0,T]}|Dm^*(t)|^p_{\mathbb{L}^2}\right]\leq\liminf_{h\to 0}\mathbb{E}^*\left[\sup_{t\in[0,T]}|m_h^*(t)|^p_{\mathbb{L}^2_{w_2}\cap\mathring{\mathbb{H}}^1}\right]<\infty,$$

for $p\in[2,\infty)$. □

Lemma 5.4. *We have the following strong convergences:*

(i) $m_h^* \times Dm_h^* \to m^* \times Dm^*$ *and* $\langle m_h^*, Dm_h^* \rangle \to 0$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$,

(ii) $m_h^* \times \left(m_h^* \times Dm_h^*\right) \to m^* \times (m^* \times Dm^*)$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_1+w_2}))$.

Proof. Part (i). Note that

$$m_h^* \times Dm_h^* - m^* \times Dm^* = (m_h^* - m^*) \times Dm_h^* + m^* \times (Dm_h^* - Dm^*)\,.$$

Then by Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T |(m_h^*(t)-m^*(t))\times Dm_h^*(t)|^2_{\mathbb{L}^2_{w_2}}\,dt\right]\\
&\le \left(\mathbb{E}^*\left[\int_0^T |(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12}\left(\mathbb{E}^*\left[\int_0^T |Dm_h^*(t)|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12},
\end{aligned}$$

where the last line converges to 0 by Lemma 5.3(ii) and (74). Similarly, by Lemma 5.3(i),

$$\mathbb{E}^*\left[\int_0^T |m^*(t)\times(Dm_h^*(t)-Dm^*(t))|^2_{\mathbb{L}^2_{w_2}}\,dt\right] \le \mathbb{E}^*\left[\int_0^T |Dm_h^*(t)-Dm^*(t)|^2_{\mathbb{L}^2_{w_2}}\,dt\right],$$

where the right-hand side converges to 0 by (76). Therefore,

$$\lim_{h\to 0}\mathbb{E}^*\left[\int_0^T |m_h^*(t)\times Dm_h^*(t) - m^*(t)\times Dm^*(t)|^2_{\mathbb{L}^2_w}\,dt\right] = 0. \tag{79}$$

Since $|m^*(t,x)| = 1$, we have $\langle m^*, Dm^*\rangle(t,x) = 0$. By the same argument as above (replacing cross product with scalar product), $\langle m_h^*, Dm_h^*\rangle \to \langle m^*, Dm^*\rangle = 0$ in $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$.

Part (ii). Note that

$$\begin{aligned}
&m_h^* \times (m_h^* \times Dm_h^*) - m^* \times (m^* \times Dm^*)\\
&= (m_h^* - m^*) \times (m_h^* \times Dm_h^*) + m^* \times (m_h^* \times Dm_h^* - m^* \times Dm^*)\,.
\end{aligned}$$

Then, with $\rho_{w_1+w_2} = \rho_{w_1}\rho_{w_2}$,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T |(m_h^*(t)-m^*(t))\times(m_h^*(t)\times Dm_h^*(t))|^2_{\mathbb{L}^2_{w_1+w_2}}\,dt\right]\\
&\le \left(\mathbb{E}^*\left[\int_0^T |(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12}\left(\mathbb{E}^*\left[\int_0^T |m_h^*(t)\rho_{w_1}^{\frac12}\times Dm_h^*(t)|^4_{\mathbb{L}^4}\right]\right)^{\frac12}\\
&\le \left(\mathbb{E}^*\left[\int_0^T |(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|^4_{\mathbb{L}^4}\,dt\right]\right)^{\frac12}\\
&\quad\times\left(\mathbb{E}^*\left[\sup_{t\in[0,T]} |m_h^*(t)\rho_{w_1}^{\frac12}|^8_{\mathbb{L}^\infty}\right]\right)^{\frac14}\left(\mathbb{E}^*\left[\left(\int_0^T |Dm_h^*(t)|^4_{\mathbb{L}^4}\,dt\right)^2\right]\right)^{\frac14},
\end{aligned}$$

where the first expectation in the last inequality converges to 0 by Lemma 5.3(ii) and the second and third expectations are finite by (72) and (74). Also,

$$\mathbb{E}^*\left[\int_0^T |m^*(t)\times(m_h^*(t)\times Dm_h^*(t) - m^*(t)\times Dm^*(t))|^2_{\mathbb{L}^2_{w_1+w_2}}\,dt\right]$$

converges to 0 Lemma 5.3(i) and part (i). Then, the strong convergence of $m_h^* \times (m_h^* \times Dm_h^*)$ follows as desired. □

Lemma 5.5. *Assume that $w_2 \geq 4w_1$. For any measurable process $\varphi \in L^4(\Omega^*; \mathbb{L}^4(0,T;\mathbb{L}^4_{w_2}))$, we have the following weak convergences (with test function φ):*

(i) $m_h^* \times (m_h^* \times Dm_h^*) \rightharpoonup m^* \times (m^* \times Dm^*)$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$,

(ii) $|m_h^* \times Dm_h^*|^2 m_h^* \rightharpoonup |m^* \times Dm^*|^2 m^*$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$,

(iii) $Dm_h^* \times (m_h^* \times Dm_h^*) \rightharpoonup Dm^* \times (m^* \times Dm^*)$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$,

(iv) $\langle m_h^*, Dm_h^* \rangle m_h^* \times Dm_h^* \rightharpoonup 0$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$,

(v) $m_h^* \times D^2 m_h^* \rightharpoonup m^* \times D^2 m^*$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$,

(vi) $m_h^* \times (m_h^* \times D^2 m_h^*) \rightharpoonup m^* \times (m^* \times D^2 m^*)$ *in* $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$.

Proof. Part (i). As in lemma 5.4(ii), we first observe that

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T \langle (m_h^* - m^*) \times (m_h^* \times Dm_h^*), \varphi \rangle_{\mathbb{L}^2_{w_2}}(t)\, dt\right] \\
(80)\qquad &\leq \left(\mathbb{E}^*\left[\int_0^T |(m_h^*(t) - m^*(t))\rho_{w_2}^{\frac{1}{2}}|^4_{\mathbb{L}^4}\, dt\right]\right)^{\frac{1}{4}} \\
&\quad\times \left(\mathbb{E}^*\left[\int_0^T |m_h^*(t) \times Dm_h^*(t)\rho_{w_2}^{\frac{1}{4}}|^2_{\mathbb{L}^2}\, dt\right]\right)^{\frac{1}{2}} \left(\mathbb{E}^*\left[\int_0^T |\varphi(t)|^4_{\mathbb{L}^4_{w_2}}\right]\right)^{\frac{1}{4}},
\end{aligned}$$

where the first expectation in the last line converges to 0 by Lemma 5.3(ii), the second and the third expectations are finite by (73) with $w_2 \geq 2w_1$ (equivalently, $\rho_{w_2} \leq \rho_{w_1}^2$) and $\varphi \in L^4(\Omega^*; L^4(0,T;\mathbb{L}^4_{w_2}))$. Since $|m^*(t,x)| = 1$ from Lemma 5.3(i), we have $m^* \times \varphi \in L^2(\Omega^*; L^2(0,T;\mathbb{L}^2_{w_2}))$. Then by Lemma 5.4(i),

$$(81)\qquad \lim_{h\to 0} \mathbb{E}^*\left[\int_0^T \langle m^* \times (m_h^* \times Dm_h^* - m^* \times Dm^*), \varphi \rangle_{\mathbb{L}^2_{w_2}}(t)\, dt\right] = 0.$$

Combining (80) and (81), we have the desired weak convergence for part (i).

