Degenerate Monge-Ampere Equations over Projective Manifolds

by

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Abstract

In this thesis, we study degenerate Monge-Ampere equations over projective manifolds. The main degeneration is on the cohomology class which is Kähler in classic cases. Our main results concern the case when this class is semi-ample and big with certain generalization to more general cases.

Two kinds of arguments are applied to study this problem. One is maximum principle type of argument. The other one makes use of pluripotential theory. So this article mainly consists of three parts. In the first two parts, we apply these two kinds of arguments separately and get some results. In the last part, we try to combine the results and arguments to achieve better understanding about interesting geometric objects. Some interesting problems are also mentioned in the last part for future consideration. The generalization of classic pluripotential theory in the second part may be of some interest by itself.

Thesis Supervisor: Gang Tian
Title: Simons Professor of Mathematics

Acknowledgments

At the very end of the final stage as a student officially, I would like to take this wonderful opportunity to show my gratefulness to all the people who have helped me along the way in the past years.

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I must still have missed quite some important people above. So to end the acknowledgements, let me say

to my families who have been supporting my own will with all they have,

to my teachers whose guidance lies on each tiny step I have taken,

to my friends whose choices have affected mine with or without being noticed,

thank you so much.

Dedicated to My Parents and Sister

Contents

Chapter 1

Introduction

1.1 Main Problem

In this thesis, we mainly want to solve Monge-Ampere equations over projective manifolds when cohomology classes are degenerate (as Kähler class).

The main concern is for the following equation over a projective manifold X:

$$(L + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

where L is a smooth closed real $(1,1)$-form and Ω is a smooth volume form over X.

In the case when $[L]$ is Kähler, classic results guarantee existence and uniqueness of a smooth solution. The degeneration we are interested in is described by nef. (i.e., numerically effective) or big of the cohomology class $[L]$. The main results are briefly described as follows:

1) For $[L]$ nef. and big, we have a unique solution which is smooth out of a subvariety with some controls about the possible singularities, for example, Lelong number being 0;

2) For $[L]$ semi-ample and big, the solution above would be bounded over X. Furthermore, the bounded solution is actually unique and continuous.

Actually, some of our discuss would work in more general cases and even for the equations which are without e^u or with e^{-u} instead on the right hand side of the equation just as well. We might go into details for similarities and differences along the way.

The whole scheme to attack our main problem is more or less just trying to apply continuity method. Various estimates need to be obtained. There are basically two kinds of arguments which will be heavily used in this work. They are both fairly classic by themselves.

One is maximum principle type of argument. Grossly speaking, for applying this argument, we need to work with smooth objects. This kind of argument is very global, and so it treats the degeneracy mentioned above in a very brutal way. Special attention should also be paid when studying smooth but degenerate objects.

The other one is "capacity argument", which might not be a standard way to call this kind of argument. Anyway, this argument sits inside the setting of pluripotential theory and moreover, the notion of capacity, as I see it, plays a very crucial role. The difference is that now we can work with objects with far less regularity. This argument also has quite some local feature which would give us some room to treat the degeneracy in a more delicate manner.

As mentioned in Abstract, the main text of this thesis can be divided into three parts. Part I consists of Chapter 2 where maximum principle type of argument is mainly used. Part II consists of Chapters 3 to 7 where capacity argument using pluripotential theory is the main tool. During the process, we also have to generalize classic pluripotential theory a little bit. This part weights the most in the whole work. Chapter 8 is the last part which contains applications and generalizations. Chapter 9 is Appendix where details omitted in the main text are provided together with discussions about some interesting and related problems which might seem to be a little too digressive to appear in the main text.

In the following, we are going to give the set-ups of two main objects which will be used to attack the main problem later. Those two kinds of arguments previously mentioned are applied in a quite dominant way respectively.

1.2 Kähler-Ricci Flow

Kähler-Ricci flow is a very interesting and classic object by itself. The most primitive version of it, Ricci flow, was introduced by R. Hamiliton in dealing with quite difficult problems in Riemannian geometry in the early eighties of the last century (in [Ha1]). Since then it has drawn a lot of intention from the mathematical world. With the effort of a huge group of brilliant mathematicians in the past about twenty years, it has become one of the most active and fruitful fields in Riemannian geometry.

The recent work of Perelman in [Per], which has at least taken a substantial step in realizing Hamilton and other people's hope of applying this geometric tool to prove topological properties of the underlying manifold, has obviously carried all mathematicians' interests about Ricci flow into a new level. There has been quite some convenient references on Ricci flow as Hamilton's survey paper ([Ha2]) and a more recent book by B. Chow and D. Knopf ([ChoKn]).

Kähler-Ricci flow is the complex version of Ricci flow (or say Ricci flow with complex structure), which was naturally introduced shortly after the appearance of Ricci flow (as in [Cao] for example). For real dimension 2 (or complex dimension 1) case, they are trivially the same. The study for this case has been very complete and satisfying from the works of B. Chow ([Cho]) and Hamilton ([Ha3]). Of course, the higher dimension case remains to be interesting in general.

More precisely, in Part II of this thesis, we consider the following evolution equation over a closed manifold X of complex dimension $n \geqslant 2$:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) + k \cdot \tilde{\omega}_t + S, \tag{1.1}$$

with initial metric $\tilde{\omega}_0 = \omega_0$. Here k is a fixed real number, S is some fixed smooth real closed $(1,1)$-form, and ω_0 is some Kähler metric on X. The requirement of $n \geqslant 2$ is just used to guarantee the nonlinearity of the equation.

Conventionally, we denote a Kähler metric by its Kähler form ω, in local complex coordinates $\{z^1, \cdots, z^n\}$,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where we use the standard convention for summation and $(g_{i\bar{j}})$ is the positive hermitian matrix valued function given by $g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$. $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, i.e., in the complex coordinates above, $\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ where $(R_{i\bar{j}})$ is the Ricci tensor of ω.

The positivity of $\tilde{\omega}_t$ can by justified by looking at the real version of the flow which is just Ricci flow with minor modification and using the corresponding result. Actually it's also quite clear once we realize that the eigenvalues for $\tilde{\omega}_t$ should vary continuously. One can also easily justify by integrating over t for the equation that the metric $\tilde{\omega}_t$ along the flow remains to be Kähler as long as the flow exists in classic sense (smoothly).

So let's formally take a look at the flow in the level of cohomology class:

$$\frac{\partial[\tilde{\omega}_t]}{\partial t} = -[\mathrm{Ric}(\Omega)] + k \cdot [\tilde{\omega}_t] + [S]$$

with $[\tilde{\omega}_0] = [\omega_0]$. Here Ω is some smooth volume form over X compatible with the complex structure (just the orientation), and

$$\mathrm{Ric}(\Omega) := -\sqrt{-1}\partial\bar{\partial}\log\big(\frac{\Omega}{(\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n}\big),$$

which will formally be denoted by $-\sqrt{-1}\partial\bar{\partial}\log\Omega$ in the rest of this work. This form actually represents the first Chern class of X (up to a conventional positive constant). Of course it is important that the underlying manifold X is Kähler in order for this to be true, but this is already assumed by the existence of Kähler metric ω_0.

Now we can solve this ordinary differential equation as follows. First rewrite it as:

$$\frac{\partial(e^{-kt}[\tilde{\omega}_t])}{\partial t} = e^{-kt}([S] - [\mathrm{Ric}(\Omega)]).$$

Then integrate over $[0, t]$ to get:

$$e^{-kt}[\tilde{\omega}_t] - [\omega_0] = ([S] - [\mathrm{Ric}(\Omega)]) \int_0^t e^{-ks} dt.$$

For the case $k = 0$:

$$[\tilde{\omega}_t] - [\omega_0] = t([S] - [\mathrm{Ric}(\Omega)]).$$

This suggests that if we are looking for a limit for $\tilde{\omega}_t$ in any sense, it's natural to require $[S] = [\mathrm{Ric}(\Omega)]$ and then the cohomology class of $\tilde{\omega}_t$ is not going to change along the flow. In fact, in this case, the limit does exist for any Kähler class and this actually provides one proof of Calabi's conjecture as in [Cao]. [1]

Now for the case $k \neq 0$:

$$e^{-kt}[\tilde{\omega}_t] - [\omega_0] = \frac{1}{k}(1 - e^{-kt})([S] - [\mathrm{Ric}(\Omega)]).$$

[1] More discussion can be found in Appendix.

So we can conclude

$$
\begin{aligned}
[\tilde{\omega}_t] &= e^{kt}[\omega_0] + \frac{1}{k}(e^{kt}-1)([S]-[\mathrm{Ric}(\Omega)]) \\
&= \frac{1}{k}([\mathrm{Ric}(\Omega)]-[S]) + \frac{e^{kt}}{k}([S]-[\mathrm{Ric}(\Omega)] + k\cdot[\omega_0]).
\end{aligned}
$$

Thus if we are looking for a limit in any sense, it is natural to require either $k<0$ or $[S]-[\mathrm{Ric}(\Omega)]+k[\omega_0]=0$. Before including this extra information in our computation, we can do more general computation for $k\neq 0$. Namely, we can reduce the equation (1.1) for Kähler metric to the level of potential with respect to some "background Kähler metric" which is changing in an explicit way, which obviously makes the problem at least looks much easier. It is also the case when $k=0$ as used in [Cao]. This is indeed one of the main differences between Kähler-Ricci flow and general Ricci flow.

Set $\omega_t = \frac{1}{k}(\mathrm{Ric}(\Omega)-S)+\frac{e^{kt}}{k}(S-\mathrm{Ric}(\Omega)+k\cdot\omega_0)$ (as "background Kähler metric"). Notice that we do not require ω_t to be positive. Then we assume: $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ where u is a real smooth function on X for each fixed t. This is reasonable since ω_t and $\tilde{\omega}_t$ are of the same $(1,1)$-class over the closed Kähler manifold X. Let's further assume that u is also smooth with respect to time t, and so we have:

$$
\frac{\partial\tilde{\omega}_t}{\partial t} = \frac{\partial\omega_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t} = e^{kt}(S-\mathrm{Ric}(\Omega)+k\cdot\omega_0) + \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}.
$$

The equation (1.1) can also be rewritten as:

$$
\begin{aligned}
\frac{\partial\tilde{\omega}_t}{\partial t} &= \sqrt{-1}\partial\bar{\partial}\log(\tilde{\omega}_t^n) + k\cdot(\omega_t+\sqrt{-1}\partial\bar{\partial}u) + S \\
&= \sqrt{-1}\partial\bar{\partial}\log(\tilde{\omega}_t^n) + (\mathrm{Ric}(\Omega)-S) + e^{kt}(S-\mathrm{Ric}(\Omega)+k\cdot\omega_0) + k\cdot\sqrt{-1}\partial\bar{\partial}u + S \\
&= \sqrt{-1}\partial\bar{\partial}\log(\tilde{\omega}_t^n) - \sqrt{-1}\partial\bar{\partial}\log\Omega + k\cdot\sqrt{-1}\partial\bar{\partial}u + e^{kt}(S-\mathrm{Ric}(\Omega)+k\cdot\omega_0).
\end{aligned}
$$

By combining the above two equations, we arrive at:

$$
\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t} = \sqrt{-1}\partial\bar{\partial}\log\frac{\tilde{\omega}_t^n}{\Omega} + k\cdot\sqrt{-1}\partial\bar{\partial}u.
$$

Notice that the term in "log" is now a global smooth function which is obviously positive over X.

So if we want to solve (1.1), it would suffice to solve the following scalar equation:

$$
\frac{\partial u}{\partial t} = \log\frac{\tilde{\omega}_t^n}{\Omega} + k\cdot u \tag{1.2}
$$

with initial value $u(0,\cdot)=0$. In fact the converse is also true which can be justified as follows.

First, the uniqueness of the solution for (1.1) and (1.2) can be seen easily. Actually for (1.1), one only needs to observe that it is just the slightly modified Ricci flow over X with initial metric being Kähler. So the uniqueness is inherited from the classic result of Ricci flow. [2] For (1.2), it's very easy to take linearization and see the parabolicity. From above we know the solutions for these two equations will correspond to the same metrics as long as the solution for (1.2) exists.

Now it only left to show the existence of the solution for (1.2) under the assumption that the solution for (1.1) exists for certain time interval. Suppose the solution u exists in $[0,T)$ while $\tilde{\omega}_t$ exists in $[0,T]$ for some $T<\infty$. Then by the uniqueness result from above, we know

$$\tilde{\omega}_t=\omega_t+\sqrt{-1}\partial\bar{\partial}u$$

in $[0,T)$. The uniform bounds for $\tilde{\omega}_t$ give bounds for the Laplacians of $u(t,\cdot)$. It only remains to get a uniform bound of u itself (i.e., the C^0 bound) in $[0,T)$ in order to prove that u can be extended smoothly to the time T. Using the unform bound of the metric in $[0,T)$, this is easy to see from the equation (1.2) itself.

Hence we can conclude the equivalence of the equations in the level of metric (1.1) and potential (1.2). Clearly similar argument works for the case with general k.

We should notice the value of u can actually be up to some function only depending on t for the same metric $\tilde{\omega}_t$. In fact the evolution equation of it, (1.2), can be modified by adding smooth function only depending on t on the right hand side without essentially changing the flow. Also we can see the choice of the smooth volume form Ω is rather superficial, namely, differential choices would lead to equivalent evolution equations in the level of potential. In fact, the volume form does not appear in the orginal flow in the level of metric, so this should be the case. Explicit computation for this would appear later in Chapter 2.

By the way, we'll only consider the nonzero constant k to be 1 or -1. This will not affect the general consideration of the equations since we can do a simple rescaling (of metric and time) to translate the equation for a general k into one of these cases. Of course this is just the standard normalization that people use for geometric consideration.

[2] We can also easily see the parabolicity of the equation (1.1) by directly taking linearization and noticing that we only need to take care of real closed $(1,1)$-forms.

Now we start to consider the cases from the cohomology consideration about the existence of limit before.

For $k = 1$ and $[S] - [\text{Ric}(\Omega)] + k[\omega_0] = 0$: We have $[\omega_0] = [\tilde{\omega}_t] = [\text{Ric}(\Omega)] - [S]$. Thus all the metrics have to be in the same Kähler class. By choosing proper Ω we can in fact set $\omega = \text{Ric}(\Omega) - S$ as before. Here we can omit the lower index t in sight of t-independence and make sure that it is indeed a Kähler metric. Now the equation becomes:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} + u.$$

This is the usual equation that people work with when studying complex manifolds with positive first Chern classes. The "+" on the right hand side before u makes the situation very different from our main concern in this work. But still it's easy to see the global existence of the solution for this evolution equation, i.e. the solution exists for $t \in [0, \infty)$. For the proof, just notice that we only need to work with any finite time interval $[0, T]$ and $T < \infty$ can be used as a constant in all the estimates. The global existence can also be by related this flow with other flows. More discussion can be found in Appendix.

Our discussion here about the $k = 1$ case is by no means very sophisticated. In fact this equation is of plenty of interests for a big group of people. A huge number of results have thrown great light to a bunch of related geometric problems. The paper by X. X. Chen and Tian ([ChxTi]) provides an excellent example in this direction, so does the more recent work of D. H. Phong and J. Sturm ([PhSt]).

The case when $k = -1$ is our main interest in this work. At this moment, we just point out that in this case, $\omega_t = (S - \text{Ric}(\Omega)) + e^{-t}(\omega_0 - S + \text{Ric}(\Omega))$ and the limiting class would be $[\omega_\infty] = S - \text{Ric}(\Omega)$. It would correspond to the class $[L]$ in the main problem.

Remark 1.2.1. *Maybe we should end this part by pointing out the reason to study Kähler-Ricci flow for solving Monge-Ampere equation. Philosophically, they share the same nonlinearity and the expressions are almost the same. More precisely, Monge-Ampere equation can be seen as the limit of Kähler-Ricci flow equation as* $t \to \infty$. [3]. *Technically, the estimates used in the study of one of them can usually be established for the other one.*

Unlike the classic case when the class $[L]$ *for the equation is Kähler, it's very necessary for us to consider the flow with changing cohomology class* $[\tilde{\omega}_t]$ *since the*

[3]One can realize this by putting the term with t-derivative to be 0 in Kähler-Ricci flow equation.

classic solution only exists when the class is Kähler.

Kähler-Ricci flow can be seen as a very delicate way to set up continuity method. We can also set up continuity method much more simple-mindedly as discussed later. But Kähler-Ricci flow with its own rich structure is a by far more interesting tool, especially when we want to further consider our main problem as I see it now.

1.3 Pluripotential Theory

Our argument in pluripotential theory basically studies the Monge-Ampere map (operator) "$v \to (\sqrt{-1}\partial\bar{\partial}v)^n$" which is essentially the left hand side of the equation we interest in. The idea is to draw information about the potential v from the property of the measure $(\sqrt{-1}\partial\bar{\partial}v)^n$ which is usually seen from the right hand side of the equation. Much less regularity is required and we are going to consider (postive) currents as in [Le].

Remark 1.3.1. *From the simple description of the idea above, we see the exact expression of the right hand side is not that important here as long as we have enough information about the measure.*

As stated below, it's merely the C^0 (or L^∞) estimate that we are searching for right now, so just as in the classic case, we only need to consider the equation whose the right hand side has no e^u term.