Part (ii).

$$\begin{aligned}
&|m_h^* \times Dm_h^*|^2 m_h^* - |m^* \times Dm^*|^2 m^* \\
&= \left(|m_h^* \times Dm_h^*|^2 - |m^* \times Dm^*|^2\right) m_h^* + |m^* \times Dm^*|^2 (m_h^* - m^*) \\
&\leq |m_h^* \times Dm_h^* - m^* \times Dm^*|\,|m_h^* \times Dm_h^* + m^* \times Dm^*|\ \ |m_h^*| + |m^* \times Dm^*|^2\ |m_h^* - m^*|.
\end{aligned}$$

Then, for the first term in the line above,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T \langle |m_h^* \times Dm_h^* - m^* \times Dm^*|\,|m_h^* \times Dm_h^* + m^* \times Dm^*|\,m_h^*, \varphi \rangle_{\mathbb{L}^2_{w_2}}(t)\, dt\right] \\
(82)\qquad &\leq \left(\mathbb{E}^*\left[\int_0^T \int_{\mathbb{R}} (|m_h^* \times Dm_h^*| + |m^* \times Dm^*|)^2\, |m_h^*|^2 |\varphi|^2(t,x)\, \rho_{w_2}(x)\, dx\, dt\right]\right)^{\frac{1}{2}} \\
&\quad\times \left(\mathbb{E}^*\left[\int_0^T |m_h^* \times Dm_h^* - m^* \times Dm^*|^2_{\mathbb{L}^2_{w_2}}(t)\, dt\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

We show that the first expectation on the right-hand side of (82) is finite. Since $w_2 \geq 4w_1$, it holds that $\rho_{w_2} \leq \rho_{w_1}^2 \rho_{w_2}^{\frac{1}{2}}$ and

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T\int_{\mathbb{R}} |m_h^*\times Dm_h^*|^2\,|m_h^*|^2|\varphi|^2(t,x)\,\rho_{w_2}(x)\,dx\,dt\right]\\
&\leq \mathbb{E}^*\left[\sup_{t\in[0,T]}|m_h^*(t)\rho_{w_1}^{\frac{1}{2}}|_{\mathbb{L}^\infty}^4\int_0^T\int_{\mathbb{R}}|Dm_h^*|^2|\varphi\rho_{w_2}^{\frac{1}{4}}|^2(t,x)\,dx\,dt\right]\\
&\leq \left(\mathbb{E}^*\left[\sup_{t\in[0,T]}|m_h^*(t)\rho_{w_1}^{\frac{1}{2}}|_{\mathbb{L}^\infty}^{16}\right]\right)^{\frac{1}{4}}\times\left(\mathbb{E}^*\left[\left(\int_0^T|Dm_h^*(t)|_{\mathbb{L}^4}^4\,dt\right)^2\right]\right)^{\frac{1}{4}}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\,dt\right]\right)^{\frac{1}{2}},
\end{aligned}$$

where three expectations in the last inequality are finite by (72), (74) and $\varphi \in L^4(\Omega^*;L^4(0,T;\mathbb{L}^4_{w_2}))$. Similarly, by Lemma 5.3(i) and (iii),

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T\int_{\mathbb{R}}|m^*\times Dm^*|^2\,|\varphi|^2(t,x)\,\rho_{w_2}(x)\,dx\,dt\right]\\
&\leq\left(\mathbb{E}^*\left[\int_0^T|Dm^*(t)|_{\mathbb{L}^4_{w_2}}^4\,dt\right]\right)^{\frac{1}{4}}\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\,dt\right]\right)^{\frac{1}{4}}<\infty.
\end{aligned}$$

Hence, the left-hand side of (82) converges to 0 as $h \to 0$ by Lemma 5.4(i). Similarly, with $|m^*(t,x)| = 1$, $\mathbb{P}^*$-a.s. we have

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T\int_{\mathbb{R}}\left\langle|m^*\times Dm^*|^2\,(m_h^*-m^*)\,(t,x),\varphi(t,x)\right\rangle\rho_{w_2}(x)\,dx\,dt\right]\\
&\leq\left(\mathbb{E}^*\left[\int_0^T|Dm^*(t)|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{2}}\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\,dt\right]\right)^{\frac{1}{4}}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T|(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac{1}{2}}|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}
\end{aligned}\tag{83}$$

where the last line converges to 0 by Lemma 5.3(ii) – (iii). Combining (82) and (83), we have the desired weak convergence for part (ii).

Part (iii). Note that

$$\begin{aligned}
&Dm_h^*\times(m_h^*\times Dm_h^*)-Dm^*\times(m^*\times Dm^*)\\
&=(Dm_h^*-Dm^*)\times(m^*\times Dm^*)+Dm_h^*\times(m_h^*\times Dm_h^*-m^*\times Dm^*).
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T\langle(Dm_h^*(t)-Dm^*(t))\times(m^*(t)\times Dm^*(t)),\varphi(t)\rho_{w_2}\rangle_{\mathbb{L}^2}\,dt\right]\\
&\leq\left(\mathbb{E}^*\left[\int_0^T|Dm^*(t)\rho_{w_2}^{\frac{1}{4}}|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\,dt\right]\right)^{\frac{1}{4}}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T|Dm_h^*(t)-Dm^*(t)|_{\mathbb{L}^2_{w_2}}^2\,dt\right]\right)^{\frac{1}{2}},
\end{aligned}$$

where the last line converges to 0 by Lemma 5.3(iii) and (76). Similarly,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T \langle Dm_h^*(t)\times(m_h^*(t)\times Dm_h^*(t)-m^*(t)\times Dm^*(t)),\varphi(t)\rho_{w_2}\rangle_{\mathbb{L}^2}\ dt\right]\\
&\le\left(\mathbb{E}^*\left[\int_0^T |Dm_h^*(t)\rho_{w_2}^{\frac14}|_{\mathbb{L}^4}^4\ dt\right]\right)^{\frac14}\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\ dt\right]\right)^{\frac14}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T |m_h^*(t)\times Dm_h^*(t)-m^*(t)\times Dm^*(t)|_{\mathbb{L}^2_{w_2}}^2\ dt\right]\right)^{\frac12},
\end{aligned}$$

which converges to 0 by (74) and Lemma 5.4(i). Together, we have

$$\lim_{h\to 0}\mathbb{E}^*\left[\int_0^T\left|\langle Dm_h^*\times(m_h^*\times Dm_h^*)-Dm^*\times(m^*\times Dm^*),\varphi\rangle_{\mathbb{L}^2_{w_2}}(t)\right|dt\right]=0.$$

Part (iv). Again, since $w_2\ge 4w_1$, we have $\rho_{w_2}^{\frac14}\le\rho_{w_1}$. Then,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T\langle\ \langle m_h^*(t),Dm_h^*(t)\rangle m_h^*(t)\times Dm_h^*(t),\varphi(t)\rho_{w_2}\rangle_{\mathbb{L}^2}\ dt\right]\\
&\le\left(\mathbb{E}^*\left[\int_0^T|m_h^*(t)\times Dm_h^*(t)\rho_{w_2}^{\frac14}|_{\mathbb{L}^4}^4\ dt\right]\right)^{\frac14}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\ dt\right]\right)^{\frac14}\left(\mathbb{E}^*\left[\int_0^T|\langle m_h^*(t),Dm_h^*(t)\rangle|_{\mathbb{L}^2_{w_2}}^2\ dt\right]\right)^{\frac12}\\
&\le\left(\mathbb{E}^*\left[\sup_{t\in[0,T]}|m_h^*(t)\rho_{w_1}|_{\mathbb{L}^\infty}^8\right]\right)^{\frac18}\left(\mathbb{E}^*\left[\left(\int_0^T|Dm_h^*(t)|_{\mathbb{L}^4}^4\ dt\right)^2\right]\right)^{\frac18}\\
&\quad\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)|_{\mathbb{L}^4_{w_2}}^4\ dt\right]\right)^{\frac14}\left(\mathbb{E}^*\left[\int_0^T|\langle m_h^*(t),Dm_h^*(t)\rangle|_{\mathbb{L}^2_{w_2}}^2\ dt\right]\right)^{\frac12}
\end{aligned}$$

where the right-hand side converges to 0 by (72) (with $\rho_{w_1}^{\frac12}\le 1$), (74) and the convergence of the scalar product in Lemma 5.4(i).

Part (v).

$$\begin{aligned}
&\langle m_h^*\times D^2m_h^*-m^*\times D^2m^*,\varphi\rangle_{\mathbb{L}^2_{w_2}}\\
&=\langle(m_h^*-m^*)\times D^2m_h^*,\varphi\rho_{w_2}\rangle_{\mathbb{L}^2}+\langle m^*\times(D^2m_h^*-D^2m^*),\varphi\rho_{w_2}\rangle_{\mathbb{L}^2}.
\end{aligned}$$

Then, for the first term on the right-hand side,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T\langle(m_h^*(t)-m^*(t))\times D^2m_h^*(t),\varphi(t)\rho_{w_2}\rangle_{\mathbb{L}^2}\ dt\right]\\
&\le\mathbb{E}^*\left[\int_0^T|(m_h^*(t)-m^*(t))\rho_{w_2}\times\varphi(t)|_{\mathbb{L}^2}\ |D^2m_h^*(t)|_{\mathbb{L}^2}\ dt\right]\\
&\le\mathbb{E}^*\left[\int_0^T|(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|_{\mathbb{L}^4}\ |\varphi(t)\rho_{w_2}^{\frac12}|_{\mathbb{L}^4}\ |D^2m_h^*(t)|_{\mathbb{L}^2}\ dt\right]\\
&\le\left(\mathbb{E}^*\left[\int_0^T|(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac12}|_{\mathbb{L}^4}^4\ dt\right]\right)^{\frac14}\times\left(\mathbb{E}^*\left[\int_0^T|\varphi(t)\rho_{w_2}^{\frac12}|_{\mathbb{L}^4}^4\ dt\right]\right)^{\frac14}
\end{aligned}$$

$$\times\left(\mathbb{E}^*\left[\int_0^T |D^2 m_h^*(t)|_{\mathbb{L}^2}^2\,dt\right]\right)^{\frac{1}{2}},$$

where the first expectation in the last inequality converges to 0 by Lemma 5.3(ii), the second expectation is finite as $\varphi\in L^4(\Omega^*;\mathbb{L}^4(0,T;\mathbb{L}^4_{w_2}))$ and the final expectation is finite by (71). Thus, the left-hand side converges to 0 as $h\to 0$. Also, $m^*\times\varphi\in L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$ and then by the weak convergence (77),

$$\lim_{h\to 0}\mathbb{E}^*\left[\int_0^T \langle m^*(t)\times(D^2 m_h^*(t)-D^2 m^*(t)),\varphi(t)\rangle_{\mathbb{L}^2_{w_2}}\,dt\right]=0.$$

Therefore,

$$\lim_{h\to 0}\mathbb{E}^*\left[\int_0^T \left|\langle m_h^*(t)\times D^2 m_h^*(t)-m^*(t)\times D^2 m^*(t),\varphi(t)\rangle_{\mathbb{L}^2_{w_2}}\right|dt\right]=0.$$

Part (vi). Similarly,

$$\begin{aligned}&\left\langle m_h^*\times\left(m_h^*\times D^2 m_h^*\right)-m^*\times\left(m^*\times D^2 m^*\right),\varphi\right\rangle_{\mathbb{L}^2_{w_2}}\\&=\langle (m_h^*-m^*)\times\left(m_h^*\times D^2 m_h^*\right),\varphi\rangle_{\mathbb{L}^2_{w_2}}+\langle m^*\times(m_h^*\times D^2 m_h^*-m^*\times D^2 m^*),\varphi\rangle_{\mathbb{L}^2_{w_2}}.\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}^*\left[\int_0^T \langle (m_h^*(t)-m^*(t))\times\left(m_h^*(t)\times D^2 m_h^*(t)\right),\varphi(t)\rangle_{\mathbb{L}^2_{w_2}}\,dt\right]\\&\le\left(\mathbb{E}^*\left[\int_0^T |(m_h^*(t)-m^*(t))\rho_{w_2}^{\frac{1}{2}}|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}\times\left(\mathbb{E}^*\left[\int_0^T |\varphi(t)\rho_{w_2}^{\frac{1}{4}}|_{\mathbb{L}^4}^4\,dt\right]\right)^{\frac{1}{4}}\\&\quad\times\left(\mathbb{E}^*\left[\int_0^T |m_h^*(t)\times D^2 m_h^*(t)\rho_{w_2}^{\frac{1}{4}}|_{\mathbb{L}^2}^2\,dt\right]\right)^{\frac{1}{2}},\end{aligned}$$

where the last line converges to 0 by Lemma 5.3(ii), $\varphi\in L^4(\Omega^*;\mathbb{L}^4(0,T;\mathbb{L}^4_{w_2}))$ and (73) with $\rho_{w_1}\ge\rho_{w_2}^{\frac{1}{4}}$. Also, we have

$$\begin{aligned}&\mathbb{E}^*\left[\int_0^T \langle m^*(t)\times(m_h^*(t)\times D^2 m_h^*(t)-m^*(t)\times D^2 m^*(t)),\varphi(t)\rho_{w_2}\rangle_{\mathbb{L}^2}\,dt\right]\\&\le\mathbb{E}^*\left[\int_0^T \langle m_h^*(t)\times D^2 m_h^*(t)-m^*(t)\times D^2 m^*(t),m^*(t)\times\varphi(t)\rangle_{\mathbb{L}^2_{w_2}}\,dt\right],\end{aligned}$$

where $m^*\times\varphi\in L^4(\Omega^*;L^4(0,T;\mathbb{L}^4_{w_2}))$ and thus by part (iii), the expectation in the last line above converges to 0. Therefore,

$$\lim_{h\to 0}\mathbb{E}^*\left[\int_0^T \langle m_h^*\times\left(m_h^*\times D^2 m_h^*\right)-m^*\times\left(m^*\times D^2 m^*\right),\varphi\rangle_{\mathbb{L}^2_{w_2}}(t)\,dt\right]=0.$$

□

Lemma 5.6. *Assume that $w_2\ge 4w_1$. Recall the definitions (61), we have*

(i) $F_{\widehat{\nu}}(m_h^*)\rightharpoonup F(m^*)$ in $L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$,

(ii) $S_{\widehat{\kappa}}(m_h^*)\rightharpoonup S(m^*)$ in $L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$,

(iii) $\kappa G(m_h^*)\to\kappa G(m^*)$ (strongly) in $L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_1+w_2}))$.

Proof. As in Lemma 5.5, let φ be an arbitrary measurable process in $L^4(\Omega^*;L^4(0,T;\mathbb{L}^4_{w_2}))$. By (6) and (7), $\kappa^2,\kappa\kappa'\in\mathbb{L}^\infty\cap\mathbb{H}^1$ and $v\in\mathcal{C}([0,T];\mathbb{L}^\infty\cap\mathbb{H}^1)$. Then for $y=\kappa^2,(\kappa^2)^-,\kappa\kappa',v$, any piecewise constant approximation (in the x-variable) z of y satisfies

$$z\to y \quad \text{in } L^4(0,T;\mathbb{L}^4_{w_2}). \tag{84}$$

For example, the approximation z can be taken to be $\widehat{y}^-$ or $\widehat{y}$. Let u be a function such that $u(m_h^*)\in L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$. Then,

$$\begin{aligned}
&\mathbb{E}^*\left[\int_0^T \langle z(t)u(m_h^*(t))-y(t)u(m^*(t)),\varphi(t)\rangle_{\mathbb{L}^2_{w_2}}\,dt\right]\\
&\le \mathbb{E}^*\left[\int_0^T \langle (z-y)\times u(m_h^*),\varphi\rangle_{\mathbb{L}^2_{w_2}}(t)\,dt\right]+\mathbb{E}^*\left[\int_0^T \langle y\left(u(m_h^*)-u(m^*)\right),\varphi\rangle_{L^2_{w_2}}(t)\,dt\right]\\
&\le \left(\mathbb{E}^*\left[\int_0^T |z(t)-y(t)|^4_{\mathbb{L}^4_{w_2}}\,dt\right]\right)^{\frac14}\left(\mathbb{E}^*\left[\int_0^T |u(m_h^*(t))|^2_{\mathbb{L}^2_{w_2}}\,dt\right]\right)^{\frac12}\left(\mathbb{E}^*\left[\int_0^T |\varphi(t)|^4_{\mathbb{L}^4_{w_2}}\,dt\right]\right)^{\frac14}\\
&\quad+\mathbb{E}^*\left[\int_0^T \langle u(m_h^*(t))-u(m^*(t)),y(t)\varphi(t)\rangle_{\mathbb{L}^2_{w_2}}\,dt\right].
\end{aligned} \tag{85}$$

If $u(m_h^*)\rightharpoonup u(m^*)$ in $L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$, then the right-hand side of (85) converges to 0 by (84) and $y\varphi\in L^4(\Omega^*;L^4(0,T;\mathbb{L}^4_{w_2}))$.

Part (i). Let $u(m_h^*)=m_h^*\times(m_h^*\times Dm_h^*)$ and let $y=v$. The result follows immediately from (85) and Lemma 5.5(v) and (vi).

Part (ii). Since $w_2\ge 4w_1$, we have $\rho_{w_2}\le\rho_{w_1}^4$, We observe that from (74) and (72) that

$$Dm_h^*\times(m_h^*\times Dm_h^*),\quad |m_h^*\times Dm_h^*|^2m_h^*,\quad \langle m_h^*,Dm_h^*\rangle m_h^*\times Dm_h^*$$

are in $L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$. Taking the following choices of u, y and z:

$$\begin{aligned}
&u(m_h^*)=(\gamma^2-1)m_h^*\times(m_h^*\times D^2m_h^*)-2\gamma m_h^*\times D^2m_h^*, && y=\kappa^2, && z=\frac12\left((\widehat{\kappa}^-)^2+\widehat{\kappa}^2\right),\\
&u(m_h^*)=\gamma^2Dm_h^*\times(m_h^*\times Dm_h^*)+|m_h^*\times Dm_h^*|^2m_h^*, && y=\kappa^2, && z=\widehat{\kappa}^2,\\
&u(m_h^*)=2\gamma\langle m_h^*,Dm_h^*\rangle\, m_h^*\times Dm_h^*, && y=\kappa^2, && z=(\widehat{\kappa}^2)^-,\\
&u(m_h^*)=\left[(\gamma^2-1)m_h^*\times(m_h^*\times Dm_h^*)-2\gamma m_h^*\times Dm_h^*\right], && y=\kappa\kappa', && z=\widehat{\kappa\kappa'},
\end{aligned}$$

and using Lemma 5.5(ii) – (vi), we follow again the argument (85) to obtain weak convergence of $S_{\widehat{\kappa}}(m_h^*)$ to $S(m^*)$ in $L^2(\Omega^*;L^2(0,T;\mathbb{L}^2_{w_2}))$.