The main result for this part is to prove the following theorem which is an improved version of what is stated in [TiZh].

Theorem 1.3.2. *Let X be a closed Kähler manifold with $dim_{\mathbb{C}}X = n \geqslant 2$. Suppose we have a holomorphic map $P : X \to \mathbb{CP}^N$ with the image $P(X)$ of the same dimension. Let ω_M [4] be any Kähler form over some neighbourhood of $P(X)$ in $\mathbb{CP}^N$. Then for the following equation of Monge-Ampere type:*

$$(P^*\omega_M + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega, \tag{1.3}$$

where Ω is a fixed smooth volume form over X and f is a nonnegative function in $L^p(X)$ for some $p > 1$ with the correct total integral over X, we have the following:

(1) (Apriori estimate) Suppose u is a weak solution in $PSH_{P^*\omega_M}(X) \cap L^\infty(X)$ of the equation with the normalization $sup_X u = 0$, then there is a constant C such that $\|u\|_{L^\infty} \leq C\|f\|^n_{L^p}$ where C only depends on P, ω_M and p;

[4] The "M" is the initial of "*model*" since ω_M can naturally be understood as the model metric of the degenerate metric which we are originally interested in.

(2) (Existence of bounded solution) There would always be such a bounded solution for the equation;

(3) (Continuity and uniqueness of bounded solution) If P is locally birational, then any bounded solution is actually the unique continuous solution.

It might be worth taking a little time to clarify some terminologies appearing in the statement.

First, u being a weak solution means both sides of the equation are equal as (Borel) measure. The meaning of the right hand side is classic with u being bounded.

In the definition of $L^p(X)$ space, we choose Ω as the volume form. The choice is clearly not so rigid.

In (3), "locally birational" means that for a small enough neighbourhood U of any point on $F(X)$, each component of $F^{-1}(U)$ would be birational to U (under F). Clearly it would be the case if F is birational itself and in fact this is the case with the most geometric interests as far as I can see.

The improvements from [TiZh] are in two places:

i) we need X to be closed Kähler instead of projective;

ii) in statement (3) about continuity, the assumption is weakened a lot.

The punchline for the proof is the generalization of an inequality between Lebesgue measure and relative capacity. A quite interesting point is that relative capacity, which has a fairly elementary definition, is quite difficulty to compute numerically on one hand, but on the other hand there have been a lot of results about the relations between it and other notions which make it perfect as a bridge to connect things up.

In the theorem above, we require the image side of P to be $\mathbb{CP}^N$ in order to make sure that the image is algebraic. This would at least make it easier for us to justify the construction used in the argument. But the manifold X does not have to be algebraic (projective). It's enough for X to be just a holomorphic object.

In the picture of our main problem, $[P^*\omega_M] = [L]$. Of course the degeneracy of the class $[L]$ should be nice enough to give us the map P.

The estimate in this theorem can be established not only for this limiting class in the set-up of continuity method but also uniformly for all the approximation classes. This is important for existence of such a solution. It also shows the flexibility of the result.

For X projective, the results in Theorem 1.3.2 except this version of continuity result as in (3) were announced and discussed in my previous preprint with my advisor. They were also presented by Tian in a talk at Imperial College in November, 2005. The general continuity was proved soon in January, 2006 after a few discussions with Professors S. Kolodziej and H. Rossi on approximating plurisubharmonic functions over singular spaces. I would like to thank them both for those very useful discussions. The current results were presented in my talk at Columbia University in February, 2006. A new result in the recent preprint by Blocki and Kolodziej allows the current generalization to a closed Kähler manifold X. I really appreciate their informing about this result. Later, we were informed that similar boundedness result should also be achieved by Philippe Eyssidieux, Vincent Guedj and Ahmed Zeriahi in the recent preprint [EyGuZe].

1.4 More History Remarks

In his original work, [Ya], Yau set up the main tool and got remarkable estimates for the direct study of Monge-Ampere equation in the case of $[L]$ being Kähler. This could be taken as the starting point for the study of this very classic nonlinear PDE equation in search for global geometric properties of the underlying manifold. Since then, this problem has been under intensive consideration for all kinds of applications and generalizations. Of course the case when $k = 1$ remains to be a very challenging and interesting problem even for the nondegenerate case.

The Kähler-Ricci flow mentioned before was essentially studied in [Cao] in the case when the limiting class is ample and the initial Kähler class coincides with the limiting class, i.e., the cohomology class is fixed along the flow.

For the canonical class $[K_X] = [L] = [\omega_\infty]$ which is nef. and big with the initial metric ω_0 being sufficiently positive, H. Tsuji studied the Kähler-Ricci flow in [Tsh1] and proved that (2.1) has a global solution $\tilde{\omega}_t$ and $\tilde{\omega}_t$ converges to a positive current which is actually a smooth Kähler-Einstein metric outside a subvariety as $t \to \infty$. [5]

But we noticed that Tsuji's basic arguments can still go through even after the extra assumption on ω_0 is removed. Our new observations for this case are that the limiting current is in fact canonical and has bounded (and actually continuous in his case) potential. Basically, we prove the following theorem which is an improved version for the main result in [TiZh]).

[5] The proof for convergence in [Tsh1] contains some unjustified statements. A uniqueness result was also claimed there whose proof doesn't seems to work very well in general.

Theorem 1.4.1. *Let X be a projective manifold with its canonical divisor K_X nef. and big (i.e., X is a smooth minimal model of general type). Then for any initial Kähler metric ω_0, the Kähler-Ricci flow (1.1) with $k = -1$ and $S = 0$ has a global solution $\tilde{\omega}_t$ for $t \in [0, \infty)$ satisfying:*

(1) $\tilde{\omega}_t \to \tilde{\omega}_\infty$ representing $-c_1(X) = K_X$ as $t \to \infty$ in the sense of current;

(2) $\tilde{\omega}_\infty$ is actually a smooth Kähler-Einstein metric outside a subvariety $S \subset X$ and $\tilde{\omega}_t|_{X\setminus S} \to \tilde{\omega}_\infty|_{X\setminus S}$ locally in C^∞-topology as $t \to \infty$;

(3) in any local complex coordinate chart, $\tilde{\omega}_\infty = \sqrt{-1}\partial\bar{\partial}\rho$ for some local continuous plurisubharmonic function ρ; [6]

(4) $\tilde{\omega}_\infty$ is canonical, i.e., independent of the choice of the initial metric ω_0.

(3) of the above theorem is proved using (the argument of) Theorem 1.3.2 which is proved by extending pluripotential theory developed by Bedford and Taylor (in [BeTa]), and Kolodziej (in [Koj1] and [Koj2]) to singular varieties. The notion of relative capacity was first introduced in [BeTa]. Our argument is a generalization of the original argument in [Koj1] and [Koj2] where he mainly interests in the case when $[L]$ is Kähler [7].

I would like to thank Prof. Kolodziej for so generous help in letting a beginner like me get some idea about pluripotential theory and understanding his orginal works.

Combining all the results, we also gives a partial answer to the following conjecture (cf. [Ti2]) which is the big picture for this whole program:

For any initial metric ω_0, the flow (2.1) has a (possibly singular) solution $\tilde{\omega}_t$ which converges to a (possibly singular) metric in a suitable sense as $t \to \infty$, moreover, this limiting metric may be singular but should be independent of the choice of the initial metric.

In fact, it was further expected that *all singularities of this limiting metric are of rational type.*

[6]In this sense, we may refer $\tilde{\omega}_\infty$ as a positive current with locally continuous potential.

[7]It's more or less like requiring the map P above to be an embedding. [Koj2] is a survey paper in this field which combines a lot of works of his.

Chapter 2

Kähler-Ricci Flow

In this chapter, we discuss the Kähler-Ricci flow introduced before when $k = -1$. The S below is a smooth real closed $(1,1)$-form which, without loss of generality, is frequently chosen to be 0.

On the level of metric, the corresponding equation is:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t + S, \tag{2.1}$$

with initial Kähler metric $\tilde{\omega}_0 = \omega_0$.

As discussed in Introduction, we can reduce it to the following equivalent equation on the level of potential:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \qquad u(0,x) = 0, \tag{2.2}$$

where $\omega_t = (1 - e^{-t})(S - \mathrm{Ric}(\Omega)) + e^{-t}\omega_0$. The relation is $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$.

2.1 Warm-up Exercise

In this section, we consider the case when $S - \mathrm{Ric}(\Omega)$ is a (Kähler) metric. Actually we can choose proper volume form Ω to achieve this if it is cohomologically possible, namely, $[S] - c_1(X)$ is a Kähler class. We are going to prove the following theorem.

Theorem 2.1.1. *In the above situation, for any initial Kähler metric ω_0, the flow (2.1) converges in C^∞-topology to the unique Kähler metric $\tilde{\omega}_\infty$ in the class $[S] - c_1(X)$ which satisfies the limiting Monge-Ampere equation*

$$-\mathrm{Ric}(\tilde{\omega}_\infty) - \tilde{\omega}_\infty + S = 0.$$

Now we have ω_t is always a Kähler metric and in fact it'll have uniformly bounded geometry for all $t \in [0, \infty)$ (i.e., uniformly bounded metric, curvature and so on).

By maximum principle, we can see $u(t, x)$ itself is uniformly bounded as long as the solution exists smoothly. In fact, suppose the solution exists for $t \in [0, s]$ [1]. Then let's consider maximal value point of u for $(t, x) \in [0, s] \times X$. At that point [2], (t_0, x_0), if $t_0 > 0$, then $\frac{\partial u}{\partial t} \geqslant 0$, and we also notice:

$$\omega_{t_0} > 0, \quad \tilde{\omega}_{t_0} = \omega_{t_0} + \sqrt{-1}\partial\bar{\partial}u(t_0, \cdot) > 0, \quad \sqrt{-1}\partial\bar{\partial}u|_{(t_0,x_0)} \leqslant 0,$$

where > 0 or $\geqslant 0$ are used to stand for positivity or nonnegativity of the real $(1, 1)$-forms as hermitian matrices. Hopefully, when the 0 on the right hand side is replaced by some real $(1, 1)$-form, the meaning is also clear.

In above, we have used the fact that $\tilde{\omega}_t$ will remain to be a Kähler metric whenever the solution of the evolution equation exists [3]. Of course this little fact is important for the study this equation and also makes sense of considering this problem in some sense. From these, we can see

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n \leqslant {\omega_t}^n$$

at (t_0, x_0). So we get $u(t_0, x_0) < C$ for some universal positive constant C from the original equation (2.2) at this point.

All the C's in this work will be some universal positive constant but they might well be different from each other. The possible dependence on some choices would be stated explicitly when it matters.

If $t_0 = 0$, this estimate is trivially true by looking at the initial value. Thus we have $u < C$ in this range. The discussion for lower bound is completely analogous. Though we have chosen some time interval $[0, s]$ at the beginning, the upper and lower estimates clearly would not be affected by the size of s. Notice that the "$-$" sign on the right hand side of (2.2) before u is crucial for this argument, which is also why it's usually called the simple sign case.

Then essentially by taking derivative with respect to t on both sides of the equation

[1] Let's emphasize that the small time existence, i.e. local existence, of the solution directly follows from the parabolicity of the equation which is another difference between Kähler-Ricci flow and general Ricci flow.

[2] Take anyone if there are more than one such points.

[3] An elementary way to see this fact here is that the volume of $\tilde{\omega}_t$ should remain positive and at the minimal value point of u for each fixed t where the solution exists, $\tilde{\omega}_t$ is positive, so it has to remain positive on X since the eigenvalues would change continuously without becoming 0.

(2.2): [4]

$$
\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u) &= \frac{\partial}{\partial t}\big(\log\frac{(\omega_t+\sqrt{-1}\partial\bar{\partial}u)^n}{\Omega}\big) \\
&= \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}) + \langle\tilde{\omega}_t, e^{-t}(S-\mathrm{Ric}(\Omega)-\omega_0)\rangle \\
&= \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}) + \langle\tilde{\omega}_t, -\tilde{\omega}_t+(S-\mathrm{Ric}(\Omega))+\sqrt{-1}\partial\bar{\partial}u\rangle \\
&= \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}+u) - n + \langle\tilde{\omega}_t, S-\mathrm{Ric}(\Omega)\rangle.
\end{aligned}
\tag{2.3}
$$

Here the notation $\langle\omega,\cdot\rangle$ means taking trace of the second term with respect to the metric ω where of course the second term is a (real) $(1,1)$-form [5]. And $\Delta_\omega v = \langle\omega, \sqrt{-1}\partial\bar{\partial}v\rangle$. All these conventions will be used frequently from now on.

Again one applies maximum principle. Consider at the minimal value point of $\frac{\partial u}{\partial t}+u(=\log\frac{\tilde{\omega}_t^n}{\Omega})$, (t_0,x_0) (for $(t,x)\in[0,s]\times X$, just as the setting before). If $t_0>0$, we have

$$
\langle\tilde{\omega}_t, S-\mathrm{Ric}(\Omega)\rangle|_{(t_0,x_0)} \leqslant n
$$

by noticing

$$
\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u)\leqslant 0, \qquad \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}+u)\geqslant 0
$$

at this point (t_0,x_0). By the classic arithmetic-geometric inequality, we arrive at

$$
\tilde{\omega}_t^n \geqslant (S-\mathrm{Ric}(\Omega))^n
$$

at that point [6]. Thus we have

$$
\frac{\partial u}{\partial t}+u = \log\frac{\tilde{\omega}_t^n}{\Omega} \geqslant max_X\{\log\frac{(S-\mathrm{Ric}(\Omega))^n}{\Omega}\} > C
$$

at (t_0,x_0). If $t_0=0$, then the estimate is still OK for a possibly larger but still universal constant C. So we have $\frac{\partial u}{\partial t}+u>C$ for all these (t,x)'s where the solution exists. Hence we can conclude

$$
\frac{\partial u}{\partial t} > -C
$$

on this range using the boundedness of u got before.

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[4] This is clearly justified since the solution $u(t,x)$ is smooth with respect to t and x basically from the form of the equation itself. Of course we are still working on the range where the solution exists.

[5] This can also be understood as doing contraction of the second term by the dual form of the metric form or just taking inner product of these two forms under the metric ω.

[6] The positivity of $S-\mathrm{Ric}(\Omega)$ (at that point) is very important here.

We can also do something else after taking t-derivative of the equation (2.2). In fact after multiplying both sides by e^t, we get:

$$\frac{\partial}{\partial t}(e^t\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(e^t\frac{\partial u}{\partial t}) + \langle\tilde{\omega}_t, S - \mathrm{Ric}(\Omega) - \omega_0\rangle. \tag{2.4}$$

Meanwhile, the equation (2.3) we used above can also be reformulated as:

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u + nt) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t} + u + nt) + \langle\tilde{\omega}_t, S - \mathrm{Ric}(\Omega)\rangle. \tag{2.5}$$

The difference (2.5)− (2.4) gives us

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u + nt - e^t\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t} + u + nt - e^t\frac{\partial u}{\partial t}) + \langle\tilde{\omega}_t, \omega_0\rangle.$$

Still by maximum principle, considering the minimal value point for the expression $\frac{\partial u}{\partial t} + u + nt - e^t\frac{\partial u}{\partial t}$ in the equation above, we can see it can not have positive t because $\langle\tilde{\omega}_t, \omega_0\rangle > 0$. Then simply by considering the initial value, we have

$$\frac{\partial u}{\partial t} + u + nt - e^t\frac{\partial u}{\partial t} \geqslant 0.$$

Using the upperbound for u, we can conclude

$$\frac{\partial u}{\partial t} < \frac{C + nt}{e^t - 1}$$

for $t > 0$ and in the range where the solution exists. Since local existence is OK which means we have (upper) bound of $\frac{\partial u}{\partial t}$ for small t (say for $0 \leqslant t < \delta$). And the inequality we just got gives uniform upper bound when $t > \frac{\delta}{2}$. Thus we obtain

$$\frac{\partial u}{\partial t} < C$$

for the range where the solution exists. [7]

Now we have already got $|u| < C$, $|\frac{\partial u}{\partial t}| < C$. Standard Laplacian estimate essentially by Yau [8] will then give

$$\langle\omega_0, \tilde{\omega}_t\rangle < C$$

[7] Let's keep in mind that what we really need for proving global existence are estimates for large t.

[8] See more details in Appendix.

in the range where the solution exists. Here the uniform bounded geometry for ω_t is very important. Remember we already have the uniform bound for the volume form $\tilde{\omega}_t^n$ since $\tilde{\omega}_t^n = e^{\frac{\partial u}{\partial t}+u}\Omega$ from the equation (2.2) itself. Thus we have bounded $\tilde{\omega}_t$ uniformly as metric.

From here, standard argument will give uniform C^k estimates for all k. Thus we see the solution will not blow up or say form any singularity along the flow. Thus the global existence of the solution (in $[0,\infty)$) would follow from local existence.

The strong convergence of the flow is actually quite easy to see in this case as follows.