Part (iii). The result follows from (6) and Lemma 5.4. □

5.2.2. *Wiener process.* Define a sequence of processes $\{\overline{M}_h\}_{h>0}$ on $(\Omega,\mathcal{F},\mathbb{P})$ by

$$\overline{M}_h(t):=\int_0^t\left(G(\overline{m}^h(s))-(\gamma P_1+P_2)\overline{m}^h(s)\right)d\widehat{W}^h(s).$$

Recall the equation of $\overline{m}^h$, we have from (62) that

$$\overline{M}_h(t)=\overline{m}^h(t)-m_0-\int_0^t\left(F_{\widehat{v}}(\overline{m}^h(s))+\frac12 S_{\widehat{\kappa}}(\overline{m}^h(s))\right)ds-R_0\overline{m}^h(t)-\int_0^t R^h\overline{m}^h(s)\,ds,$$

where the operator R^h is given by

$$R^h u := -\widehat{v}(\gamma P_1 + P_2)u + Q_1 u + \alpha Q_2 u + \frac{1}{4}\left((\widehat{\kappa}^2)^- + \widehat{\kappa}^2\right)\left[2\gamma Q_1 u - (\gamma^2-1)Q_2 u\right]$$
$$+\frac{1}{2}\left(\widehat{\kappa}^2\left(P_3 + \gamma^2 P_4\right)u - 2\gamma(\widehat{\kappa}^2)^- P_5 u\right) + \frac{1}{2}\widehat{\kappa\kappa'}\left[2\gamma P_1 - (\gamma^2-1)P_2\right]u.$$

Similarly, define a sequence of processes $\{M_h^*\}_{h>0}$ on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ by

$$M_h^*(t) := m_h^*(t) - m_0 - \int_0^t \left(F_{\widehat{v}}(m_h^*(s)) + \frac{1}{2}S_{\widehat{\kappa}}(m_h^*(s))\right)\,ds - R_0 m_h^*(t) - \int_0^t R^h m_h^*(s)\,ds.$$

Lemma 5.7. *For each $t \in (0,T]$, we have the following weak convergence in $L^2(\Omega^*; \mathbb{H}^{-1}_{w_1})$:*

$$M_h^*(t) \rightharpoonup M^*(t) := m^*(t) - m_0 - \int_0^t \left(F(m^*(s)) + \frac{1}{2}S(m^*(s))\right)\,ds.$$

Proof. Recall that $\mathbb{H}^1_{w_1}$ is compactly embedded in $\mathbb{L}^2_{w_2}$. Let $t \in (0,T]$ and $\varphi \in L^2(\Omega^*; \mathbb{H}^1_{w_1})$. By Remark 5.2, the two sets of remainders

$$\{R_0\overline{m}^h, R_1\overline{m}^h, P_1\overline{m}^h, \dots, P_5\overline{m}^h, Q_1\overline{m}^h, Q_2\overline{m}^h\},$$
$$\{R_0 m_h^*, R_1 m_h^*, P_1 m_h^*, \dots, P_5 m_h^*, Q_1 m_h^*, Q_2 m_h^*\},$$

have the same laws for $\overline{m}^h, m_h^* \in L^2(0,T; \mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^2) \cap \mathcal{C}([0,T]; \mathbb{L}^2_{w_1} \cap \mathring{\mathbb{H}}^1)$. Then, by Lemma 4.2,

$$\lim_{h\to 0}\mathbb{E}^*\left[{}_{\mathbb{H}^{-1}_{w_1}}\left\langle R_0 m_h^*(t) + \int_0^t R^h m_h^*(s)\,ds, \varphi\right\rangle_{\mathbb{H}^1_{w_1}}\right]$$
$$= \lim_{h\to 0}\mathbb{E}^*\left[\langle R_0 m_h^*(t), \varphi\rangle_{\mathbb{L}^2_{w_2}} + \int_0^t \left\langle R^h m_h^*(s), \varphi\right\rangle_{\mathbb{L}^2_{w_2}}\,ds\right] = 0.$$

By the pointwise convergence (67) of m_h^* in $\mathcal{C}([0,T]; \mathbb{H}^{-1}_{w_1})$ and Lemma 5.6(i) – (ii),

$$\lim_{h\to 0}\mathbb{E}^*\left[{}_{\mathbb{H}^{-1}_{w_1}}\langle M_h^*(t), \varphi\rangle_{\mathbb{H}^1_{w_1}}\right]$$
$$= \lim_{h\to 0}\mathbb{E}^*\left[{}_{\mathbb{H}^{-1}_{w_1}}\langle m_h^*(t) - m_0, \varphi\rangle_{\mathbb{H}^1_{w_1}} - \int_0^t \left\langle F_{\widehat{v}}(m_h^*(s)) + \frac{1}{2}S_{\widehat{\kappa}}(m_h^*(s)), \varphi\right\rangle_{\mathbb{L}^2_{w_2}}\,ds\right]$$
$$- \lim_{h\to 0}\mathbb{E}^*\left[{}_{\mathbb{H}^{-1}_{w_1}}\left\langle R_0 m_h^*(t) + \int_0^t R^h m_h^*(s), \varphi\right\rangle_{\mathbb{H}^1_{w_1}}\right]$$
$$= \mathbb{E}^*\left[{}_{\mathbb{H}^{-1}_{w_1}}\langle M^*(t), \varphi\rangle_{\mathbb{H}^1_{w_1}}\right].$$

□

Lemma 5.8. *The process W^* is a Q-Wiener process on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, and $W^*(t) - W^*(s)$ is independent of the σ-algebra generated by $m^*(r)$ and $W^*(r)$ for $r \in [0,s]$.*

Proof. See [3, Lemma 5.2(i)] (using Lemma 5.1). □

Lemma 5.9. *For each $t \in [0,T]$,*

$$M^*(t) = \int_0^t G(m^*(s))\, dW^*(s).$$

Proof. Fix h and $t \in (0,T]$. For each $n \in \mathbb{N}$, define the partition $\{s_i^n = \frac{iT}{n} : i = 0,\dots,n\}$. Define

$$\begin{aligned}
\delta \widehat{W}^h(t,s_i^n) &:= \widehat{W}^h(t \wedge s_{i+1}^n) - \widehat{W}^h(t \wedge s_i^n),\\
\delta \widehat{W}_h^*(t,s_i^n) &:= \widehat{W}_h^*(t \wedge s_{i+1}^n) - \widehat{W}_h^*(t \wedge s_i^n),\\
\delta W^*(t,s_i^n) &:= W^*(t \wedge s_{i+1}^n) - W^*(t \wedge s_i^n),
\end{aligned}$$

where $\widehat{W}_h^*(s)$ is the piecewise constant approximation of $W_h^*(s)$ (as in (55)) for every $s \in [0,T]$. As in (68), we also have

$$\widehat{W}_h^* \to W^* \text{ in } \mathcal{C}([0,T];L^2(\mathbb{R})), \quad \mathbb{P}^*\text{-a.s.} \tag{86}$$

Consider the following two $\mathbb{L}^2_{w_2}$-valued random variables:

$$\begin{aligned}
\overline{Y}_{h,n}(t) &:= \overline{M}_h(t) - \sum_{i=0}^{n-1} \Big(G(\overline{m}^h(s_i^n)) - (\gamma P_1 + P_2)\overline{m}^h(s_i^n) \Big)\, \delta \widehat{W}^h(t,s_i^n),\\
Y_{h,n}^*(t) &:= M_h^*(t) - \sum_{i=0}^{n-1} \big(G(m_h^*(s_i^n)) - (\gamma P_1 + P_2)\, m_h^*(s_i^n) \big)\, \delta \widehat{W}_h^*(t,s_i^n).
\end{aligned}$$

Following Remark 5.2, $\overline{Y}_{h,n}$ and $Y_{h,n}^*$ have the same distribution. As $n \to \infty$,

$$\overline{Y}_{h,n}(t) \to \overline{M}_h(t) - \int_0^t \Big(G(\overline{m}^h(s)) - (\gamma P_1 + P_2)\overline{m}^h(s) \Big)\, d\widehat{W}^h(s) = 0 \text{ in } L^2(\Omega;\mathbb{L}^2_{w_2}).$$

This implies that $Y_{h,n}^*(t)$ also converges to 0 in $L^2(\Omega^*;\mathbb{L}^2_{w_2})$ as $n \to \infty$. Thus,

$$M_h^*(t) = \int_0^t \big(G(m_h^*(s)) - (\gamma P_1 + P_2)\, m_h^*(s) \big)\, d\widehat{W}_h^*(s), \quad \mathbb{P}^*\text{-a.s.}$$

We observe that

$$\begin{aligned}
&\mathbb{E}^*\left[M_h^*(t) - \int_0^t G(m^*(s))\, dW^*(s) \right]\\
&= \mathbb{E}^*\left[\int_0^t \left(G(m_h^*(s)) - (\gamma P_1 + P_2) m_h^*(s) - \sum_{i=0}^{n-1} G(m_h^*(s_i^n)) \mathbb{1}_{(s_i^n, s_{i+1}^n]}(s) \right) d\widehat{W}_h^*(s) \right]\\
&\quad + \mathbb{E}^*\left[\sum_{i=0}^{n-1} G(m_h^*(s_i^n))\, \delta \widehat{W}_h^*(t,s_i^n) - \sum_{i=0}^{n-1} G(m^*(s_i^n))\, \delta W^*(t \wedge s_i^n) \right]\\
&\quad + \mathbb{E}^*\left[\int_0^t \left(\sum_{i=0}^{n-1} G(m^*(s_i^n)) \mathbb{1}_{(s_i^n, s_{i+1}^n]}(s) - G(m^*(s)) \right) dW^*(s) \right]\\
&= \mathbb{E}^*[J_0^h] + \mathbb{E}^*[J_1^h] + \mathbb{E}^*[J_2].
\end{aligned}$$

For J_0^h and J_2, let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that

$$\left(\mathbb{E}^*\left[\int_0^t \left|G(m^*(s)) - \sum_{i=0}^{n-1} G(m^*(s_i^n))\mathbb{1}_{(s_i^n,s_{i+1}^n]}(s)\right|^2_{\mathbb{H}^{-1}_{w_1}} ds\right]\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}. \tag{87}$$

Since $\widehat{W}_h^*$ and $\widehat{W}^h$ have the same laws on $\mathcal{C}([0,T];H^2(\mathbb{R}))$, we have

$$\begin{aligned}
\left(\mathbb{E}^*\left[|J_0^h|^2_{\mathbb{H}^{-1}_{w_1}}\right]\right)^{\frac{1}{2}} \leq & \left(\mathbb{E}^*\left[\int_0^t \sum_j q_j^2 \left|\widehat{f}_j G(m_h^*(s)) - \widehat{f}_j G(m^*(s))\right|^2_{\mathbb{H}^{-1}_{w_1}} ds\right]\right)^{\frac{1}{2}} \\
& + \left(\mathbb{E}^*\left[\int_0^t \sum_j q_j^2 \left|\widehat{f}_j \left(G(m^*(s)) - \sum_{i=0}^{n-1} G(m^*(s_i^n))\mathbb{1}_{(s_i^n,s_{i+1}^n]}(s)\right)\right|^2_{\mathbb{H}^{-1}_{w_1}} ds\right]\right)^{\frac{1}{2}} \\
& + \left(\mathbb{E}^*\left[\int_0^t \sum_j q_j^2 \left|\sum_{i=0}^{n-1} \left(\widehat{f}_j G(m^*(s_i^n)) - \widehat{f}_j G(m_h^*(s_i^n))\right)\mathbb{1}_{(s_i^n,s_{i+1}^n]}(s)\right|^2_{\mathbb{H}^{-1}_{w_1}} ds\right]\right)^{\frac{1}{2}} \\
& + \left(\mathbb{E}^*\left[\int_0^t \sum_j q_j^2 \left|\widehat{f}_j(\gamma P_1 + P_2) m_h^*(s)\right|^2_{\mathbb{H}^{-1}_{w_1}} ds\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

Recall that $\mathbb{L}^2_w \hookrightarrow \mathbb{H}^{-1}_{w_1}$ for all $w > w_1$. Let $w = w_1 + w_2$. As $h \to 0$, the first and the third term on the right-hand side converges to 0 by Lemma 5.6(iii), the fourth term converges to 0 by Lemma 4.2 and (6), and the second term is less than $\frac{\varepsilon}{2}$ by (87). Hence, for a sufficiently small h, we have

$$|J_0^h|_{L^2(\Omega^*;\mathbb{H}^{-1}_{w_1})} < \frac{\varepsilon}{2}.$$

Similarly, $|J_2|_{L^2(\Omega^*;\mathbb{H}^{-1}_{w_1})} < \frac{\varepsilon}{2}$.

For J_1^h, we have

$$\begin{aligned}
\mathbb{E}^*\left[|J_1^h|^2_{\mathbb{H}^{-1}_{w_1}}\right] \leq & \,\mathbb{E}^*\left[\left|\sum_{i=0}^{n-1} \left(G(m_h^*(s_i^n)) - G(m^*(s_i^n))\right)\delta W^*(t,s_i^n)\right|^2_{\mathbb{H}^{-1}_{w_1}}\right] \\
& + \mathbb{E}^*\left[\left|\sum_{i=0}^{n-1} G(m_h^*(s_i^n))\left(\delta\widehat{W}_h^*(t,s_i^n) - \delta W^*(t,s_i^n)\right)\right|^2_{\mathbb{H}^{-1}_{w_1}}\right].
\end{aligned}$$

Since W^* is a Q-Wiener process, the first term on the right-hand side converges to 0 by Lemma 5.6(iii), Also, the second term converges to 0 by the pointwise convergence (68) (or (86)) and the result $G(m_h^*)\rho_{\frac{w}{2}} \in L^2(\Omega^*;L^2(0,T;\mathbb{L}^\infty))$, which can be deduced from the estimates (70), (71) and (72).

Therefore, for any sufficiently small h,

$$\mathbb{E}^*\left[\left|M_h^*(t) - \int_0^t G(m^*(s))\, dW^*(s)\right|^2_{\mathbb{H}^{-1}_{w_1}}\right] < \varepsilon^2.$$

Using Lemma 5.7 and the uniqueness of weak limit, the proof is concluded. □

We are ready to prove the main theorem.

5.2.3. *Proof of Theorem 2.4.* By Lemmata 5.7 and 5.9, m^* satisfies the (12) in $\mathbb{H}^{-1}_{w_1}$. Moreover, using Lemma 5.3(i), we can simplify F, S and G:

$$\begin{aligned} F(m^*) &= -v\left(Dm^* + \gamma m^* \times Dm^*\right) - m^* \times D^2 m^* + \alpha D^2 m^* + \alpha |Dm^*|^2 m^*, \\ S(m^*) &= \kappa^2\left((1-\gamma^2)D^2 m^* - 2\gamma^2 |Dm^*|^2 m^* - 2\gamma m^* \times D^2 m^*\right) \\ &\quad + \kappa\kappa'\left((1-\gamma^2)Dm^* - 2\gamma m^* \times Dm^*\right), \\ G(m^*) &= -Dm^* + \gamma m^* \times Dm^*, \end{aligned}$$

and each of them is in $L^2(\Omega^*; L^2(0,T;\mathbb{L}^2))$, hence the equality (12) holds in $\mathbb{L}^2$. Recall the properties of m^* shown previously in (78) and Lemma 5.3, we have now verified that m^* is a solution of (8) in the sense of Definition 2.1. It only remains to show that $m - m_0 \in \mathcal{C}^\alpha([0,T];\mathbb{L}^2)$. For $s,t \in [0,T]$ and $p \in [1,\infty)$, there exists a constant C that may depend on p, T, C_κ such that

$$\begin{aligned} &\mathbb{E}^*\left[|m^*(t) - m^*(s)|^{2p}_{\mathbb{L}^2}\right] \\ &\le |t-s|^p \mathbb{E}^*\left[\left(\int_s^t \left|F(m^*(r)) + \frac{1}{2}S(m^*(r))\right|^2_{\mathbb{L}^2} dr\right)^p\right] \\ &\quad + \mathbb{E}^*\left[\left(\int_s^t \sum_j q_j^2 |f_j G(m^*)(r)|^2_{\mathbb{L}^2} dr\right)^p\right] \\ &\le C|t-s|^p \mathbb{E}^*\left[\sup_{r\in[0,T]} |Dm^*(r)|^{2p}_{\mathbb{L}^2} + \left(\int_s^t \left(|Dm^*(r)|^4_{\mathbb{L}^4} + |D^2 m^*(r)|^2_{\mathbb{L}^2}\right) dr\right)^p\right], \end{aligned}$$

where the expectation on the right-hand side is finite. Then by Kolmogorov's continuity criterion, $m^*(t) - m_0 \in \mathcal{C}^\alpha([0,T];\mathbb{L}^2)$, $\mathbb{P}^*$-a.s. for $\alpha \in (0, \frac{1}{2})$.