Recall the equation (2.4):

$$\frac{\partial}{\partial t}(e^t\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(e^t\frac{\partial u}{\partial t}) + \langle\tilde{\omega}_t, S - \mathrm{Ric}(\Omega) - \omega_0\rangle.$$

From the uniform bound of $\tilde{\omega}_t$ as metric, we now have

$$\frac{\partial}{\partial t}(e^t\frac{\partial u}{\partial t}) > \Delta_{\tilde{\omega}_t}(e^t\frac{\partial u}{\partial t}) - C.$$

So $\frac{\partial}{\partial t}(e^t\frac{\partial u}{\partial t} + Ct) > \Delta_{\tilde{\omega}_t}(e^t\frac{\partial u}{\partial t} + Ct)$. The same argument as before will tell us that the maximal value point of $e^t\frac{\partial u}{\partial t} + Ct$ can only have time $t = 0$. Thus we have $e^t\frac{\partial u}{\partial t} + Ct > -C$. Hence one can conclude that globally

$$\frac{\partial u}{\partial t} > -Ce^{-\frac{t}{2}}.$$

Similarly from $\frac{\partial}{\partial t}(e^t\frac{\partial u}{\partial t}) < \Delta_{\tilde{\omega}_t}(e^t\frac{\partial u}{\partial t})+C$, we can get $\frac{\partial u}{\partial t} < Ce^{-\frac{t}{2}}$ globally on $[0,\infty)\times X$. Combining these two inequalities, we arrive at

$$|\frac{\partial u}{\partial t}| < Ce^{-\frac{t}{2}}.$$

This easily gives the smooth convergence of the flow as $t \to \infty$ since it guarantees this for the C^0-norm, and so for all the norms for derivatives from the boundedness of those norms and the interpolation inequalities. [9]

The limiting Kähler metric,

$$\tilde{\omega}_\infty = \omega_\infty + \sqrt{-1}\partial\bar{\partial}u(\infty,\cdot) = S - \mathrm{Ric}(\Omega) + \sqrt{-1}\partial\bar{\partial}u(\infty,\cdot)$$

[9] In fact, we can prove the exponential convergence for any $C^k(X)$-norm. The method will be used in Appendix to get similar result for "$c_1 = 0$" case.

would clearly satisfy the limiting equation

$$0 = -\mathrm{Ric}(\tilde{\omega}_\infty) - \tilde{\omega}_\infty + S \quad \textit{or} \quad 0 = \log \frac{\tilde{\omega}_\infty^n}{\Omega} - u(\infty, \cdot)$$

over X by the strong convergence.

The uniqueness of a solution for this limiting equation is a classic result which indeed follows from maximum principle argument in a rather trivial way.

In the case of X having positive canonical class, we can just set $T = 0$ and $-\mathrm{Ric}(\Omega) > 0$ for some proper smooth volume form Ω. The limit Kähler metric is the unique Kähler-Einstein metric. The uniqueness follows directly from maximum principle just as for the general case above.

This case has been treated in [Cao] if one further assumes that the initial class is proper chosen so that the cohomology class $[\omega_t]$ remains to be the same along the flow. But now we have seen no matter which Kähler class we start with, the flow will lead us to the unique Kähler-Einstein metric. This philosophy is much more important for later concern when we don't have the "proper" Kähler class to start with.

2.2 Nef. and Big Class

Now we start to consider the degenerate case. In other words, the limiting class (as $t \to \infty$) $[S] - c_1(X) = [S] + K_X$ is no longer Kähler. In the following, we mainly focus on the case when it is nef. and big. Now we also assume X to be projective. The meaning of these two terminologies will be clear in the discussion. The following theorem is what we are going to prove.

Theorem 2.2.1. *In the situation above, for any initial Kähler metric ω_0, the Kähler-Ricci flow (2.1) exists for all time, i.e., $t \in [0, \infty)$. It converges as $t \to \infty$ out of Y, a subvariety of X, locally in C^∞-topology to a smooth Kähler metric over $X \setminus Y$ which satisfies the same limiting equation as before in this range. The weak limit over X is a closed positive $(1,1)$-current with some controls for the singularities along S. The limit still does not depend on the choice of the initial metric.*

The uniqueness of the limit for choices of initial metric will be proved in the next section. It should also be pointed out that our arguments can also work in more general cases as we'll see below. Further study of those cases are very interesting problems.

2.2.1 Long Time Existence

In this subsection, we will prove that (2.2) has a global solution with the assumption that $[S]+K_X$ is nef. (i.e. numerically effective). For simpilcity, we'll take $S=0$ for simplicity which clearly won't affect the argument at all. So now we assume that K_X is nef. instead.

At this moment, we can still only require X to be Kähler (and closed), so this notion of numerically effective should be a natural generalization of the usual notion for the case when X is projective (or algebraic), i.e., we just require the property that $K_X+\epsilon[\omega_0]$ is a Kähler class for any $\epsilon>0$. Here we use the initial Kähler class $[\omega_0]$ for convenience, but of course this should clearly be true for any Kähler class.

Thus for any fixed $\epsilon>0$, we can choose a real closed $(1,1)$-form ψ_ϵ such that $[\psi_\epsilon]=K_X$ and $\psi_\epsilon+\epsilon\cdot\omega_0\geqslant 0$ with $\{(\psi_\epsilon+\epsilon\cdot\omega_0)^n=0\}\neq\varnothing$. Moreover, we can take $\mathrm{Ric}(\Omega_\epsilon)=-\psi_\epsilon$ for some smooth volume form Ω_ϵ which is unique up to a positive constant. Since $\psi_\epsilon+\epsilon\cdot\omega_0\geqslant 0$, we have $\psi_\epsilon+a\cdot\omega_0>0$ for $\forall a>\epsilon$.

Set $\omega_t=\psi_\epsilon+e^{-t}(\omega_0-\psi_\epsilon)$ and $\tilde{\omega}_t=\omega_t+\sqrt{-1}\partial\bar{\partial}u$. We have the following equation which is just (2.2) with the special choice of volume form above:

$$\frac{\partial u}{\partial t}=\log\frac{\tilde{\omega}_t^n}{\Omega_\epsilon}-u,\qquad u(0,\cdot)=0. \tag{2.6}$$

In sight of $\omega_t=(1-e^{-t})(\psi_\epsilon+\frac{e^{-t}}{1-e^{-t}}\omega_0)$, we set $T_\epsilon=\log(\frac{1+\epsilon}{\epsilon})$ such that $\frac{e^{-T_\epsilon}}{1-e^{-T_\epsilon}}=\epsilon$. Lets' first show the solution for (2.6) exists for $t\in[0,T_\epsilon)$.

From the above choices, ω_t is a Kähler metric for t in this range. Thus we see ω_t has uniformly bounded geometry for $\forall t\in[0,s](\subset[0,T_\epsilon))$.

The parabolicity of the equation provides the local existence and uniqueness of the solution. So just as in the previous section, it only remains to get uniform estimates of u for $t\in[0,s]$ where $s<T_\epsilon$ under the assumption that the solution exists in this range in order to get the existence of the solution for $t\in[0,T_\epsilon)$. So let's fix such an s for now. In fact the following argument is very similar to what is in the previous section. Notice that we can use s and ϵ as positive constants in the following estimates.

By maximum principle, considering the (local in t) maximal and minimal value points of u, we can get

$$-C_{\epsilon,s}<u<C_\epsilon,$$

where the lower indices s and ϵ indicate the dependence on s and ϵ respectively. The upper bound actually does not depend on s.

Take derivative with respect to t for (2.6) to get:

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}) - e^{-t}\langle \tilde{\omega}_t, \omega_0 - \psi_\epsilon \rangle - \frac{\partial u}{\partial t},$$

which can be reformulated to get the following two equations:

$$\frac{\partial}{\partial t}(e^t \frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(e^t \frac{\partial u}{\partial t}) - \langle \tilde{\omega}_t, \omega_0 - \psi_\epsilon \rangle, \tag{2.7}$$

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t} + u) - n + \langle \tilde{\omega}_t, \psi_\epsilon \rangle. \tag{2.8}$$

The difference of these two equalities above gives:

$$\frac{\partial}{\partial t}\big(e^t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u\big) = \Delta_{\tilde{\omega}_t}\big(e^t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u\big) + n - \langle \tilde{\omega}_t, \omega_0 \rangle. \tag{2.9}$$

There is also a slightly modified difference $(1+\epsilon)\cdot(2.8) - \epsilon \cdot (2.7)$:

$$\frac{\partial}{\partial t}\big((1+\epsilon)(\frac{\partial u}{\partial t} + u) - \epsilon e^t \frac{\partial u}{\partial t}\big) = \Delta_{\tilde{\omega}_t}\big((1+\epsilon)(\frac{\partial u}{\partial t} + u) - \epsilon e^t \frac{\partial u}{\partial t}\big) - (1+\epsilon)n + \langle \tilde{\omega}_t, \psi_\epsilon + \epsilon\omega_0 \rangle. \tag{2.10}$$

From (2.9), noticing $\langle \tilde{\omega}_t, \omega_0 \rangle > 0$, by maximum principle and noticing $e^t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u - nt = 0$ when $t = 0$, one gets that

$$e^t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u - nt \leqslant 0.$$

Now we combine it with local existence and the uniform upper bound for u to conclude that

$$\frac{\partial u}{\partial t} < C_\epsilon.$$

From (2.10), noticing $\langle \tilde{\omega}_t, \psi_\epsilon + \epsilon\omega_0 \rangle \geqslant 0$, by maximum principle, we get that

$$(1+\epsilon)(\frac{\partial u}{\partial t} + u) - \epsilon e^t \frac{\partial u}{\partial t} + (1+\epsilon)nt \geqslant min_{t=0}\{\frac{\partial u}{\partial t}\} = -C_\epsilon$$

which gives $(1+\epsilon-\epsilon e^t)\frac{\partial u}{\partial t} \geqslant -C_\epsilon - (1+\epsilon)u - (1+\epsilon)nt > -C_\epsilon$. Since $1+\epsilon-\epsilon e^t \geqslant 1+\epsilon-\epsilon e^s > 0$ for $t \in [0,s]$, we can conclude

$$\frac{\partial u}{\partial t} > -C_{\epsilon,s}.$$

Until now we have got all the C^0 estimates needed. Now the existence of solution for (2.6) for $t \in [0, s]$ follows from the standard argument using Laplacian estimate just as before. More details can be found in Appendix. Hence we get the existence of solution in $[0, T_\epsilon)$.

The global existence of the solution for (2.2) is easy to see by considering the relations between all the equations (2.6) for different ϵ's as follows.

Actually from the equivalence of the equations (2.1) and (2.2) as mentioned in Introduction, we can get the existence of the solution for (2.1) for any time interval by taking sufficiently small $\epsilon > 0$ and using the existence of the solution for correspondent (2.6) in $[0, T_\epsilon)$. Thus we have the global existence of the solution for the metric flow (2.1) in $[0, \infty)$, and so for any potential flow (2.6). In fact we can see this more concretely without mentioning (2.1) at all.

Consider (2.6) for some $\delta > 0$ other than the fixed ϵ before. We have $\psi_\delta = \psi_\epsilon + \sqrt{-1}\partial\bar{\partial} f$ for some smooth real function f over X. Since $-\mathrm{Ric}(\Omega_\epsilon) = \psi_\epsilon$, we have $-\mathrm{Ric}(e^f \Omega_\epsilon) = \psi_\delta$. Thus one can take $\Omega_\delta = e^f \Omega_\epsilon$. Now the new "$\omega_t$" is

$$\eta_t = \psi_\delta + e^{-t}(\omega_0 - \psi_\delta) = \omega_t + (1 - e^{-t})\sqrt{-1}\partial\bar{\partial} f.$$

The corresponding equation (2.6) for δ is:

$$\frac{\partial v}{\partial t} = \log \frac{(\eta_t + \sqrt{-1}\partial\bar{\partial} v)^n}{e^f \Omega_\epsilon} - v, \qquad v(0, \cdot) = 0.$$

Define $\tilde{u} = v + (1 - e^{-t}) f$. We have $\tilde{u}(0, \cdot) = v(0, \cdot) = 0$ (the same initial value) and:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial v}{\partial t} + e^{-t} f \\ &= \log \frac{(\eta_t + \sqrt{-1}\partial\bar{\partial} v)^n}{e^f \Omega_\epsilon} - v + e^{-t} f \\ &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial} \tilde{u})^n}{\Omega_\epsilon} - v - f + e^{-t} f \\ &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial} \tilde{u})^n}{\Omega_\epsilon} - \tilde{u}. \end{aligned}$$

From uniqueness of the solution for (2.6), $\tilde{u}$ is just the original solution u.

The above explicitly gives the relation between the solutions of (2.6) for different ϵ's explicitly which makes it obvious how to use all these equations for different ϵ's to get the global existence of the solution for each one of them. This global existence in potential level is not so trivial if we only consider those equations separately. The discussion above also tells us that when K_X is nef., (2.2) can be solved globally no matter which Ω is chosen. One should notice the volume form Ω is also involved in the definition of ω_t there. [10] We can summarize all the discussion in this subsection in the following proposition.

Proposition 2.2.2. *Over a closed Kähler manifold X with complex dimension greater or equal to 2, Kähler-Ricci flow (2.1) (or (2.2)) exists uniquely and globally for any initial Kähler metric provided $[S] + K_X$ is numerically effective.*

Remark 2.2.3. *In fact, the argument here can be used to prove that the Kähler-Ricci flow exists as long as the class remains to be Kähler. We'll prove this later in this chapter in a general setting.*

2.2.2 Local Convergence

In this subsection, we discuss the convergence of the Kähler metrics ($\tilde{\omega}_t$ for each time slice) along the flow. As before, we still do the argument on the level of potential. Now X is assumed to be a projective manifold with K_X being nef. and big. This is the simplification we are going to consider. As in the previous part, we can do exactly the same thing for the class $[S] + K_X$ when it is nef. and big.

We've already known the flow exists globally from the last part. Because the limiting class K_X may not be positive in general, we can't expect that the limit can be really a (smooth) metric, which from the way we obtain the limit means that the estimates for u uniformly over $[0, \infty) \times X$ should be out of reach. Thus we have to make a choice of losing some globalness. But if we want to get a limit as $t \to \infty$ in any sense, the globalness of the estimates for all time is better to be preserved. So naturally we choose to lose the globalness over X and try to see what we can say locally on the manifold.

Remark 2.2.4. *Actually this idea of localizing the extimates on X was introduced by Tsuji in [Tsh1] where he used a clever way to get the bigness of K_X expressed in the equation and involved in the application of maximum principle. One of the main goals for the rest of this subsection is to clarify some details about his argument there.*

At the beginning, we'll make some auxiliary assumptions on our choices for the flow which make our picture of convergence more clear. They are not essential and will be removed at the end.

[10] Since we have to start with some Kähler metric ω_0, any choice of Ω will provide some time interval for existence of solution from the argument in the first part of this section. But we know the solution of the specific potential flow actually exists forever.

Let $\omega_\infty = -\mathrm{Ric}(\omega_0{}^n) = -\mathrm{Ric}(\omega_0)$ with some initial metric ω_0 satisfying $\omega_0 - \omega_\infty > 0$. Now $\omega_t = \omega_\infty + e^t(\omega_0 - \omega_\infty)$ and then $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. Hence (2.2) becomes

$$\frac{\partial u}{\partial t} = \log\frac{\tilde{\omega}_t^n}{\omega_0{}^n} - u, \qquad u(0,\cdot) = 0.$$

Taking t-derivative for the above equation as usual, we get

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}) - e^{-t}\langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial u}{\partial t}.$$

The initial value for $\frac{\partial u}{\partial t}$ is also 0 from our choice of the volume form Ω. Since $\omega_0 - \omega_\infty > 0$ by assumption, simply by maximum principle, we have $\frac{\partial u}{\partial t} \leqslant 0$ globally and in fact $\frac{\partial u}{\partial t} < 0$ for $t > 0$. This also tells us $u < 0$ for $t > 0$. Thus we have got the global upper bounds:

$$\frac{\partial u}{\partial t} \leqslant 0, \qquad u \leqslant 0.$$

Moreover, we know u is (strictly) decreasing along the flow, which is obviously helpful in search of convergence. In fact, this alone would tell us that the limit of u would be a plurisubharmonic function with respect to ω_∞ if there is some very weak lower bound for u (even just for a point on X) [11].

We also have as before:

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t} + u) - n + \langle\tilde{\omega}_t, \omega_\infty\rangle. \tag{2.11}$$

Since the potential flow equation (2.2) can be reformulated as

$$\frac{\partial u}{\partial t} + u = \log\frac{\tilde{\omega}_t^n}{\omega_0{}^n},$$

this equation above describes the change of volume along the flow in a straightforward way.

Now we start to get the bigness of K_X involved. The following lemma (see for example [Ka1] and [Ka3]) tells us how to use it.

Lemma 2.2.5. *Over a projective manifold X, if a divisor L is nef. and big, then $L - \epsilon E$ is Kähler for some effective integral divisor E and $\epsilon \in (0, a)$ for some $a > 0$.*

The proof of this result essentially makes use of the openness of the big cone for the projective manifold X. L being nef. means it's in the closure of the positive cone

[11] Otherwise, we can't exclude the case that the limit might be identically $-\infty$.

(as in [Kl]). The idea is quite obvious when one has the picture of those cones in mind. In fact we can choose the divisor E to be big. Notice the projectivity of X is fairly much rooted here. By abusing of notation, the divisor also stands for the correspondent cohomology class below.

On the level of form, we can have $\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log h_{E,\epsilon} > 0$ for $\epsilon \in (0, a)$ where locally $h_{E,\epsilon}$ stands for the square of the norm for a local holomorphic nowhere 0 section of the holomorphic line bundle correspondent to E with respect to some bundle hermitian metric which might well depend on ϵ. The second part of the summation is nothing but the curvature form of line bundle E (with respect to this hermitian metric) multiplied by $-\epsilon$.

Use σ to denote the canonical global holomorphic section for the line bundle E which vanishes along E in the following. We now have a more global expression of this inequality as:

$$\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0,$$

where $|\cdot|$ denotes the bundle norm [12]. Be careful that strictly speaking this inequality above should be understood to make sense only over $X \setminus \{\sigma = 0\}$. It should be understood as the expression before using $h_{E,\epsilon}$ if one wants to make sense over the whole manifold X. But the function $|\sigma|^2$ is smooth over X and takes value in some finite interval $[0, C_\epsilon]$.

We can reformulate the equation before about volume evolution (2.11) as follows:

$$\frac{\partial}{\partial t}\big(\log\frac{\tilde{\omega}_t^n}{|\sigma|^{2\epsilon}{\omega_0}^n}\big) = \Delta_{\tilde{\omega}_t}\big(\log\frac{\tilde{\omega}_t^n}{|\sigma|^{2\epsilon}{\omega_0}^n}\big) - n + \langle\tilde{\omega}_t, \omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle.$$

Indeed there are no changes for both sides of the equation. But this current equation should also be considered only over $X \setminus \{\sigma = 0\}$.

For any fixed t, $\log(\frac{\tilde{\omega}_t^n}{|\sigma|^{2\epsilon}{\omega_0}^n})$ will blow up to $+\infty$ along $\{\sigma = 0\}$. Thus if we consider local minimal value point of it, using estimates local in time, we can see the point exists in $X \setminus \{\sigma = 0\}$ where this equation actually makes sense. So we can proceed as before in applying maximum principle. At that point, we have:

$$\langle\tilde{\omega}_t, \omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle \leqslant C_\epsilon.$$

Notice that it is possible for the point to have $t = 0$. So we use C_ϵ on the right hand side instead of n. Anyway, we have $\tilde{\omega}_t^n \geqslant C_\epsilon(\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2)^n$. Hence we

[12]For simplicity, we omit the lower indices ϵ and E.

conclude that at that point,

$$\frac{\tilde{\omega}_t^n}{|\sigma|^{2\epsilon}\omega_0{}^n} \geqslant \frac{C_\epsilon(\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2)^n}{|\sigma|^{2\epsilon}\omega_0{}^n} > C_\epsilon,$$

where the numerator in the middle term is the volume form for a metric over X. By the choice of the point, we get that globally over $[0,\infty) \times X$,

$$\frac{\tilde{\omega}_t^n}{|\sigma|^{2\epsilon}\omega_0{}^n} > C_\epsilon$$

since though we choose the point as a local (for time) minimal, the bound we get is uniform for any chosen time interval. Thus we arrive at:

$$\frac{\partial u}{\partial t} + u > -C_\epsilon + \epsilon\log|\sigma|^2.$$

Together with the upper bounds $\frac{\partial u}{\partial t} \leqslant 0$ and $u \leqslant 0$, we get the global degenerate lower bounds:

$$\frac{\partial u}{\partial t} > -C_\epsilon + \epsilon\log|\sigma|^2, \qquad u > -C_\epsilon + \epsilon\log|\sigma|^2.$$

Up to now, we already know there is a pointwise limit of u as $t \to \infty$ out of $\{\sigma = 0\}$ which is even global over X if one allows $-\infty$ as a legal value. Of course we should expect something better than that.

In the following we'll use the standard Laplacian estimate in a slightly modified (degenerate) way. For any fixed $\epsilon \in (0, a)$, let's set:

$$\omega_{t,\epsilon} = \omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 + e^{-t}(\omega_0 - \omega_\infty).$$

Then we have $\tilde{\omega}_t = \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}(u - \epsilon\log|\sigma|^2)$. One should undertand them over $X \setminus \{\sigma = 0\}$ in this form, but they have global smooth extensions to the whole of X by replacing the singular-looking term $-\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ by the genuine curvature form.

The point for doing this is that now $\omega_{t,\epsilon}$ takes value in the segment between $\omega_0 + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ and $\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ for $t \in [0,\infty)$. The later one is a metric for the fixed ϵ by our choice. The former one is also a metric as we assume $\omega_0 - \omega_\infty > 0$. Hence we have uniform control for all $\omega_{t,\epsilon}$ as metric.

Over $X \setminus \{\sigma = 0\}$, (2.2) can be rewritten as:

$$(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}(u - \epsilon\log|\sigma|^2))^n = e^{\frac{\partial u}{\partial t} + u + \log\frac{{\omega_0}^n}{{\omega_{t,\epsilon}}^n}}{\omega_{t,\epsilon}}^n.$$

Now we can get the standard inequality for Laplacian estimate in the following form after using the uniform upper bounds for $\frac{\partial u}{\partial t}$, u and the uniform control of the metric $\omega_{t,\epsilon}$ as metric. The inequality is for all $(t,x) \in [0,\infty) \times (X \setminus \{\sigma = 0\})$: [13]

$$\begin{aligned} & e^{C_\epsilon(u-\epsilon\log|\sigma|^2)}(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})\big(e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle\big) \\ > & -C_\epsilon + (C_\epsilon\frac{\partial u}{\partial t} - C_\epsilon)\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle + C_\epsilon\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle^{\frac{n}{n-1}} \\ > & -C_\epsilon + (C_\epsilon\log|\sigma|^2 - C_\epsilon)\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle + C_\epsilon\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle^{\frac{n}{n-1}}, \end{aligned}$$

where $\frac{\partial u}{\partial t} > -C_\epsilon + \epsilon\log|\sigma|^2$ is used to get the second "$>$". Notice that we can have this lower bound for any $\epsilon \in (0,a)$. But we use the same ϵ as above for simplicity.

Unfortunately we don't have the uniform control (from below) for all the coeffcients in the last expression which is important for the classic way of applying maximum principle. But in fact we can get away with this as follows.

Consider the local maximal value point of $e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle$. Easy to see this point exists and in fact is in $X \setminus \{\sigma = 0\}$ since this expression vanishes on $\{\sigma = 0\}$ and is continuous everywhere. The case when the point has $t = 0$ can be included in the final esimate trivially as usual. Now at that point which has time $t > 0$, we have:

$$\begin{aligned} 0 & > -C_\epsilon + (C_\epsilon\log|\sigma|^2 - C_\epsilon)\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle + C_\epsilon\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle^{\frac{n}{n-1}} \\ & = -C_\epsilon + C_\epsilon\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle\big(\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle^{\frac{1}{n-1}} + C_\epsilon\log|\sigma|^2 - C_\epsilon\big). \end{aligned}$$

Thus we should have $\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle < C_\epsilon(C_\epsilon - \log|\sigma|^2)^{n-1}$ since $|\sigma| \in [0, C_\epsilon]$. So we can have at that point:

$$e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon},\tilde{\omega}_t\rangle \leqslant C_\epsilon(C_\epsilon - \log|\sigma|^2)^{n-1}e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}.$$

The whole point of using maximum principle is trying to get a universal control for the right hand side of this expression which then makes it legal to extend the estimate to everywhere. Actually we only have to observe the following to achieve this.

The lower bound for u is: $u > -C_\delta + \delta\log|\sigma|^2$ for any $\delta \in (0,a)$. Though the norm

[13] The computation for getting this inequality is pointwise which makes it justified for our situation here.

$|\cdot|$ may depend on δ, the difference is only a global smooth nowhere 0 function on X, so in fact here we can forget about the difference between the norms as we also have a constant C_δ anyway. Thus let's say we keep the norm $|\cdot|$ to be the same. Since $u > -C_\delta + \delta\log|\sigma|^2$ for some fixed $\delta \in (0, \epsilon)$, we get

$$u - \epsilon\log|\sigma|^2 > -C_\delta + (\delta - \epsilon)\log|\sigma|^2,$$

which gives the following

$$e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)} < C_{\epsilon,\delta} \cdot |\sigma|^{C_\epsilon(\epsilon-\delta)}.$$

Thus at the local maximal value point considered above, we have:

$$e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon}, \tilde{\omega}_t\rangle \leqslant C_{\epsilon,\delta}(C_\epsilon - \log|\sigma|^2)^{n-1}|\sigma|^{C_\epsilon(\epsilon-\delta)} < C_{\epsilon,\delta}$$

where the last step is from the trivial fact mentioned before: $|\sigma| \in [0, C_\epsilon]$. So now we get globally that

$$e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_0, \tilde{\omega}_t\rangle \leqslant C_{\epsilon,\delta}.$$

We can fix the choice of δ with respect to ϵ (for example $\delta = \frac{\epsilon}{2}$) without losing any essential information, so $C_{\epsilon,\delta}$ can be replaced by C_ϵ for simplicity. Rewrite the estimate to get more explicit information of $\tilde{\omega}_t$ below using $u \leqslant 0$:

$$\langle\omega_0, \tilde{\omega}_t\rangle \leqslant C_\epsilon|\sigma|^{-2\epsilon C_\epsilon}. \tag{2.12}$$

Remark 2.2.6. *Recall that the meaning of the positive constant C_ϵ in the power of $|\sigma|$ is essentially $max_{\{\forall x\in X, k\neq l\}}|(R_{\omega_{\infty,\epsilon}})_{k\bar{k}l\bar{l}}|$ where $\omega_{\infty,\epsilon} = \omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$. Let's consider the power of $|\sigma|^{-2}$ which is $\epsilon \cdot C_\epsilon$. If $\omega_\infty > 0$ (or K_X being Kähler), we can let ϵ tend to 0 and see $\epsilon \cdot C_\epsilon$ also goes to 0, which should give the global uniform estimate which is of course consistent with the case considered in the previous subsection. Clearly one also has to make sure all the other constants C_ϵ's won't blow up to $+\infty$ as $\epsilon \to 0$ which is indeed the case. Basically the argument goes through without getting the term $\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|$ involved. This tells us that this power $\epsilon \cdot C_\epsilon$ is related to the degenaracy of K_X as a Kähler class which is naturally what we should expect, and so is the Laplacian estimate got above. The size of it greatly affects the possible control of the metrics $\tilde{\omega}_t$ and it should be fairly computable in practice.*

Combining this Laplacian estimate with the volume estimate before:

$$\tilde{\omega}_t^n > C_\epsilon |\sigma|^{2\epsilon} {\omega_0}^n, \tag{2.13}$$

for any $\epsilon \in (0, a)$, we have the uniform control of $\tilde{\omega}_t$ as metric over any compact subset of $X \setminus \{\sigma = 0\}$. Of course we have already got the t-global C^0-estimate for u for such subsets.

For the higher order estimates, we can either go directly by using the computation by Yau for third order and use Schauder estimates to iterate, or save the trouble by applying the general theory about uniformly parabolic nonlinear equation. Anyway we can have uniform higher order estimates in any compact subset out of $\{\sigma = 0\}$. More details can be found in Appendix.

Now it's routine to conclude that $u(t, \cdot)$ converges in C^∞-topology for any compact subset out of $\{\sigma = 0\}$ as $t \to \infty$ as follows.

First by Ascoli-Arzela's Theorem and diagonalizing argument, we have the convergence of $u(t, \cdot)$ for a sequence of t's going to infinity locally out of $\{\sigma = 0\}$ in C^∞-topology. Then the decreasing of $u(t, \cdot)$ with respect to t will make sure the convergence is locally in local C^0-norm for the whole flow, i.e., $t \to \infty$. Now using the uniform local bounds of the higher order derivatives and the interpolation inequalities (see for example [GiTr]), we can get the local convergence in C^∞-topology as $t \to \infty$. Simply from the equation itself, we also have such convergence for $\frac{\partial u}{\partial t}$ which means it should actually go to 0 locally smoothly by the convergence of u. Thus we have in $X \setminus \{\sigma = 0\}$:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial} u_\infty)^n = e^{u_\infty} {\omega_0}^n. \tag{2.14}$$

where $u(\infty, \cdot)$ stands for the limit which is smooth out of $\{\sigma = 0\}$ from above. On the whole of X, we only have pointwise convergence for u which is simply from the monotonicity of u with respect to t and the degenerated lower bound of u by allowing $-\infty$ to be a legal value. But that alone will provide some global information for the limit as mentioned before. We'll give more details now.

Recall we have Kähler metrics along the flow

$$\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial} u(t, \cdot)$$

where $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$. Locally on X we can have $\omega_t = \sqrt{-1}\partial\bar{\partial}(\varphi_\infty + e^{-t}(\varphi_0 - \varphi_\infty))$ where φ_0, φ_∞ are local potentials for ω_0, ω_∞ which might not be a metric, but still real, closed, and of type $(1, 1)$. We can easily make sure that $\varphi_0 - \varphi_\infty > 0$ by adding constants. Thus this local potential for ω_t is decreasing with respect to t just as $u(t,\cdot)$ does. Thus we have decreasing local potential $\varphi_\infty + e^{-t}(\varphi_0 - \varphi_\infty) + u(t,\cdot)$ for metric $\tilde{\omega}_t$ whose limit will be a plurisubharmonic function locally on X by the properties of plurisubharmonic functions [14]. We also know it's $\varphi_\infty + u_\infty$ almost everywhere in the open subset of X we started with. Again by the properties of plurisubharmonic functions [15], we have known the function everywhere from that. Thus in fact there should be no ambiguity about the limit u_∞ over the whole of X once we get to know the regular part of it. Of course this coincides with the decreasing limit. But as for now, there could be places where u_∞ takes $-\infty$. It'll be taken care of later.

Moreover, we see $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$ is a real closed $(1,1)$-current which represents the same cohomology class as ω_∞ just by noticing the limiting function u_∞ is a real-valued L^1 function over X from the bounds on u along the flow. Of course we can also see the limiting current is actually a Kähler metric in $X \setminus \{\sigma = 0\}$ by the estimates above for metrics along the flow, and it is Kähler-Einstein in the same range, namely $X \setminus \{\sigma = 0\}$, from the limiting equation (2.16) it satisfies there if we are considering the flow with $S = 0$.

Remark 2.2.7. *The holomorphic section σ used above comes from an effective integral divisor E which is not necessary unique. We can choose different E's to study the same equation (2.2). The limit $u(\infty,\cdot)$ is unique at least in the sense of pointwise convergence. So all the extra information we can draw from the above argument is for the same limit. For example, we can say $u(\infty,\cdot)$ is smooth in the complement of the intersection of all the σ's and satisfies the limiting equation in the same range. Of course, we only need finitely many σ's.*

In the terminology of algebraic geometry, the intersection above is called the stable base locus set (for $[S]+K_X$). And it can be easily seen from above that if this set is empty, then we actually have a smooth limit from the uniform bounds of u which we can get by combining the estimates for finitely many σ's and so $[S]+K_X$ is indeed Kähler. Thus it can be taken as a characterization set for the positivity (ampleness) of the class $[S]+K_X$. [16] *Actually it has been proved in [Nak] by algebraic geometry argument that this set is just the classic characterization set of the ampleness of the class $[S]+K_X$, which is the union of the varieties along which the class is degenerate (see in [Kl]), when the class is rational, nef. and big to start with.*

[14] See for example in [De1], and this is what we called plurisubharmonic with respect to ω_∞ before.

[15] Basically, this is the "essentially upper semi-continuity" as in [Le]. We'll have more discussion about this later.

[16] The other direction is rather trivial.

Finally there is just a little mess we need to clean up for this convergence result. As mentioned at the beginning, we used additional assumption about the initial Kähler metric ω_0 and also we use the volume form ${\omega_0}^n$. It's time to see all these are not necessary.

Let's restate the flow equation in the level of potential (2.2):

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \qquad u(0,\cdot) = 0,$$

where $\omega_t = \omega_\infty + e^t(\omega_0 - \omega_\infty)$ with $\omega_\infty = -\mathrm{Ric}(\Omega)$. Simply by maximum principle, we can get $u < C$. Recall that we also have $(e^t - 1)\frac{\partial u}{\partial t} - u - nt \leqslant 0$, so $\frac{\partial u}{\partial t} \leqslant \frac{u+nt}{e^t-1}$ for (t,x) with $t > 0$. Combining these with the local estimate for small time, we have $\frac{\partial u}{\partial t} < Ce^{-\frac{t}{2}}$ globally. The degenerated volume estimate is essentially not benefited from the additional assumptions, and so will not be affected after removing them.

It can be seen that the Laplacian estimates earlier in this subsection are only affected in the following way. $\omega_{t,\epsilon}$ still takes value in the segment between $\omega_0 + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ and $\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ when t varies from 0 to ∞. The later one is a metric for the fixed ϵ. But the former one may NOT be now. And in fact as the norm $|\cdot|$ for the line bundle E might well depend on ϵ, we might not be able to say for sufficiently small ϵ, both of the end-points are metrics. It seems we are in trouble to find uniformly bounded background metrics which is important for Laplacian estimate. But remember that we no longer worry about the global existence of the solution of the flow. It's safe for us to restrict our attention for sufficiently large t [17]. And also this should clearly be enough for considering the limit of solution as t goes to infinity. This trivial observation allows us to only consider $\omega_{t,\epsilon}$ for t close to ∞ where we know it's a metric. In other words, we can now use the argument of Laplacian estimate before for the range $(t,x) \in [S_\epsilon, \infty) \times X$ where we have uniform control for $\omega_{t,\epsilon}$ as metric. Then everything goes through and we can still make the same conclusion for $\langle\omega_0, \tilde{\omega}_t\rangle$ and the higher derivatives can be estimated just as before.