5.2.4. *Proof of Theorem 2.5.* Let (m_1, W) and (m_2, W) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be two solutions of (8) in the sense of Definition 2.1. Let $u = m_1 - m_2$ and $w > 0$. Applying Itô's lemma to $\frac{1}{2}|u(t)|^2_{\mathbb{L}^2_w}$,

$$\begin{aligned} \frac{1}{2}|u(t)|^2_{\mathbb{L}^2_w} &= |u(0)|^2_{\mathbb{L}^2_w} + \int_0^t \langle u(s), F(m_1(s)) - F(m_2(s))\rangle_{\mathbb{L}^2_w} dt \\ &\quad + \frac{1}{2}\int_0^t \langle u(s), S(m_1(s)) - S(m_2(s))\rangle_{\mathbb{L}^2_w} dt \\ &\quad + \frac{1}{2}\int_0^t \sum_j q_j^2 \left|f_j(G(m_1(s)) - G(m_2(s)))\right|^2_{\mathbb{L}^2_w} dt \\ &\quad + \int_0^t \langle u(s), (G(m_1(s)) - G(m_2(s)))\, dW(s)\rangle_{\mathbb{L}^2_w} \\ &= |u(0)|^2_{\mathbb{L}^2_w} + \int_0^t [U_1(s) + U_2(s) + U_3(s)]\, ds + U_4(t), \end{aligned} \tag{88}$$

An estimate on U_1:

$$\begin{aligned} U_1(s) &= \langle u, F(m_1) - F(m_2)\rangle_{\mathbb{L}^2_w}(s) \\ &= \left\langle u, v\left(-Du + \gamma u \times Dm_1 + \gamma m_2 \times Du\right) - u \times D^2 m_1 - m_2 \times D^2 u\right\rangle_{\mathbb{L}^2_w}(s) \\ &\quad + \alpha\left\langle u, \langle D(m_1+m_2), Du\rangle m_2 + D^2 u + |Dm_1|^2 u\right\rangle_{\mathbb{L}^2_w}(s) \\ &= \langle u, v(-Du + \gamma m_2 \times Du)\rangle_{\mathbb{L}^2_w}(s) + \left\langle u, -m_2 \times D^2 u\right\rangle_{\mathbb{L}^2_w}(s) \end{aligned}$$

$$+\alpha\langle u,\langle D(m_1+m_2),Du\rangle m_2\rangle_{\mathbb{L}^2_w}(s)+\alpha\left\langle D^2u,u\right\rangle_{\mathbb{L}^2_w}(s)+\alpha\left\langle u,|Dm_1|^2u\right\rangle_{\mathbb{L}^2_w}(s).$$

Then, for an arbitrary $\varepsilon>0$,

$$\begin{aligned}\langle u,v(-Du+\gamma m_2\times Du)\rangle_{\mathbb{L}^2_w}(s)&=\left\langle u\rho_w^{\frac{1}{2}},v(-Du+\gamma m_2\times Du)\rho_w^{\frac{1}{2}}\right\rangle_{\mathbb{L}^2}(s)\\&\le\frac{1}{2\varepsilon^2}C_v^2(1+\gamma^2)|u(s)|^2_{\mathbb{L}^2_w}+\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w},\end{aligned}\tag{89}$$

and

$$\begin{aligned}\left\langle u,-m_2\times D^2u\right\rangle_{\mathbb{L}^2_w}(s)&=-\left\langle D^2u,u\times m_2\rho_w\right\rangle_{\mathbb{L}^2}(s)\\&=\langle Du,Du\times m_2\rho_w+u\times D(m_2\rho_w)\rangle_{\mathbb{L}^2}(s)\\&=\langle Du,u\times Dm_2\rangle_{\mathbb{L}^2_w}+\left\langle Du,u\times m_2\rho_w'\rho_w^{-1}\right\rangle_{\mathbb{L}^2_w}(s)\\&\le\frac{1}{2\varepsilon^2}\left(|Dm_2(s)|^2_{\mathbb{L}^\infty}+w^2\right)|u(s)|^2_{\mathbb{L}^2_w}+\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w}.\end{aligned}\tag{90}$$

Similarly,

$$\alpha\langle u,\langle D(m_1+m_2),Du\rangle m_2\rangle_{\mathbb{L}^2_w}(s)\le\frac{1}{2\varepsilon^2}\alpha^2|Dm_1(s)+Dm_2(s)|^2_{\mathbb{L}^\infty}|u(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2}\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w},\tag{91}$$

and

$$\begin{aligned}\alpha\left\langle u,D^2u\right\rangle_{\mathbb{L}^2_w}(s)&=\alpha\left\langle u\rho_w,D^2u\right\rangle_{\mathbb{L}^2}(s)\\&=-\alpha\langle Du,D(u\rho_w)\rangle_{\mathbb{L}^2}(s)\\&=-\alpha|Du(s)|^2_{\mathbb{L}^2_w}-\alpha\left\langle Du,u\rho_w'\rho_w^{-1}\right\rangle_{\mathbb{L}^2_w}(s)\\&=-\alpha|Du(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2}\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2\varepsilon^2}\alpha^2w^2|u|^2_{\mathbb{L}^2_w}.\end{aligned}\tag{92}$$

Also,

$$\alpha\left\langle u,|Dm_1|^2u\right\rangle_{\mathbb{L}^2_w}(s)\le\alpha|Dm_1(s)|^2_{\mathbb{L}^\infty}|u(s)|^2_{\mathbb{L}^2_w}.\tag{93}$$

Hence,

$$U_1(s)\le\psi_1(s)|u(s)|^2_{\mathbb{L}^2_w}+(3\varepsilon^2-\alpha)|Du(s)|^2_{\mathbb{L}^2_w},$$

for the process ψ_1 given by

$$\begin{aligned}\psi_1(s)=&\frac{1}{2\varepsilon^2}\left(C_v^2(1+\gamma^2)+|Dm_2(s)|^2_{\mathbb{L}^\infty}+w^2+\alpha^2|Dm_1(s)+Dm_2(s)|^2_{\mathbb{L}^\infty}+\alpha^2w^2\right)\\&+\alpha|Dm_1(s)|^2_{\mathbb{L}^\infty}.\end{aligned}\tag{94}$$

For $i=1,2$, there exists a constant $C>0$ such that

$$\mathbb{E}^*\left[\int_0^T|Dm_i|^2_{\mathbb{L}^\infty}(t)\,dt\right]\le\mathbb{E}^*\left[\int_0^T|Dm_i|^2_{\mathbb{H}^1}(t)\,dt\right]<\infty,\tag{95}$$

which implies $\int_0^T\psi_1(t)\,dt<\infty$, $\mathbb{P}$-a.s.

<u>An estimate on U_2:</u>

$$\begin{aligned}U_2(s)&=\frac{1}{2}\langle u,S(m_1)-S(m_2)\rangle_{\mathbb{L}^2_w}(s)\\&=\frac{1}{2}\left\langle u,\kappa^2\left[(1-\gamma^2)D^2u-2\gamma(u\times D^2m_1+m_2\times D^2u)\right]\right\rangle_{\mathbb{L}^2_w}(s)\end{aligned}$$

$$
\begin{aligned}
&-\gamma^2\left\langle u,\kappa^2\left[\langle D(m_1+m_2),Du\rangle m_2+|Dm_1|^2u\right]\right\rangle_{\mathbb{L}^2_w}(s)\\
&+\frac{1}{2}\left\langle u,\kappa\kappa'\left[(1-\gamma^2)Du-2\gamma(u\times Dm_1+m_2\times Du)\right]\right\rangle_{\mathbb{L}^2_w}(s)\\
=&\frac{1}{2}\left\langle u,\kappa\kappa'\left[(1-\gamma^2)Du-2\gamma m_2\times Du\right]\right\rangle_{\mathbb{L}^2_w}(s)\\
&+\frac{1}{2}(1-\gamma^2)\left\langle u,\kappa^2D^2u\right\rangle_{\mathbb{L}^2_w}(s)-\gamma\left\langle u,\kappa^2m_2\times D^2u\right\rangle_{\mathbb{L}^2_w}(s)\\
&-\gamma^2\left\langle u,\kappa^2\langle D(m_1+m_2),Du\rangle m_2\right\rangle_{\mathbb{L}^2_w}(s)-\gamma^2\left\langle u,\kappa^2|Dm_1|^2u\right\rangle_{\mathbb{L}^2_w}(s).
\end{aligned}
$$

Again, for $\varepsilon>0$,

$$
\begin{aligned}
&\frac{1}{2}\left\langle u,\kappa\kappa'\left[(1-\gamma^2)Du-2\gamma m_2\times Du\right]\right\rangle_{\mathbb{L}^2_w}(s)\\
&\leq\frac{1}{4\varepsilon^2}C_\kappa^4\left((1-\gamma^2)^2+4\gamma^2\right)|u(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2}\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w}.
\end{aligned}
$$

As in (90) and (91),

$$
\begin{aligned}
-\gamma\left\langle u,\kappa^2m_2\times D^2u\right\rangle_{\mathbb{L}^2_w}(s)&=-\gamma\left\langle D^2u,\kappa^2u\times m_2\rho_w\right\rangle_{\mathbb{L}^2}(s)\\
&=\gamma\left\langle Du,Du\times m_2\kappa^2\rho_w+u\times D(m_2\kappa^2\rho_w)\right\rangle_{\mathbb{L}^2}(s)\\
&=\gamma\left\langle Du,\kappa^2u\times Dm_2+\kappa^2u\times m_2\rho_w'\rho_w^{-1}+2\kappa\kappa'u\times m_2\right\rangle_{\mathbb{L}^2_w}(s)\\
&\leq\frac{1}{2\varepsilon^2}\gamma^2C_\kappa^4\left(|Dm_2(s)|^2_{\mathbb{L}^\infty}+w^2+4\right)|u(s)|^2_{\mathbb{L}^2_w}+\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w},
\end{aligned}
$$

and

$$
-\gamma^2\left\langle u,\kappa^2\langle D(m_1+m_2),Du\rangle m_2\right\rangle_{\mathbb{L}^2_w}\leq\frac{1}{2\varepsilon^2}\gamma^4C_\kappa^4|Dm_1(s)+Dm_2(s)|^2_{\mathbb{L}^\infty}|u(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2}\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w}.
$$

Also,

$$
-\gamma^2\left\langle u,\kappa^2|Dm_1|^2u\right\rangle_{\mathbb{L}^2_w}(s)\leq 0,\quad\forall s\in[0,T].
$$

For the remaining term in U_2, we use integration-by-parts as in (92):

$$
\begin{aligned}
&\frac{1}{2}(1-\gamma^2)\left\langle u,\kappa^2D^2u\right\rangle_{\mathbb{L}^2_w}\\
&=\frac{1}{2}(\gamma^2-1)\left\langle Du,D(\kappa^2u\rho_w)\right\rangle_{\mathbb{L}^2}\\
&=\frac{1}{2}(\gamma^2-1)\left[\left\langle Du,\kappa^2Du\right\rangle_{\mathbb{L}^2_w}(s)+\left\langle Du,\kappa^2u\rho_w'\rho_w^{-1}+2\kappa\kappa'u\right\rangle_{\mathbb{L}^2_w}(s)\right]\\
&\leq-\frac{1}{2}\sum_j q_j^2|f_jDu(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2}\gamma^2C_\kappa^2|Du(s)|^2_{\mathbb{L}^2_w}+\frac{1}{4\varepsilon^2}C_\kappa^4\left(1-\gamma^2\right)^2(w^2+4)\,|u(s)|^2_{\mathbb{L}^2_w}\\
&\quad+\frac{1}{2}\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w}.
\end{aligned}
$$

Thus,

$$
U_2(s)\leq\psi_2(s)|u(s)|^2_{\mathbb{L}^2_w}+\frac{5}{2}\varepsilon^2|Du(s)|^2_{\mathbb{L}^2_w}+\frac{1}{2}\gamma^2C_\kappa^2|Du(s)|^2_{\mathbb{L}^2_w}-\frac{1}{2}\sum_j q_j^2|f_jDu(s)|^2_{\mathbb{L}^2_w},
$$

where

$$\psi_2(s) = \frac{1}{4\varepsilon^2} C_\kappa^4 (1-\gamma^2)^2 (w^2+5) + \frac{1}{2\varepsilon^2}\left(\gamma^2(w^2+6) + \gamma^2 |Dm_2(s)|^2_{\mathbb{L}^\infty} + \gamma^4 |Dm_1(s)+Dm_2(s)|^2_{\mathbb{L}^\infty}\right), \tag{96}$$

and by (95), $\int_0^T \psi_2(t)\,dt < \infty$, $\mathbb{P}$-a.s.

An estimate on U_3:

$$U_3(s) = \frac{1}{2}\sum_j q_j^2 \left| f_j\left(G(m_1)-G(m_2)\right)\right|^2_{\mathbb{L}^2_w}(s)$$
$$= \frac{1}{2}\sum_j q_j^2 \left| f_j\left(-Du(s) + \gamma u(s)\times Dm_1(s) + \gamma m_2(s)\times Du(s)\right)\right|^2_{\mathbb{L}^2_w}(s),$$

where for every $j \geq 1$,

$$\begin{aligned}
& f_j^2 \left|G(m_1)-G(m_2)\right|^2(s,x) \\
&= f_j^2 \left|-Du + \gamma u \times Dm_1 + \gamma m_2 \times Du\right|^2(s,x) \\
&= f_j^2 \left(|Du|^2 + \gamma^2 |m_2\times Du|^2 + \gamma^2 |u\times Dm_1|^2 + 2\gamma \langle -Du + \gamma m_2 \times Du, u \times Dm_1\rangle\right)(s,x) \\
&\leq (1+\gamma^2)|f_j Du(s,x)|^2 + q_j^2 \gamma^2\ |Dm_1(s)|^2_{\mathbb{L}^\infty}\ |f_j u(s,x)|^2 \\
&\quad + \frac{2}{\varepsilon^2}\gamma^2\left(1+\gamma^2\right)|Dm_1(s)|^2_{\mathbb{L}^\infty}\ |f_j^2 u(s,x)|^2 + \varepsilon^2 |Du(s,x)|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
U_3(s) &\leq \frac{1}{2}(1+\gamma^2)\sum_j q_j^2 |f_j Du(s)|^2_{\mathbb{L}^2_w} + \gamma^2 C_\kappa^2 \left(\frac{1}{2} + \frac{1}{\varepsilon^2}(1+\gamma^2)C_\kappa^2\right)\ |Dm_1(s)|^2_{\mathbb{L}^\infty} |u(s)|^2_{\mathbb{L}^2_w} \\
&\quad + \frac{1}{2}\varepsilon^2 |Du(s)|^2_{\mathbb{L}^2_w} \\
&\leq \psi_3(s)|u(s)|^2_{\mathbb{L}^2_w} + \frac{1}{2}\sum_j q_j^2 |f_j Du(s)|^2_{\mathbb{L}^2_w} + \frac{1}{2}\gamma^2 C_\kappa^2 |Du(s)|^2_{\mathbb{L}^2_w} + \frac{1}{2}\varepsilon^2 |Du(s)|^2_{\mathbb{L}^2_w},
\end{aligned}$$

where the second term on the right-hand side cancels with the corresponding term in $U_2(s)$ and $\psi_3(s) = \gamma^2 C_\kappa^2 \left(\frac{1}{2} + \frac{1}{\varepsilon^2}(1+\gamma^2)C_\kappa^2\right)|Dm_1(s)|^2_{\mathbb{L}^\infty}$ is similarly integrable $\mathbb{P}$-a.s.