Now let's consider $v = u + e^{-\frac{t}{4}}$. $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \frac{1}{4}e^{-\frac{t}{4}}$ which will be negative for t sufficiently large.

From the estimates for u, we still have $v > -C_\epsilon + \epsilon\log|\sigma|^2$ for any $\epsilon \in (0,a)$ and all the estimates for space derivatives. Thus for v we can have the convergence as $t \to \infty$ which can be easily translated to the convergence for u. And of course we still have the plurisubharmonicity for the limit [18]. This should give us everything up

[17] The control we need for the rest part of t will follow from the local estimates if we know the length of the finite t-interval which we want to ignore.

[18] In fact, it's easy to see $u_\infty = v_\infty$.

to now after removing the additional assumptions.

We can actually save the trouble above by proving a degenerate exponential lower bound for $\frac{\partial u}{\partial t}$ below.

Let's recall the following equations:

$$\frac{\partial}{\partial t}\big(e^t\frac{\partial u}{\partial t}\big) = \Delta_{\tilde{\omega}_t}\big(e^t\frac{\partial u}{\partial t}\big) + \langle \tilde{\omega}_t, \omega_\infty - \omega_0\rangle,$$

$$\frac{\partial}{\partial t}\big(e^t\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u\big) = \Delta_{\tilde{\omega}_t}\big(e^t\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u\big) + n - \langle \tilde{\omega}_t, \omega_0\rangle.$$

Multiply the second one with some constant $-A$ for $A \in (0,1)$ which will be fixed shortly and then sum up the two equations to get:

$$\frac{\partial}{\partial t}\big((1-A)e^t\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial t}+Au\big) = \Delta_{\tilde{\omega}_t}\big((1-A)e^t\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial t}+Au\big)-An+\langle \tilde{\omega}_t, \omega_\infty-(1-A)\omega_0\rangle.$$

We can rewrite this equation as below over $X \setminus \{\sigma = 0\}$

$$\begin{aligned}
&\frac{\partial}{\partial t}\big((1-A)e^t\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + Au - \epsilon\log|\sigma|^2 + Ant\big)\\
&= \Delta_{\tilde{\omega}_t}\big((1-A)e^t\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + Au - \epsilon\log|\sigma|^2 + Ant\big)\\
&\quad + \langle \tilde{\omega}_t, \omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 - (1-A)\omega_0\rangle
\end{aligned}$$

where $\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ is a Kähler metric over X as before. Thus by taking $0 < A < 1$ close enough to 1, by maximum principle, we have

$$(1-A)e^t\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + Au - \epsilon\log|\sigma|^2 + Ant \geqslant -C - \epsilon\log|\sigma|^2 \geqslant -C$$

since the minimum value point will never be in the set $\{\sigma = 0\}$. And this bound would be global for X. Then by the uniform upper bounds of $\frac{\partial u}{\partial t}$ and u, we can get

$$\frac{\partial u}{\partial t} \geqslant (-C - Ct)e^{-t} + e^{-t}C_\epsilon\log|\sigma|^2 \geqslant -Ce^{-\frac{t}{2}} + e^{-t}C_\epsilon\log|\sigma|^2.$$

This is the degenerate lower estimate we want. Clearly it can be done for any such σ and we only need finite many of them for the stable base locus set. Clearly it gives better description about the local convergence. [19]

[19] The search of local exponential convergence for C^k-norm is not successful as before. The global argument used before can't be carried through since the uniform bound for metrics along the flow is degenerate. Local argument considering Dirichlet problem, which is used to get the local higher order

In one word, we get the convergence of the flow equation (2.2) in some (local) sense and the limit satisfies:

$$(-\mathrm{Ric}(\Omega)+\sqrt{-1}\partial\bar{\partial}u_{\omega_0,\Omega,\infty})^n = e^{u_{\omega_0,\Omega,\infty}}\Omega$$

out of some small set. Here we include the initial Kähler metric ω_0 and the smooth volume form Ω as lower indices for the limiting function to indicate the apriori dependence on them from the discussion above. But we should notice that the small set which is the stable base locus set of $[S]+K_X$ does not depend on choices. Now we can conclude the following proposition. The statement is for the case $S=0$ which is of most geometric interests. The statement for general case is not so different except for the Kähler-Einstein part.

Proposition 2.2.8. *Over a smooth projective manifold X with complex dimension greater or equal to 2, for any initial data, Kähler-Ricci flow (2.1) converges locally in C^∞-topology to a smooth Kähler-Einstein metric out of a subvariety of X, with some control of (possible) singularities along E as above, provided K_X is nef. and big. The subvariety mentioned is actually the stable base locus set of K_X.*

Moreover, the limit can be extended to X a positive $(1,1)$-current and we have the pointwise convergence of the flow on the level of potential.

Actually, we can draw more information about the solution for the flow equation itself from the convergence result. In order to do this, we should notice the essential decreasing of the volume along the flow from the following computation. First recall the equation below:

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t}) - e^{-t}\langle\tilde{\omega}_t,\omega_0-\omega_\infty\rangle - \frac{\partial u}{\partial t}.$$

Taking another t-derivative on both sides and noticing summation of the first two terms on the right hand side is just $\langle\tilde{\omega}_t,\frac{\partial\tilde{\omega}_t}{\partial t}\rangle$, we get:

$$\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial t^2}) = \Delta_{\tilde{\omega}_t}(\frac{\partial^2 u}{\partial t^2}) + e^{-t}\langle\tilde{\omega}_t,\omega_0-\omega_\infty\rangle - (\frac{\partial\tilde{\omega}_t}{\partial t},\frac{\partial\tilde{\omega}_t}{\partial t})_{\tilde{\omega}_t} - \frac{\partial^2 u}{\partial t^2}.$$

Sum up these two equation above to get:

$$\frac{\partial}{\partial t}\big(\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u)\big) \leqslant \Delta_{\tilde{\omega}_t}\big(\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u)\big) - \frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u).$$

derivative bounds, doesn't look good as it'll need the exponential convergence of the corresponding boundary term which involves space derivatives (in other words, bounding the term after multiplying by e^t on the boundary first).

That tells us:

$$\frac{\partial}{\partial t}\big(e^t\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u)\big)\leqslant \Delta_{\tilde{\omega}_t}\big(e^t\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u)\big).$$

Maximum principle argument gives

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}+u)\leqslant Ce^{-t}$$

which tells the essential decreasing of $\frac{\partial u}{\partial t}+u$ and also of the volume form of $\tilde{\omega}_t$ as $\tilde{\omega}_t^n=e^{\frac{\partial u}{\partial t}+u}\Omega$. Then it is easy to see the (pointwise) limit of $\frac{\partial u}{\partial t}+u$ exists over X. The limit would be u_∞ out of the stable base locus set and also upper semi-continuous over X. Thus it should be bigger or equal to u_∞ on X which could apriori take value $-\infty$ at some points in the stable base locus set. Here we have used the essentially upper semi-continuity of u_∞.

We can see the limits are actually the same as follows. If at some point in X, u_∞ is $-\infty$, then the limit of $\frac{\partial u}{\partial t}+u$ would also have to be $-\infty$ from the upper estimate of $\frac{\partial u}{\partial t}$. If u_∞ is a finite value at any point of X, then we know the limit of $\frac{\partial u}{\partial t}+u$ would be no smaller than u_∞ which means it's also finite and the limit of $\frac{\partial u}{\partial t}$ is nonnegative. But the limit of $\frac{\partial u}{\partial t}$ would have to be nonpositive if it exists from the estimate, so $\frac{\partial u}{\partial t}$ converges to 0. This should not be surprising from the essential negativity of $\frac{\partial u}{\partial t}$ and the boundedness assumption of the limit for u.

By integrating the estimate above, we arrive at:

$$\frac{\partial u}{\partial t}+u\geqslant u_\infty-Ce^{-t}$$

over X. If one can have a uniform lower bound for u for all space and time (and thus for u_∞), then it's also true for $\frac{\partial u}{\partial t}$. As seen later using pluripotential theory, we actually have the boundedness of u when $[\omega_\infty]$ is semi-ample and big (which is Tsuji's case for the canonical class of a smooth minimal model of general type). [20] And in this case, from discussion above, we know $\frac{\partial u}{\partial t}$ converges to 0 pointwisely over X.

2.3 Uniqueness Result

In this section, we study the uniqueness of the limit from the previous section. There are two kinds of uniqueness to consider. One is the uniqueness as limit for the

[20] So in some sense, the little careless point about Laplacian estimate in [Tsh1] would actually be OK. But obviously, more complicated C^0 (or L^∞) estimate is involved and the assumption is more restrictive than necessary as shown before.

Kähler-Ricci flow under all the choices. The other is the uniqueness as solution for the limiting equation, i.e., the degenerate Monge-Ampere equation which is the main interest for this work.

2.3.1 Uniqueness as Limit of Flow

With the same assumptions as in the previous section, we want to prove the limit for (2.1) will not depend on the choice of the initial data. For (2.2), we also expect to see the choice of the volume form Ω will change the limit (of potential) in an explicit way.

In fact the computation about difference choices of volume form Ω is already contained in the proof of global exsitence of the solution for flow. Of course for this part we can fix the initial Kähler metric ω_0. Recall for $\Omega_2 = e^f\Omega_1$, we have $u_2 = u_1 - (1-e^{-t})f$ where u_1 and u_2 are the correspondent solutions for (2.2). Then obviously we have $u_{2,\infty} = u_{1,\infty} - f$ which would imply that we have the same limiting (singular) metric. This can also be seen from the uniqueness result for the solution of (2.1) since we are starting from the same ω_0 and it's indeed the same flow.

Now we consider the dependence on the initial Kähler metric which is a more essential problem. We fix Ω for the equation on the level of potential.

First we observe an integral equality for all such limits, u_∞, which is rather trivial from what we know about the convergence but plays a quite important for the uniqueness argument. Recall that we have the estimates for any fixed $\epsilon \in (0, a)$:

$$-C_\epsilon + \epsilon\log|\sigma|^2 < u < C_\epsilon, \qquad -C_\epsilon + \epsilon\log|\sigma|^2 < \frac{\partial u}{\partial t} < C_\epsilon.$$

Also we know $u \to u_\infty$ and $\frac{\partial u}{\partial t} \to 0$ as $t \to \infty$ smoothly in any compact subsets out of some small set (say $\{\sigma = 0\}$). Thus we can easily see that the convergences are also in L^p-norm for $\forall 1 \leqslant p < \infty$ by noticing the contribution for the neighbourhood of $\{\sigma = 0\}$ can be controlled very well. [21].

Remark 2.3.1. *We need a smooth volume form on X to make sense of the L^p-norm, and the choice clearly doesn't matter. u_∞ is clearly in L^p-spaces as it's smooth out of $\{\sigma = 0\}$ and with possibly* log*-singularities along $\{\sigma = 0\}$. In fact what we need here is $e^{\frac{\partial u}{\partial t}+u} \to e^{u_\infty}$ in L^1-norm as $t \to \infty$. So we don't even need the "*log*" lower bounds and the upper bounds will be enough.*

[21] In fact by the almost everywhere pointwise convergence, Dominated Convergence Theorem will do the job.

Consider the flow equation in the form $(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t}+u}\Omega$ which has both sides smooth for each time slice along the flow. Integrate over X to see:

$$\int_X e^{\frac{\partial u}{\partial t}+u}\Omega = \int_X (\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = \int_X {\omega_t}^n.$$

Now take the limit as $t \to \infty$ for both sides to get:

$$\int_X e^{u_\infty}\Omega = \int_X {\omega_\infty}^n. \tag{2.15}$$

This is the integral equality we want. Notice there is little difference between the two $\int_X$'s since the right hand side one is to integrate over something smooth which is not the case for the left hand side where you can think about the integration is only for the regular part or Lebesgue integration.

A byproduct of this equality is $\int_X {\omega_\infty}^n > 0$ which means $([S] + K_X)^n > 0$. The above argument for this actually works for general nef. and big line bundle, L, which gives $[L]^n > 0$. Of course this fact is well known from simple algebraic geometry argument.

In fact we can go a little further by using the limiting equation to get

$$\int_X (\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = \int_X {\omega_\infty}^n.$$

At this moment, we should really clarify the meaning of left hand side which is the integration for the regular part.

Remark 2.3.2. *If we can have the boundedness of u_∞, then $(\omega_\infty+\sqrt{-1}\partial\bar{\partial}u_\infty)^n$ makes sense over X as a (Borel) measure by plurisubharmonicity* [22]*. Then the integration on the left hand side above can be thought of as over X. In fact the weak convergence of the flow metrics to the limiting current also follows from classic pluripotential theory. This can be seen as some motivation for the other part of this work.*

Now let's see how this integral equality is going to help in proving the uniqueness of the limit for any choice of initial Kähler metric, ω_0. Basically, it would imply "=" from "$\geqslant$".

First it is easy to see that it suffices to prove the limits are the same in the case when the two initial metrics are comparable, i.e., when one is "$>$" than the other. The reason is that provided this is true, then for any two Kähler metrics ω_1 and ω_2,

[22] Actually the boundedness of u might not be that necessary only for this as we can see later. But it will definitely make the whole result much more interesting.

we can have $u_{\omega_1,\infty} = u_{\omega_1+\omega_2,\infty} = u_{\omega_2,\infty}$. So now let's suppose ω_0 and ω are two Kähler metrics on X, and the initial metrics we are considering are ω_0 and $\omega_0 + \omega$. Then the correspondent equations on the level of potential would be:

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \qquad u(0,\cdot) = 0,$$

$$\frac{\partial v}{\partial t} = \log\frac{(\omega_t + e^{-t}\omega + \sqrt{-1}\partial\bar{\partial}v)^n}{\Omega} - v, \qquad v(0,\cdot) = 0,$$

where $\omega_t = S - \mathrm{Ric}(\Omega) + e^{-t}(\omega_0 - S + \mathrm{Ric}(\Omega))$. Take the difference to get:

$$\frac{\partial(u-v)}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}u + e^{-t}\omega + \sqrt{-1}\partial\bar{\partial}(v-u))^n} - (u-v),$$

with $(u-v)(0,\cdot) = 0$. Applying maximum principle, we can see $u - v \leqslant 0$. Hence we have $u_\infty \leqslant v_\infty$ over X by pointwise convergence. Since $\int_X e^{u_\infty}\Omega = \int_X e^{v_\infty}\Omega$, we can conclude $u_\infty = v_\infty$ almost everywhere. In fact they ought to be the same on the regular part. Then by plurisuharmonicity of the limits, we see $u_\infty = v_\infty$ over X.

Thus we have got the uniqueness of limit for the flow and the following result is proved.

Proposition 2.3.3. *With the same assumption as before, the limits of the solutions of (2.2) for all choices satisfy (2.15), and the limits of the solutions of (2.1) for different choices of initial Kähler metrics are the same.*

Now I want to mention a possible alternative way of proving this result above and the little obstacle which may well come from my own shallow knowledge about nonlinear PDE's.

In fact as we can see from our argument before, for equation (2.2), taking derivative makes it linear-looking. So we imagine that in order to compare the limits from two (comparable) initial Kähler metrics, it might help to condsider all the limits with all the metrics between them as the initial metric and study the change by considering the derivative with respect to the parameter for the initial metrics. More precisely, using the same set-up as above, for any $\epsilon \in [0,1]$, let's consider the following family of equations:

$$\frac{\partial u^\epsilon}{\partial t} = \log\frac{(\omega_t + \epsilon e^{-t}\omega + \sqrt{-1}\partial\bar{\partial}u^\epsilon)^n}{\Omega} - u^\epsilon, \qquad u^\epsilon(0,\cdot) = 0.$$

The upper index ϵ in u^ϵ indicates that it corresponds to the initial metric $\omega_0 + \epsilon\omega$. Similar for the metrics below. Now as suggested above, by formally taking

ϵ-derivative, we get the following equation: (formally set $\beta^\epsilon = \frac{\partial u^\epsilon}{\partial \epsilon}$)

$$\frac{\partial \beta^\epsilon}{\partial t} = \Delta_{\tilde{\omega}_t^\epsilon}\beta^\epsilon + \langle \tilde{\omega}_t^\epsilon, e^{-t}\omega\rangle - \beta^\epsilon, \qquad \beta^\epsilon(0,\cdot) = 0.$$

From this equation, for each fixed ϵ, since we have all the metrics $\tilde{\omega}_t^\epsilon$ which is smooth for time and space. We can solve for β^ϵ. Then by maximum principle as usual, we see $\beta^\epsilon \geqslant 0$. Thus we can get the uniqueness of the limit as before if we can justify that this solution β^ϵ here is nothing but $\frac{\partial u^\epsilon}{\partial \epsilon}$. Notice we might still have to make sense of this term. Easy to see it would suffice to have the smooth ϵ-dependence of the solutions u^ϵ which is of course true for families of linear equations when the parameter is involved in a nice and smooth way.