We have

$$U_1(s) + U_2(s) + U_3(s) \leq \left(\psi_1(s)+\psi_2(s)+\psi_3(s)\right)|u(s)|^2_{\mathbb{L}^2_w} + \left(6\varepsilon^2 + \gamma^2 C_\kappa^2 - \alpha\right)|Du(s)|^2_{\mathbb{L}^2_w}.$$

We can choose a sufficiently small $\varepsilon > 0$ such that under the assumption (20),

$$\left(6\varepsilon^2 + \gamma^2 C_\kappa^2 - \alpha\right) < 0,$$

which implies

$$U_1(s) + U_2(s) + U_3(s) \leq \left(\psi_1(s)+\psi_2(s)+\psi_3(s)\right)|u(s)|^2_{\mathbb{L}^2_w} = \psi(s)\ |u(s)|^2_{\mathbb{L}^2_s}.$$

Therefore, by (88),

$$\frac{1}{2}d|u(t)|^2_{\mathbb{L}^2_w} \leq \psi(t)|u(t)|^2_{\mathbb{L}^2_w}\,dt + \langle u(t), [G(m_1(t)) - G(m_2(t))]\,dW(t)\rangle_{\mathbb{L}^2_w}.$$

Define the process Y by

$$Y(t) := \frac{1}{2}|u(t)|^2_{\mathbb{L}^2_w} e^{-2\int_0^t \psi(s)\, ds}, \quad t \in [0,T].$$

Then,

$$\begin{aligned} dY(t) &= \left\langle \frac{1}{2} d|u(t)|^2_{\mathbb{L}^2_w}, e^{-2\int_0^t \psi(s)\, ds} \right\rangle + \left\langle \frac{1}{2}|u(t)|^2_{\mathbb{L}^2_w}, de^{-2\int_0^t \psi(s)\, ds} \right\rangle \\ &\quad + \left\langle \frac{1}{2} d|u(t)|^2_{\mathbb{L}^2_w}, de^{-2\int_0^t \psi(s)\, ds} \right\rangle \\ &\leq e^{-2\int_0^t \psi(s)\, ds} \langle u(t), [G(m_1(t)) - G(m_2(t))]\, dW(t)\rangle_{\mathbb{L}^2_w}. \end{aligned}$$

Since $|u(t)|_{\mathbb{L}^\infty} \leq 2$ $\mathbb{P}$-a.s. and there exists a constant C such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left(|Dm_1(t)|^2_{\mathbb{L}^2} + |Dm_2(t)|^2_{\mathbb{L}^2}\right)\right] \leq C,$$

the process

$$M(t) := \int_0^t e^{-2\int_0^s \psi(r)\, dr} \langle u(s), [G(m_1(s)) - G(m_2(s))]\, dW(s)\rangle_{\mathbb{L}^2_w}.$$

is a martingale, and then

$$\mathbb{E}[Y(t)] \leq Y(0) + \mathbb{E}[M(t)] = Y(0), \quad t \in [0,T].$$

By the definition of $Y(t)$, if $Y(0) = m_1(0) - m_2(0) = 0$, then

$$|u(t)|^2_{\mathbb{L}^2_w} = 0, \quad \mathbb{P}\text{-a.s.}$$

for $t \in [0,T]$, proving pathwise uniqueness of the solution of (8). By the Yamada-Watanabe Theorem, the uniqueness in law follows.

APPENDIX A.

A.1. **Some calculations in discrete spaces.**

(a) discrete integration-by-parts:

$$\sum_{x\in\mathbb{Z}_h} \langle u(x), \partial^h w(x)\rangle = -\sum_{x\in\mathbb{Z}_h} \langle \partial^h u^-(x), w(x)\rangle,$$

for $u, w \in \mathbb{H}^1_h = \{v \in \mathbb{L}^2_h : |\partial^h v|_{\mathbb{L}^2_h} < \infty\}$ with appropriate decay properties. In particular,

$$\left\langle \partial^h u, \partial^h w \right\rangle_{\mathbb{L}^2_h} = -\left\langle \Delta^h u, w \right\rangle_{\mathbb{L}^2_h}.$$

(b) discrete expansion of $\langle u, \Delta^h u\rangle$ and $\langle u, \partial^h u\rangle$: for any u satisfying that $|u(x)| = 1$ for all $x \in \mathbb{Z}_h$,

$$\begin{aligned} \langle u(x), \Delta^h u(x)\rangle &= -\frac{1}{2}\left(|\partial^h u(x)|^2 + |\partial^h u^-(x)|^2\right) \leq 0, \\ \langle u(x), \partial^h u(x)\rangle &= -\frac{h}{2}|\partial^h u(x)|^2 \leq 0. \end{aligned} \tag{97}$$

(c) product rule:

$$\partial^h(fu) = (\partial^h f)u(x) + f^+\partial^h u = (\partial^h f)u^+ + f\partial^h u(x)$$

for f scalar-valued and u vector-valued; similarly for f and u both scalar-valued, and for $\langle f,u\rangle$ and $u\times u$ when f,u are vector-valued.

(d) L^2_h-norm of $\Delta^h u$:

$$|\Delta^h u|_{\mathbb{L}^2_h} = |\partial^h(\partial^h u)^-|_{\mathbb{L}^2_h} = |\partial^h(\partial^h u)|_{\mathbb{L}^2_h}.$$

Lemma A.1 ([17, Chapter 1, Theorem 3])**.** *For $u^h:\mathbb{Z}_h\to\mathbb{R}$,*

$$\left|(\partial^h)^k u^h\right|_{\mathbb{L}^p_h} \le C|u^h|_{\mathbb{L}^2_h}^{1-\frac{k+\frac{1}{2}-\frac{1}{p}}{n}}\left|(\partial^h)^n u^h\right|_{\mathbb{L}^2_h}^{\frac{k+\frac{1}{2}-\frac{1}{p}}{n}},$$

for $p\in[2,\infty]$, $k\in[0,n)$ and C is a constant independent of u^h.

A.2. **Some tightness results.**

Lemma A.2 ([6, Theorem 2.1])**.** *Let $B_0\subset B\subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B. Let $p\in(1,\infty)$ and $\alpha\in(0,1)$ be given. Let X be the space*

$$X = L^p(0,T;B_0)\cap W^{\alpha,p}(0,T;B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

Lemma A.3 ([6, Theorem 2.2])**.** *If $B_1\subset\widetilde{B}$ are two Banach spaces with compact embedding, and the real numbers $\alpha\in(0,1)$, $p>1$ satisfy $\alpha p>1$, then the space $W^{\alpha,p}(0,T;B_1)$ is compactly embedded into $\mathcal{C}([0,T];\widetilde{B})$.*

Lemma A.4 ([15, Corollary 19])**.** *Let I be an either bounded or unbounded interval of $\mathbb{R}$. Let E be a Banach space. Suppose $s\ge r$, $p\le q$ and $s-\frac{1}{p}\ge r-\frac{1}{q}$ for $0<r\le s<1$ and $1\le p\le q\le\infty$. Then,*

$$W^{s,p}(I;E)\hookrightarrow W^{r,q}(I;E).$$

In addition, we verify the continuous embedding

$$W^{\alpha,4}(0,T;\mathbb{L}^2_w)\hookrightarrow W^{\beta,2}(0,T;\mathbb{L}^2_w),\quad w\ge 1,\ \alpha\in\left(\frac{1}{4},\frac{1}{2}\right),\ \beta=\alpha-\frac{1}{4}. \tag{98}$$

Indeed, for $u\in W^{\alpha,4}(0,T;\mathbb{L}^2_w)$,

$$\begin{aligned}
|u|_{W^{\beta,2}(0,T;\mathbb{L}^2_w)} &= \left[\int_0^T |u(t)|^2_{\mathbb{L}^2_w}\,dt + \int_0^T\int_0^T \frac{|u(t)-u(s)|^2_{\mathbb{L}^2_w}}{|t-s|^{1+2\beta}}\,dt\,ds\right]^{\frac{1}{2}}\\
&\le \left[\left(\int_0^T |u(t)|^4_{\mathbb{L}^2_w}\,dt\right)^{\frac{1}{2}} T^{\frac{1}{2}} + \left(\int_0^T\int_0^T \frac{|u(t)-u(s)|^4_{\mathbb{L}^2_w}}{|t-s|^{2+4\beta}}\,dt\,ds\right)^{\frac{1}{2}} T\right]^{\frac{1}{2}}\\
&\le \left[2T\int_0^T |u(t)|^4_{\mathbb{L}^2_w}\,dt + 2T^2\int_0^T\int_0^T \frac{|u(t)-u(s)|^4_{\mathbb{L}^2_w}}{|t-s|^{2+4\beta}}\,dt\,ds\right]^{\frac{1}{4}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(2\max\{T,T^2\}\right)^{\frac{1}{4}} \left[\int_0^T |u(t)|^4_{\mathbb{L}^2_w}\, dt + \int_0^T \int_0^T \frac{|u(t)-u(s)|^4_{\mathbb{L}^2_w}}{|t-s|^{1+4\alpha}}\, dt\, ds\right]^{\frac{1}{4}} \\
&= \left(2\max\{T,T^2\}\right)^{\frac{1}{4}} |u|_{W^{\alpha,4}(0,T;\mathbb{L}^2_w)},
\end{aligned}
$$

where the second inequality holds by $\sqrt{a}+\sqrt{b} \leq \sqrt{2a+2b}$ for $a,b \geq 0$.

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