At the first sight, it might look promising to set $v^\epsilon = \int_0^\epsilon \beta^s ds + u^0$ which is smooth for t and x and try to prove it actually satisfies the above family of equations , but in fact this doesn't seem to work as $\tilde{\omega}_t^\epsilon$, which contains u^ϵ, is deeply involved in the equation for β^s and the regularity of it with respect to ϵ is somehow the problem here.

Anyway, it should be pointed out that the ϵ above plays a very different role from the time parameter t whose derivative gets involved in the evolution equation which makes it easy to deduce the regularity of the solution with respect to time t. The ϵ is really just a fixed value for each equation, so the regularity of all the solutions with respect to it would probably require some explicit construction of the solutions as in the case of linear differential equations. It's most likely that we need a regularity result about implicit function theorem (or fixed point theorem) for maps with parameters between function spaces. It looks reasonable in sight of the classic uniqueness result for each equation.

2.3.2 Uniqueness for Limiting Equation

Recall that the limit for the flow equation (2.2) satisfies, out of the stable base locus set of $[S] + K_X$, the limiting equation:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{u_\infty}\Omega, \tag{2.16}$$

where $\omega_\infty = S - \mathrm{Ric}(\Omega)$. We still assume that X is projective with $[S] + K_X$ being nef. (i.e., numerically effective) and big. Once again, the discussion below will be for the case when $S = 0$ for simplicity, but it still works for the general situation.

In case when K_X is a Kähler class, we can take ω_∞ to be a Kähler metric and find the limiting equation is just the classic Monge-Ampere equation on a closed Kähler manifold. As we mentioned before, u_∞ will then be smooth on the whole of X and obviously is just the classic solution for this Monge-Ampere equation. We can easily see that the converse is also true below, i.e., if u_∞ got before is smooth over X, we can see K_X is Kähler.

We know that out of the stable base locus set, $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$ will be a Kähler metric from our estimates. Moreover, since u_∞ is smooth on the whole of X by assumption, we know $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$ is a smooth closed real $(1,1)$-form over X. And of course u_∞ will satisfy the limiting equation on the whole of X by continuity which says that the nth-power of this $(1,1)$-form is actually a smooth volume form. Since the eigenvalues of the $(1,1)$-form at one point (indeed a lot of them which only need to be out of the stable base locus set) are all positive, that should also be the case for any point on X, and so this $(1,1)$-form is actually a Kähler metric form. So we conclude that K_X is a positive class.

Combining all these, we know that in the case of K_X being nef. and big, positivity of K_X, emptiness of the stable base locus set of K_X and smoothness of u_∞ over X would be equivalent to each other.

From now on, we focus on the case when K_X is nef. and big but not positive. Some singularities should be expected for u_∞. A major difference from the classic case is that we can't have ω_∞ be a genuine metric. As a matter of fact, we can have a nonnegative ω_∞ to represent K_X [23] which can be seen as a degenerate metric. That's why we call (2.16) as degenerate Monge-Ampere equation. Existence and uniqueness of solution are usually concerned for any kind of PDE. We'll give some general discussion for this degenerate Monge-Ampere equation below.

Since we can't expect smooth solution, it's important to specify what kind of singularities is allowed for the solution. From the flow argument in the preceding sections, we get a solution u_∞ for (2.16) which is smooth and really satisfies (2.16) in the usual sense out of the stable base locus set of K_X, and we also have some controls of the singularities along the stable base locus set. So it seems quite natural to only consider solutions for (2.16) with all the similar properties as u_∞, which takes care of the existence of such solutions, and try to get the uniqueness of such solutions.

For now, there doesn't seem to be too many methods that can be applied here other than the very classic ones using integration. Let's recall one of them first in

[23] We need to use the result from algebraic geometry about semi-ampleness of nef. and big K_X as in [Ka2] which is not true for a general class.

case of K_X being positive. [24] If we have two solutions u and v, i.e.,

$$\omega_u{}^n = (\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

and also for v. Then we have the following computation:

$$\begin{aligned}
0 &\leqslant \int_X (u-v)(e^u - e^v)\Omega \\
&= \int_X (u-v)(\omega_u{}^n - \omega_v{}^n) \\
&= \int_X (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)(\omega_u{}^{n-1} + \omega_u^{n-2}\omega_v + \cdots + \omega_v^{n-1}) \\
&= -\int_X \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)(\omega_u{}^{n-1} + \cdots + \omega_v{}^{n-1}) \leqslant 0.
\end{aligned}$$

We'll frequently forget the "$\wedge$" when the meaning is clear. From above we can see $u = v$ and that's the desired uniqueness.

Now in our situation, using the same set-up as above, noticing it is no longer true that $\omega_\infty > 0$ and u and v are smooth, we still have the computation above for the first few steps since the integrability is available from our estimates. In fact, let's say we are integrating over $X \setminus \{\sigma = 0\}$. But now we have to be more careful when applying Stokes' Theorem as follows:

$$\begin{aligned}
&\int_{X\setminus\{\sigma=0\}} (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)(\omega_u{}^{n-1} + \cdots + \omega_v{}^{n-1}) \\
&= lim_{\epsilon\to 0}\int_{\{|\sigma|\geqslant\epsilon\}} (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)(\omega_u{}^{n-1} + \cdots + \omega_v{}^{n-1}) \\
&= lim_{\epsilon\to 0}\int_{\{|\sigma|\geqslant\epsilon\}} \big(d((u-v)\sqrt{-1}\bar{\partial}(u-v)(\omega_u{}^{n-1} + \cdots + \omega_v{}^{n-1})) \\
&\qquad\qquad - \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)(\omega_u + \cdots + \omega_v{}^{n-1})\big) \\
&= lim_{\epsilon\to 0}\big(\int_{\{|\sigma|=\epsilon\}} (u-v)\sqrt{-1}\bar{\partial}(u-v)(\omega_u{}^{n-1} + \cdots + \omega_v{}^{n-1}) - \\
&\qquad \int_{\{|\sigma|>\epsilon\}} \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)(\omega_u{}^{n-1} + \cdots + \omega_v{}^{n-1})\big).
\end{aligned}$$

In above we have chosen proper sequence of ϵ to approach 0 so that $\{|\sigma| = \epsilon\}$'s are smooth by Sard's Theorem, which justifies the application of Stokes' Theorem. Notice that $\{|\sigma| = \epsilon\}$ is oriented by inward normal direction (towards $\{\sigma = 0\}$). We also

[24] Then $\omega_\infty > 0$ and the solution of potential is smooth and corresponds to a K-E metric.

know that for the final step, only the limit of the whole expression in $(\cdots)$ exists, i.e. it's not justified to take the limit for each term separately. But we do know the second term is nonnegative for any ϵ. So if by any means we can see "the upper limit of the first term is nonpositive" (A), we can continue the above by "$\leqslant 0$" which would give $u = v$ over $X \setminus \{\sigma = 0\}$. This together with a little argument using properties of plurisuharmonic functions will give $u = v$ on X.

For the statement A, the most natural way of justifying it would be to prove the following:

$$lim_{\epsilon\to 0}\int_{\{|\sigma|=\epsilon\}}(u-v)\sqrt{-1}\bar{\partial}(u-v)({\omega_u}^{n-1}+\cdots+{\omega_v}^{n-1}) = 0\cdots\cdots(*).$$

Now we have to recall the estimates required to hold for u (and also v) which are essentially the ones we can prove for the solution as the limit of the flow:

$$-C_\delta + \delta \log|\sigma|^2 < u < C_\delta, \qquad \langle\omega, \omega_\infty + \sqrt{-1}\partial\bar{\partial}u\rangle < C_\delta|\sigma|^{-2\delta C_\delta}$$

for all $\delta \in (0, a)$ where ω is any fixed metric. Here we only consider these two inequalities out of $\{\sigma = 0\}$. And we have also had a little discussion about the power of $|\sigma|$ in the second inequality. From all these and by interpolation inequalities [25], we have all the controls for the terms appearing in the limit. It's quite clear that if we can take the power mentioned above, $-2\delta C_\delta$, to be sufficiently close to 0 for a properly chosen $\delta > 0$, then we see $(*)$ is justified. This would be a sufficient condition for the uniqueness of the solution we are considering. In fact the required conditions for the solution can be reduced to only for this "δ" which is sufficiently small in $(0, a)$ and makes the uniqueness argument above work, but from the way we constructed the limit solution from the flow, it'll actually have the estimates for all δ's sufficiently small (in $(0, a)$).

As mentioned before, there could be some other help we can get for proving uniqueness as follows. Let's point out that simple observation again. The holomorphic section σ may not be unique, and there is nothing preventing us from using the good ones, i.e., those sections which make the above uniqueness argument work. If none of them can do, we might also use more than one of them simultaneously. More precisely, for two such sections σ_1 and σ_2, we can have as before for the limit of the

[25] We can take the constants universal for compact subsets out of $\{\sigma = 0\}$ since they are basically about diameter for a fixed metric which is uniform for our case (see in [GiTr] for example).

flow:

$$-C_i + \epsilon_i \log|\sigma_i|^2 < u_\infty < C_i, \quad \langle \omega, \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty \rangle < C_i|\sigma_i|^{-\alpha_i}$$

for $i = 1,\ 2$ where α_i's are proper positive constants. It should be emphasized that for $i = 1, 2$ respectively, we get these two estimates by all the estimates along the flow. and so they holds out of $\{\sigma_i = 0\}$ respectively [26]. But if we want to combine the estimates for $i = 1,\ 2$ together, we should have them on the whole of X in some sense, otherwise we can only consider out of $\{\sigma_1 = 0\} \cup \{\sigma_2 = 0\}$ which doesn't look too good.

In fact for the first inequality, remember u_∞ actually makes sense on the whole of X as a pointwise limit, so we can have over X:

$$C_i|\sigma|^{2\epsilon_i} \leqslant e^{u_\infty} \leqslant C_i$$

where the first "$\leqslant$" can't be replaced by "$<$" since u_∞ can be $-\infty$ apriori. Thus we have

$$C(|\sigma_1|^{2\epsilon_1} + |\sigma_2|^{2\epsilon_2}) \leqslant e^{u_\infty} \leqslant C$$

over X by taking linear combination. Clearly this is better than each one of those two at least for degeneracy consideration.

Now for the second one, let's first write them down as

$$\frac{1}{\langle \omega, \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty \rangle} > C_i|\sigma_i|^{\alpha_i}$$

out of $\{\sigma_i = 0\}$ for $i = 1, 2$. Then we see

$$\frac{1}{\langle \omega, \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty \rangle} > C(|\sigma_1|^{\alpha_1} + |\sigma_2|^{\alpha_2}), \quad i.e.,$$

$$\langle \omega, \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty \rangle < \frac{C}{|\sigma_1|^{\alpha_1} + |\sigma_2|^{\alpha_2}}$$

out of $\{\sigma_1 = 0\} \cap \{\sigma_2 = 0\}$. [27]. Of course the limiting metric makes sense in this range by the discussion before. In the case when the divisors $\{\sigma_1 = 0\}$ and $\{\sigma_2 = 0\}$ are of normal crossing, we can use Stokes' Theorem around $\{\sigma_1 = 0\} \cap \{\sigma_2 = 0\}$ which is of complex codimension 2. We can have a better chance for the boundary term

[26] As we can change the constants, using "$<$" instead of "$\leqslant$" is OK as everything involved is finite in the ranges considered here.

[27] For the complement of $\{\sigma_1 = 0\} \cup \{\sigma_2 = 0\}$, we get by taking linear combination. It's rather trivial for the rest part.

to have limiting contribution 0. If that's the case, we can restrict our consideration to this kind of solutions and get uniqueness result [28]. Obviously we can have trivial generalization of this method for even more sections. But clearly in surface case, we should not expect this to be of any help.

The above consideration is basically to use the local estimates for the limiting metric to prove (∗), but in fact we still have a little global information from the estimates for the limiting solution which has already been mentioned before, namely,

$$\int_{X\setminus\{\sigma=0\}}(\omega_\infty+\sqrt{-1}\partial\bar{\partial}u_\infty)^n=\int_{X\setminus\{\sigma=0\}}{\omega_\infty}^n,$$

where obviously we can take $\int_X$ for the right hand side. Thus we get:

$$\begin{aligned}0&=lim_{\epsilon\to 0}\int_{\{|\sigma|\geqslant\epsilon\}}\big((\omega_\infty+\sqrt{-1}\partial\bar{\partial}u_\infty)^n-\omega_\infty^n\big)\\&=lim_{\epsilon\to 0}\int_{\{|\sigma|\geqslant\epsilon\}}\sqrt{-1}\partial\bar{\partial}u_\infty\big((\omega_\infty+\sqrt{-1}\partial\bar{\partial}u_\infty)^n+\cdots+{\omega_\infty}^n\big)\\&=lim_{\epsilon\to 0}\int_{\{|\sigma|=\epsilon\}}\sqrt{-1}\bar{\partial}u_\infty\big((\omega_\infty+\sqrt{-1}\partial\bar{\partial}u_\infty)^n+\cdots+{\omega_\infty}^n\big).\end{aligned}$$

Notice we don't have this from the local estimates and it should provide us with extra global information about this limiting solution. But until now, I can't see how this can help us in the uniqueness argument.

By the way, in the case of dimension $n=2$, our original computation for comparing two solutions can be reformulated as follows:

$$\begin{aligned}0&\leqslant\int_{X\setminus\{\sigma=0\}}(u-v)({\omega_u}^2-{\omega_v}^2)\\&=\int_{X\setminus\{\sigma=0\}}(u-v)\sqrt{-1}\partial\bar{\partial}(u-v)(\omega_u+\omega_v)\\&=\int_{X\setminus\{\sigma=0\}}\big(\partial((u-v)\sqrt{-1}\bar{\partial}(u-v)(\omega_u+\omega_v))\\&\qquad-\sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)(\omega_u+\omega_v)\big).\end{aligned}$$

We don't have the integrability for each term of the last step. But we do know the second term has positive (maybe $+\infty$) integral. So again if we want to say the whole last expression is nonpositive, it has only left to consider the other term which can

[28]Existence is still by construction of u_∞ using flow method before.

be written down as (up to constant $\frac{1}{2}$):

$$\sqrt{-1}\partial\bar{\partial}(u-v)^2(\omega_u+\omega_v)=\Delta_{\omega_u}(u-v)^2\cdot{\omega_u}^2+\Delta_{\omega_v}(u-v)^2\cdot{\omega_v}^2.$$

Then the integration over X looks to be 0 if we forget about the singularities. If we want to treat it rigorously, we still have to use the estimates before.

There is another kind of argument used for proving the uniqueness (for general n) in classic case which essentially uses the following computations:

$$\Delta_{\omega_v}(u-v)^2=2(u-v)\Delta_{\omega_v}(u-v)+2|\partial(u-v)|^2_{\omega_v},$$

$$\Delta_{\omega_v}(u-v)=\langle\omega_v,\sqrt{-1}\partial\bar{\partial}(u-v)\rangle=\langle\omega_v,\omega_u\rangle-n\geqslant n\cdot\frac{{\omega_u}^n}{{\omega_v}^n}-n=ne^{u-v}-n.$$

For our situation, only considering the regular part of the solutions, the second one tells us $u-v\geqslant 0\Longrightarrow\Delta_{\omega_v}(u-v)>0$. Thus we can imagine that "the proper knowledge (i.e. nonpositivity) for the integral of the left hand side of the first equation over $\{u>v\}$"(B) can give that $u-v$ is a nonnegative constant there. Then a little argument using the continuity of u and v in the regular part can give us that $u\geqslant v$ or $u\leqslant v$ in the whole regular part. Hence the uniqueness is proved by the integral equality used before.

It seems what we need is also proper estimates to justify the statement B, but remember here we are considering the sets in the domain $\{u\geqslant v\}$ which can be quite complicated because we can only argue in the regular part. And we can not quite say that we could consider $\{u\geqslant v\}$ and $\{u\leqslant v\}$ together since different metrics are used which breaks the symmetry. This shows some advantage of the expression earlier for the case $n=2$.

Remark 2.3.4. *The requirement here for singular solutions can be translated to the assumption of the solution being inside some proper Sobolev space, $W^{2,q}$ for some $q>1$ sufficiently large,* [29] *which by standard Sobolev embedding results would be contained in some Hölder space, $C^{0,\alpha}$, for some proper $\alpha>0$. This simple observation would help us later in different context.*

Anyway, the main philosophy for this section is trying to say the limit of the flow is the unique solution. And all the discussion above is trying to see whether

[29] We can use Calderon-Zygmund estimate to get control for all second order derivatives from Laplacian estimate at this place.

the properties it satisfies can guarantee the uniqueness of the solutions with those properties.

2.4 Big Class and Finite Time Singularity

We can also say something about the case when $[S] + K_X$ is merely big. In this case we should expect the (regular) flow to end in finite time or say there would be finite time singularity.

First, we can prove that the flow exists in the maximal time interval $[0, T)$, where

$$T := sup\{t|(e^{-t} - 1)c_1(X) + e^{-t}[\omega_0] \text{ is ample}\}.$$

Here we have taken $S = 0$ as for the main part of the previous discussion, but clearly there is nothing special about it (unless explicitly stated).

If K_X is also numerically effective, then $T = \infty$ which is the case being considered before. So now we focus on the case $T < \infty$. The proof is basically the same in spirit and the details have appeared in [TiZh] in the general case of $T \leqslant \infty$. So we just include the difference of the case of $T < \infty$ from the case of $T = \infty$ in a very concise way for completeness.

For any small $\epsilon > 0$, we can choose $T_\epsilon > 0$ such that $T_\epsilon + \epsilon < T$ and a real closed $(1,1)$ form ψ_ϵ such that $[\psi_\epsilon] = K_X$ and $\psi_\epsilon + a_\epsilon \cdot \omega_0 > 0$, where $a_\epsilon = \frac{1}{e^{T_\epsilon+\epsilon}-1}$. Choose a smooth volume form Ω_ϵ such that $\mathrm{Ric}(\Omega_\epsilon) = -\psi_\epsilon$. This Ω_ϵ is unique up to multiplication by a positive constant. Set $\omega_t = \psi_\epsilon + e^{-t}(\omega_0 - \psi_\epsilon)$ and $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. Then u can be chosen to satisfy (2.2) with Ω replaced by Ω_ϵ just as (2.6). As before, we shall first show the solution for (2.6) exists for $t \in [0, T_\epsilon]$. Observe that ω_t is a Kähler metric for $t \in [0, T_\epsilon]$ with uniformly bounded geometry. So by exactly the same argument as before, we get the existence of solution in $[0, T_\epsilon]$. We can have the same explicit relation between solutions of (2.6) associated to different ϵ's and it would allow us to glue together all these solutions for (2.6) associated to different ϵ's to get a global solution up to the time T. In fact, the potential flow can be solved in the maximal time interval $[0, T)$ no matter which Ω is chosen. We can summarize the above discussion in the following. [30]

Proposition 2.4.1. *Let X be a closed Kähler manifold. Then the Kähler-Ricci flow (2.1) (or (2.2)) with initial metric ω_0 has a unique smooth solution on $[0, T)$, where T*

[30] This result below, after simple rescaling, gives an affirmative answer to one of the problems listed in [FeIlKn].

is the supremum of t such that $(1-e^{-t})K_X + e^{-t}[\omega_0]$ is a Kähler class. In particular, if K_X is numerically effective, the solution exists for all time.

Of course, we still have the essential decreasing of $\frac{\partial u}{\partial t} + u$ and also of the volume form of $\tilde{\omega}_t$.

The convergence of the Kähler metrics along the flow can be achieved with just a little modification. The limiting class $[\omega_T]$ won't be positive from the definition of T and the fact that $T < \infty$, so we should expect local convergence.

The uniform upper bounds for u and $\frac{\partial u}{\partial t}$ are obtained as before. Now the following equation

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}) - n + \langle \tilde{\omega}_t, \omega_T \rangle$$

can be used to get degenerate lower bound for u with $[\omega_T]$ replacing $[\omega_\infty] = K_X$. Clearly the lemma used there can also be applied for this class $[\omega_T]$. Notice we can't get the lower bound for $\frac{\partial u}{\partial t}$ in $t \in [0, T)$ since the coefficient is now $1 - e^{t-T}$ instead of 1. In order to get a similar lower bound for $\frac{\partial u}{\partial t}$, we can still use the bigness of K_X as Tsuji did in [Tsh1] by studying the equation about volume evolution (2.11) which is the only equation used to get degenerate lower bounds of u and $\frac{\partial u}{\partial t}$ for $T = \infty$ case. The following result tells us how to use the bigness of K_X, which is a generalization of the lemma used before and can be found in [Ka1] and [Tsh2]. The proof is also essentially contained in [Ka3].

Lemma 2.4.2. *Let L be a divisor in a projective manifold X. If L is big, then there is an effective divisor E such that $L - \epsilon E$ is Kähler for $\epsilon \in (a, b)$ where $0 \leqslant a < b < \infty$.*

It's rather similar to the lemma used before. Recall the proof for that one essentially makes use of the openness of the big cone for the projective manifold X which clearly contains the positive cone. Even if L is not nef., one can still use the openness of the cone for big divisors to prove the lemma above. However, the constant ϵ may not be as close to 0 as one wants. In this case, this result is called Kodaira's Lemma as in [Tsh2].

Now in order to get a lower bound for $\frac{\partial u}{\partial t}$, we will apply the lemma above to $K_X = [\omega_\infty]$ and use the equation (2.11) which was used before exactly for this purpose. We can get a similar lower bound for $\frac{\partial u}{\partial t}$ with a constant ϵ which may not be as close to 0 as we want. Also for the choice of the divisor E, it's more restrictive for $[\omega_\infty]$ than for $[\omega_T]$, which is very clear from the geometry of the cones and the positions of $[\omega_T]$ and $K_X = [\omega_\infty]$. Of course, we can also have lower bound for u in this way, but it's not as good as what we have previously got by using the new equation.

Remark 2.4.3. *Let's point out that now the lower bounds for* u *and* $\frac{\partial u}{\partial t}$ *have different degeneration in general which might cause trouble for further consideration of this problem. For example, the degeneration of limiting volume at time* T *might not correspond to the map constructed later from the class* $[\omega_T]$ *in surface case.*

Anyway, this gives enough C^0 estimates. Laplacian estimate would be the same as before except that the section σ (or say the divisor E) used would have to be for the big class $[\omega_\infty]$, which then would work for $[\omega_T]$ and so for the background metric $\omega_{t,\epsilon}$ and lower bound for u, since we'll also have to use the lower bound for $\frac{\partial u}{\partial t}$ during the process. Let's also point out that in the final estimate for the Laplacian, the ϵ in the power can still be as close to 0 as possible since it's from ω_T and u.

Combining this with the known volume estimate:

$$\tilde{\omega}_t^n > C_\epsilon |\sigma|^{2\epsilon} {\omega_0}^n$$

where this ϵ may not be as close to 0 as possible. we have a uniform bound on $\tilde{\omega}_t$ in any given compact subset of $X \setminus \{\sigma = 0\}$. Here $[\omega_T]$ is not ample, the constant C_ϵ may blow up to ∞ as ϵ tends to 0. The higher order derivative estimates for u outside $\{\sigma = 0\}$ still follow from the standard theory on Monge-Ampere equations or Calabi's third order estimates as shown in [Ya] just as before.

We can conclude that $u(t, \cdot)$ converges in C^∞-topology for any compact subset out of $\{\sigma = 0\}$ as $t \to T$ in exactly the same way. The limit u_T is smooth outside $\{\sigma = 0\}$. Moreover, we have

$$(\omega_T + \sqrt{-1}\partial\bar{\partial} u_T)^n = e^{u_T + \frac{\partial u}{\partial t}|_T} \Omega, \qquad \text{on } X \setminus \{\sigma = 0\}, \tag{2.17}$$

where $\frac{\partial u}{\partial t}|_T$ denotes the limit of $\frac{\partial u}{\partial t}$. The positive limiting current $\omega_T + \sqrt{-1}\partial\bar{\partial} u_T$ is actually a Kähler metric in $X \setminus \{\sigma = 0\}$ by the above estimates for u. But now we do not have any reason to expect that $\frac{\partial u}{\partial t}$ would go to 0 locally as $t \to T$. In fact, we can still get a degenerate exponential lower bound for $\frac{\partial u}{\partial t}$ just as before. But the e^{-t} will not give anything special as $t \to T < \infty$.

Remark 2.4.4. E *may not be unique. We can choose different* E*'s to study (2.2). However, the limit* $u(T, \cdot)$ *is unique for this equation. This implies that* u_T *is smooth outside the intersection of all such* E*'s. Let's emphasize this* E *should be for the class* K_X *(not* $[\omega_T]$*) since we need the lower bound for* $\frac{\partial u}{\partial t}$*. With a slight abusing of notion, we still call such an intersection the stable base locus of* K_X *(or* $[S] + K_X$ *in general).*

Now let us summarize the above discussion in the following theorem for the case of $S = 0$, i.e., considering canonical class K_X. There is no difference for general situation here.

Theorem 2.4.5. *Suppose X is a projective manifold with big canonical bundle K_X and ω_0 is a given Kähler metric. Let T be defined before and $T < \infty$. Then the Kähler-Ricci flow (2.1) has a unique solution with initial data ω_0 on $[0, T)$ which converges as $t \to T$ to a positive $(1,1)$-current satisfying: this limiting current, which represents the cohomology class of K_X, is a smooth Kähler metric outside the stable base locus set of K_X and the solution of (2.1) converges to this limiting (singular) metric in the local C^∞-topology for this open subset. Moreover, in a suitable sense, the flow can be extended to the time T and we have the pointwise convergence of the flow on the level of potential. Though the limiting current may be singular along the stable base locus of K_X, its Lelong number vanishes everywhere and the potential function for the limiting current lies in any L^p-spaces for $p < \infty$ at this moment.*

There are some other differences between the cases $T < \infty$ and $T = \infty$.

First, for the complex dimension 2 case, consider the canonical class K_X. When $T < \infty$, if the initial Kähler metric represents a rational class, then the limiting class $[\omega_T]$ is still rational and indeed semi-ample. And (with a proper choice of the initial metric which will be described below) the map from X to some $\mathbb{CP}^N$ using some multiple of K_X would be a blowing-down map which crushes some rational (-1)-curves and the image would be smooth. When $T = \infty$, the map would crush (-2)-curves instead and the image would have rational double points. The details for this situation would appear later in application of our main results.

Now let's give some details for $T < \infty$ case. Clearly we have

$$\big(K_X + \frac{e^{-T}}{1 - e^{-T}}[\omega_0]\big) \cdot C = 0$$

where C is a irreducible complex curve. If $[\omega_0]$ is a rational Kähler class, then the coefficient $\frac{e^{-T}}{1-e^{-T}}$ is rational and so is the whole class. This argument clearly works for other class $([S] + K_X)$.

Remark 2.4.6. *For higher dimension, if one just uses the classic characterization of ampleness in [Kl] as above, then one might only get the coefficient is integral over $\mathbb{Q}$. As pointed out to me by Zuoliang Hou, in considering K_X $(S = 0)$, there is a classic result in algebraic geometry, Rationality Theorem (as in [KorMo]), which guarantees the rationality of the class $[\omega_T]$. But we do not have it for a general class $([S] + K_X)$.*

Applying a result in [Ka1], in considering canonical class K_X, we then have that $[\omega_T]$ is semi-ample. Some large multiple of this class (holomorphic line bundle) will have enough holomorphic sections to give a map from X to some $\mathbb{CP}^N$. In fact, as $[\omega_T] - E$ is ample for some rational class E, we can see the map could be birational. In the following, we justify that it's just a map which blows down some (disjoint) (-1)-curves. [31]

Let's first find more information about the curve C before. Obviously, $K_X \cdot C < 0$. Since $K_X - \epsilon E > 0$ (i.e., Kähler) for some $\epsilon > 0$ and E an integral effective divisor (curve), $0 < K_X \cdot C - \epsilon E \cdot C$ which tells $E \cdot C < 0$, so $C \cdot C < 0$ as the intersection of two difference irreducible curves is always nonnegative. Then by the adjunction formula $K_X \cdot C = K_C \cdot C - C \cdot C$, we have $0 > K_C \cdot C = 2g_C - 2$ where $g_C \geqslant 0$ is the algebraic genus of C. So we see $g_C = 0$ which tells that C is a rational curve (i.e., isomorphic to $\mathbb{CP}^1$). Using again the adjunction formula, we see $0 > -2 - C \cdot C$. Since $C \cdot C < 0$, so $C \cdot C = -1$ and $K_X \cdot C = -1$. Conversely, by adjunction formula, we see any (-1)-curve would have -1 intersection with K_X. But it may not be true that all of them have $[\omega_T] \cdot C = 0$. And the number of (-1)-curves in X would have to be finite by topological consideration of β-number. All of them should be disjoint from one to the other simply because (any divisor representative of) K_X, which is big, should always contain some positive rational multiple of each of them as $K_X \cdot C < 0$, then we can easily contradicts any positive intersection between any two of these by $K_X \cdot C < 0$.

The union of those C's which have 0 intersection with $[\omega_T]$ would be the stable base locus set of $[\omega_T]$ from the result in [Nak] mentioned before. Since it'll be the intersection of finitely many curves E's such that $[\omega_T] - \frac{1}{M}E > 0$ for sufficiently large enough integer M, we can see $[\omega_T] - \frac{1}{M}\sum_{[\omega_T]\cdot C_i = 0} C_i$ is positive for M large enough as follows.

We just need to see it intersects any irreducible curve D positively. For one of those C's, it's not a problem. For a curve D with $[\omega_T] \cdot D > 0$, clearly, D is not in one of those E's which we still denote by E, and $([\omega_T] - \frac{1}{M}E) \cdot D > 0$. It's easy to see $([\omega_T] - \frac{1}{M}\sum_{[\omega_T]\cdot C_i = 0} C_i) \cdot D > ([\omega_T] - \frac{1}{M}E) \cdot D$ since D intersects every irreducible curve in E positively. Hence it's done.

So we know the map from large multiple of $[\omega_T]$ would indeed be birational out of those C's. As $[\omega_T] \cdot C = 0$, any holomorphic section, s, would have its 0 locus set not intersecting C or containing C. Thus for any two such sections s_1 and s_2, $s_1 = a \cdot s_2$ for some $a \in \mathbb{C}$ over C. So we know the map would crush each C to a point. But it's

[31] See for example [BaPetVa] for more systematic treatment for complex surfaces.

possible that different C's can be crushed to the same point. This already gives very restricted information about the map and the possible singularities of the image.

Finally, we want to say each C got mapped to a different point which will give the blowing-down picture claimed before. This basically requires that for each one of those C's, there is a holomorphic section whose 0 locus set contains only it (and no other C's). At this moment, it looks difficult for me to justify this. But there is a trivial case that guarantees this which is when we only have one such curve C. Notice that different (-1)-curve would represent different cohomology class which trivial comes from the self-intersection -1 and intersection 0 between different ones. So we can choose the initial metric properly so that $[\omega_T] \cdot C = 0$ for just one such C. It's clear that this choice of ω_0 would be proper for all the time in the following sense. Suppose we can continue the flow on the image which is a smooth manifold with the initial class to be the hyperplane class which corresponds to the class $[\omega_T]$. Then at the possible finite time when singularities occur, we still only encount only one (-1)-curve. Though this curve may not be (-1)-curve in the original manifold X, it's still decided by cohomology information of X. So a proper choice of $[\omega_0]$ would always be proper in the above sense.

Another difference for $T < \infty$ case is, in general, when $[\omega_T]$ has a nonnegative representative [32], we can have the boundedness of the potential u in $[0, T)$ simply by maximum principle argument as follows.

Recall the equation:

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}) = \Delta_{\tilde{\omega}_t}(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}) - n + \langle \tilde{\omega}_t, \omega_T \rangle.$$

The assumption above tells $\omega_T + \sqrt{-1}\partial\bar{\partial} f \geqslant 0$. Thus we can easily get:

$$\frac{\partial}{\partial t}\big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t} + nt - f\big) \geqslant \Delta_{\tilde{\omega}_t}\big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t} + nt - f\big).$$

Then maximum principle gives $\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t} + nt - f \geqslant -C$. Since $t \in [0, T)$ where $T < \infty$, this gives $u \geqslant (e^{t-T} - 1)\frac{\partial u}{\partial t} - Ct - C \geqslant -C$.

$T < \infty$ is essential for the discussion above. But notice we do not have uniform boundedness for $\frac{\partial u}{\partial t}$. In fact, if that's the case, we would automatically have the continuity of the limit of potential as $t \to T$.

Later, using pluripotential theory argument, we'll see that in the case of $T = \infty$,

[32]Semi-ampleness of the class will do, but this is more general-looking.

when the limiting class is semi-ample, we have the boundedness of u and $\frac{\partial u}{\partial t}$. But this would not justify the continuity of the limit of u as $t \to \infty$, which can be proved by later argument using pluripotential theory together with some classic results in several complex variables [33].

Remark 2.4.7. *The most natural and interesting problem for the case $T < \infty$ now would be how to continue the flow in a proper sense to infinity and get certain meaningful limit. We hope to address this problem in the future.*

2.5 Other Set-ups of Continuity Method

Remember that our main goal is to solve for the limiting equation of the Kähler-Ricci flow:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u \Omega$$

where ω_∞ is a real closed $(1,1)$-form with $[\omega_\infty] = K_X$ being nef. and big, and Ω is a smooth volume form on X with $\mathrm{Ric}(\Omega) = -\omega_\infty$ [34]. X is a projective manifold of complex dimension $n \geqslant 2$.

We have already found a solution by considering it as the limit for the Kähler-Ricci flow equation:

$$\frac{\partial v}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}v)^n}{\Omega} - v, \qquad v(0,\cdot) = 0$$

where $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ with some fixed (initial) Kähler metric ω_0.

Of course we have $\omega_t \to \omega_\infty$ as $t \to \infty$. Intuitively, we can think about the flow method to get such a solution for the degenerate Monge-Ampere equation (just the limiting equation) as using a family of changing background forms correspondent to Kähler classes, ω_t, to approach the target form ω_∞ with the desirable cohomology information, correspondingly the modified family of metrics $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u(t,\cdot))$ will approach the desirable (singular) metric, which satisfies some natural equation and has the right cohomology information, $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$. This is just a lengthy description about continuity method applied in this situation.

We have used a pretty delicate way of modifying the original family of forms, namely, the modified family forming a smooth evolving flow (the Kähler-Ricci flow).

[33] The continuity for u_T above can be proved similarly in the case when the class $[\omega_T]$ is semi-ample and big.

[34] This is rather superficial as we can see from the discussion before. In fact, we can consider $[\omega_\infty] = [S] + K_X$ without any modification of the previous argument. This class $[S] + K_X$ would be the class $[L]$ in the main problem which does not have to be K_X. But we'll still consider this case in the following for simplicity as usual.

But only for the concern of the limiting equation, it is not necessary to make the modified family of metrics so nice. Actually if we do things more brutally, it'll sometimes reduce the technical difficulties as explained below. By all means, we should expect the solution thus got to be the same one as before which is indeed the case.

2.5.1 Only Perturbing Background Class

In this subsection, we use perturbation methods which only change the left hand side of the equation. Basically, we modify ω_∞ by some linear term to get a Kähler class.

Since K_X is nef., the possibly most natural family of changing forms we should think of would be $\{\omega_\infty + \epsilon\omega\}_{\epsilon\in(0,1]}$ where ω is a fixed Kähler metric. Clearly the perturbed classes $[\omega_\infty + \epsilon\omega]$'s are all Kähler. We require $\omega \geqslant 0$ for the argument below at first for simplicity and it'll be removed later. But we definitely need $[\omega]$ to be a Kähler class. The modified family of metrics would be $\{\tilde{\omega}_\epsilon = \omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon\}_{\epsilon\in(0,1]}$ satisfying

$$(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega.$$

The existence and uniqueness of the modified metrics are classic results. Notice that each ω_ϵ may not be positive, but it can be made positive by adding some $\sqrt{-1}\partial\bar{\partial}f_\epsilon$ which may also depend on ϵ, so now we can apply classic results by changing u_ϵ to $u_\epsilon - f_\epsilon$. In spirit, we have the counterpart of global existence in the flow case for free using this simple-minded perturbation.

Our mission now is to study the limiting situation of u_ϵ and also $\tilde{\omega}_\epsilon$ as $\epsilon \to 0$. All the discussion below is for each u_ϵ separately. But we want the estimates to have some uniformality which also reminds us about the arguments for flow about global estimates for all time.

By maximum principle, we have $u_\epsilon < C$ uniformly for all $\epsilon \in (0, 1]$. [35]

Now we can localize the estimates in the same spirit as before using the bigness of K_X [36]. Let's digress a little bit here to see the necessity of the bigness assumption of the (limiting) class.

X is projective, so there is an integral Kähler class. Thus rational Kähler classes

[35] At the maximal value point, $\sqrt{-1}\partial\bar{\partial}u_\epsilon \leqslant 0$ and $\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon > 0$, thus $\omega_\epsilon \geqslant \tilde{\omega}_\epsilon > 0$, and so ${\omega_\epsilon}^n \geqslant \tilde{\omega}_\epsilon^n$. Now ϵ is fixed in this maximum principle argument, so we only have to take maximum over X. The life is slightly easier.

[36] We do not want to use the fact that K_X would be semi-ample since the argument is supposed to be true for any other nef. and big class.

are dense in the positive cone. All the cones will be in the $\mathbb{R}$ picture. By a classic result of algebraic geometry as in [Kl], nef. cone is the closure of positive cone. The projectivity of X makes sure that the big cone is open which is essentially proved using similar argument as in [Ka3] by considering exact sequence of sheaves. And it's also proved there that if a rational class $[L]$ is nef., then the bigness of L is equivalent to $[L]^n > 0$ where n is the complex dimension of X.

Let's also take a look at the case when $[L]$ is irrational. Suppose it's nef. and big. Construct classes $[L] + \sum_j a_j^k E_j$ to be rational and positive where positive real numbers $a_j^k \to 0$ as $k \to \infty$, E_j's are integral and effective, and the summation is finite. This is can be done by using $[L]$ is nef. and big and noticing the density of rational positive classes. Now as it's clear that each $[L] + \sum_j a_j^k E_j$ has "no fewer holomorphic sections" than $[L]$ though the counting is not that direct, and so the coefficient for the highest order term, which is $([L] + \sum_j a_j^k E_j)^n$ by positivity of the class, has to be not smaller than some fixed positive number from the bigness of $[L]$. Clearly the limit of $([L] + \sum_j a_j^k E_j)^n$ is $[L]^n$ as $k \to \infty$, hence $[L]^n > 0$. In fact this direction has also been proved before using the limit of flow we constructed.

Now assume $[L]$ is nef. with $[L]^n > 0$. Because it's not rational, in order to count the number of sections, we have to take the integral part of the muitiple of it. Then if one wants to apply the proof for rational case in [Ka3], we need to use a integral class H which is positive enough to dominate what is removed from $m[L]$ to make it integral. It's definitely the case if we can fix a representation of $[L]$ by a finite linear combination of effective integral divisors with positive coefficients when taking the integral part of any $m[L]$.

Anyway, consider a rational class $[L]$, which is of most geometric interests, in the main equation

$$(L + \sqrt{-1}\partial\bar{\partial}u)^n = e^u \Omega$$

where $L = \omega_\infty$ above. Suppose L is nef. for now. If it's not big, then the integration of the left hand side would be 0 in any reasonable sense, so is the right hand side for a solution u. Thus we can expect nothing but $-\infty$ as a solution which is clearly not so interesting. [37] This shows the naturality of the bigness assumption in an intuitive way.

As before, we assume $\omega_\infty + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0$ for $\delta \in (0, a)$ [38]. Now we fix any $\delta \in (0, a)$. Basically let's use the following expression of the original perturbed

[37] Of course it's meaningful to study the way of collapsing.

[38] The norm $|\cdot|$ may well depends on δ. But in fact for the following we only consider for each δ and will not use two simultaneously at all, so we can just ignore this.

equation over $X \setminus \{\sigma = 0\}$:

$$(\omega_\epsilon + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \delta\log|\sigma|^2))^n = e^{u_\epsilon}\Omega.$$

The minimum value point of $u_\epsilon - \delta\log|\sigma|^2$ clearly exists out of $\{\sigma = 0\}$ since u_ϵ is smooth. Thus at that point, we have $u_\epsilon > -C_\delta$ since $\omega_\epsilon + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ are uniform as metric for all $\epsilon \in (0, 1]$ [39], and so at that point

$$u_\epsilon - \delta\log|\sigma|^2 > -C_\delta - \delta\log|\sigma|^2 > -C_\delta.$$

This tells that over X, we have the degenerated but uniform lower bounds

$$u_\epsilon > -C_\delta + \delta\log|\sigma|^2.$$

Now rewrite the equation over $X \setminus \{\sigma = 0\}$ as follows

$$(\omega_{\epsilon,\delta} + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \delta\log|\sigma|^2))^n = e^{u_\epsilon + \log\frac{\Omega}{{\omega_{\epsilon,\delta}}^n}}{\omega_{\epsilon,\delta}}^n$$

with $\omega_{\epsilon,\delta} = \omega_\epsilon + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$.

Using the uniformality of $\omega_{\epsilon,\delta}$ as metric (for fixed δ) and the uniform upper bound for u_ϵ above, the standard computation of Laplacian estimate for this equation above gives that over $X \setminus \{\sigma = 0\}$ [40]:

$$e^{C_\delta(u_\epsilon - \delta\log|\sigma|^2)}\Delta_{\tilde{\omega}_\epsilon}(e^{-C_\delta(u_\epsilon - \delta\log|\sigma|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle) > -C_\delta - C_\delta\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle + C_\delta\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle^{\frac{n}{n-1}}.$$

Obviously, it still makes sense to talk about the maximal value point of the term, $e^{-C_\delta(u_\epsilon - \delta\log|\sigma|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle$ (for each ϵ), and it is actually out of $\{\sigma = 0\}$. Then at that point, we have $\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle < C_\delta$, and so

$$e^{-C_\delta(u_\epsilon - \delta\log|\sigma|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle < C_\delta e^{-C_\delta(u_\epsilon - \delta\log|\sigma|^2)} < C_\delta$$

where we have used the estimate $u_\epsilon - \delta\log|\sigma|^2 > -C_\delta$ for the last step. [41] We can

[39] The assumption $\omega \geqslant 0$ makes life easier here. But in fact, as we only need to consider ϵ sufficiently small, this assumption is not that important.

[40] See Appendix for more details about the computation.

[41] One might want to use the lower estimate for u_ϵ with different δ's to get something more interesting as in the discussion for the flow. But that won't be the case here as this is just an estimate for one point. The good part is that we do NOT have to use different δ's to carry through the argument in comparison to the flow case.

now have $e^{-C_\delta(u_\epsilon-\delta\log|\sigma|^2)}\langle\omega_{\epsilon,\delta},\tilde\omega_\epsilon\rangle < C_\delta$ for the whole of X, which can be rewritten as

$$\langle\omega_{\epsilon,\delta},\tilde\omega_\epsilon\rangle < C_\delta e^{C_\delta(u_\epsilon-\delta\log|\sigma|^2)} < C_\delta|\sigma|^{-2\delta C_\delta}.$$

We can use a fixed metric for $\omega_{\epsilon,\delta}$ since it is uniform as metric. And if we check the meaning for the power of $|\sigma|$ in the estimate above, it should be almost the same as the one for the corresponding estimate for the flow case, namely, representing the degeneracy of K_X as Kähler class.

Unitl now we have got the uniform volume lower bound and Laplacian upper bound for all $\tilde\omega_\epsilon$ with $\epsilon\in(0,1]$ for any compact subset out of $\{\sigma=0\}$. Then we can say all the higher order derivative estimates are available from standard arguments.

By Ascoli-Arzela's Theorem, we can then find a sequence of ϵ's such that the correspondent $\tilde\omega_\epsilon$ converges in C^∞-topology locally out of $\{\sigma=0\}$ [42]. We can already see that in the regular part, the limit would satisfy the equation for $\epsilon=0$, i.e., the main equation we want to study.

Since all the properties, including the regular part on X and the global integral equality except for the global "plurisubharmonicity"[43], of the limiting solution coming from the flow method, can be established for this new limit by basically the same discussion as before because the estimates are the same, we'd better justify that they are in fact the same solution. Otherwise the uniqueness of such solutions discussed before will be totally out of luck. Fortunately, they are indeed the same from easy maximum principle argument as follows.

At the first look, the new limit might even depend on the sequence chosen for ϵ, but in fact it won't as we can have the convergence for $\epsilon\to 0$ which would follow from the monotonicity of u_ϵ proved below. Actually we are proving something a little more general. Consider two smooth closed $(1,1)$-forms $\omega_1\geqslant\omega_2$, and suppose we have smooth functions u_1, u_2 satisfying:

$$(\omega_1+\sqrt{-1}\partial\bar\partial u_1)^n = e^{u_1}\Omega, \qquad (\omega_2+\sqrt{-1}\partial\bar\partial u_2)^n = e^{u_2}\Omega$$

with $\omega_1+\sqrt{-1}\partial\bar\partial u_1$ and $\omega_2+\sqrt{-1}\partial\bar\partial u_2$ being metrics. Take quotient to get:

$$\frac{\big(\omega_2+\sqrt{-1}\partial\bar\partial u_2+(\omega_1-\omega_2)+\sqrt{-1}\partial\bar\partial(u_1-u_2)\big)^n}{(\omega_2+\sqrt{-1}\partial\bar\partial u_2)^n} = e^{u_1-u_2}.$$

By maximum principle, considering the minimal value point of u_1-u_2 and noticing

[42] Just as before, a diagonalization argument is involved.

[43] It means plurisubharmonic with respect to ω_∞.

$\omega_1 - \omega_2 \geqslant 0$, we conclude $u_1 \geqslant u_2$. This is the desired monotonicity which gives the decreasing convergence of u_ϵ to the limit as ϵ decreasing to 0 since $\omega \geqslant 0$ and will justify the local convergence in C^∞-topology as $\epsilon \to 0$.

At this point, let's digress a little to give similar consideration about the dependence of u_ϵ on ϵ which has very much the same flavor as the formal discussion for the flow.

Formally, take the derivative with respect to ϵ for the perturbed family of equations, $\log \frac{(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n}{\Omega} = u_\epsilon$ [44] to get:

$$\Delta_{\tilde{\omega}_\epsilon}\left(\frac{\partial u_\epsilon}{\partial \epsilon}\right) + \langle \tilde{\omega}_\epsilon, \omega \rangle = \frac{\partial u_\epsilon}{\partial \epsilon}.$$

Then formally by maximum principle, we can see $\frac{\partial u_\epsilon}{\partial \epsilon} > 0$ which gives the monotonicity.

But just as in the flow case, we have to justify all these. Obviously, it'll be enough to have the smoothness of u_ϵ with respect to ϵ. But it doesn't look so trivial to me for now. [45]

In fact by this monotonicity and the global integral equality discussed before, we can easily see the new limit won't depend on the choice of the perturbing Kähler metric ω. We can also get the global plurisuharmonicity of this limit by considering the limit globally from a pointwise decreasing convergent sequence of plurisubharmonic functions.

There is still a choice for Ω which should also affect the form of ω_∞. It's easy to see this will not affect the solution in the level of metric since the change of u_ϵ is quite explicit and the equation is indeed the same.

Now there is a minor issue about the perturbation ω. In fact we only need it to be a smooth real closed $(1,1)$-form representing a Kähler class. We still have the approximations u_ϵ and $\tilde{\omega}_\epsilon$ as before. For the estimates, we only have to see that for

[44] It's just a handy reformulation of $(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega$ for taking derivative.

[45] Over a closed manifold X, for the equation $(\omega + \sqrt{-1}\partial\bar{\partial}u_\delta)^n = F_\delta \omega^n$ with ω being a fixed Kähler metric and F_δ being a smooth positive function which is also smooth with respect to δ, then classic theory from functional analysis would tell us that u_δ would be smooth with respect to δ after being properly normalized. The main idea is to consider the map from u_δ to F_δ which is between function spaces. But for our case above, ω is changing, so we have to consider a family of such maps between function spaces. Fixed point theorem is involved in this context and limited regularity on parameter ϵ can be guaranteed as I see it now. This problem is in the same spirit as what occurs in the flow case.

fixed δ, we still have uniform metrics $\omega_\epsilon + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ for sufficiently small $\epsilon > 0$.

The only difference could be the monotonicity for u_ϵ with respect to ϵ since now $\alpha\omega$ may not be positive, and so we do not see the monotonous convergence of u_ϵ as $\epsilon \to 0$ from the above argument. But we still have that the sequence limit won't depend on the choice of sequence because we can still use the monotonicity result above to compare u_ϵ with v_ϵ correspondent to some big positive perturbation (very positive "ω") by noticing that the monotonicity argument before still works. Again by the global integral equality, we see the limits are the same out of the stable base locus set.

But if we feel satisfied about the discussion in the previous paragragh, then we have trouble to get the global plurisubharmonicity for the limit with perturbation not being nonnegative. And in fact it would be hard to define the limit globally on X like this. Of course since we know that we can extend the limit from the regular part to the whole of X when using nonnegative perturbation and the extension is clearly unique if we require plurisubharmonicity, we should feel comfortable about the global meaning of the limit as a plurisubharmonic function. However, this is clearly not so satisfying. Actually we can still directly get all the information as before when ω is not necessarily positive as follows.

Take $\omega_1 = \omega + \sqrt{-1}\partial\bar{\partial}f > 0$ for some smooth function $f \leqslant 0$ over X. Then we can rewrite the perturbed equations as

$$(\omega_\infty + \epsilon\omega_1 + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \epsilon f))^n = e^{u_\epsilon}\Omega$$

where u_ϵ is clearly the solution for the original perturbed equation. Now consider $\epsilon > \delta > 0$ and take quotient of the two correspondent equations to get:

$$\big(\omega_\infty + \delta\omega_1 + \sqrt{-1}\partial\bar{\partial}(u_\delta - \delta f) + (\epsilon - \delta)\omega_1 + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - u_\delta - (\epsilon - \delta)f)\big)^n$$

$$= e^{u_\epsilon - u_\delta}(\omega_\infty + \delta\omega_1 + \sqrt{-1}\partial\bar{\partial}(u_\delta - \delta f))^n.$$

By maximum principle, considering the minimal value point of $u_\epsilon - u_\delta - (\epsilon - \delta)f$, we get that at that point, $u_\epsilon - u_\delta \geqslant 0$, and so $u_\epsilon - u_\delta - (\epsilon - \delta)f \geqslant 0$. The last inequality would be true over X. So we arrive at

$$u_\epsilon - \epsilon f \geqslant u_\delta - \delta f.$$

This is enough for concluding the convergence of u_ϵ as $\epsilon \to 0$. Moreover, this monotonicity is obviously enough to conclude the plurisubharmonicity of the limit by the usual argument.

Finally let's show the limit is actually just the same as what is previously got from the flow. Since we have already seen these two limits will not depend on all the choices respectively, the favorable situation can be chosen to compare them: [46]

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u_t)^n = e^{\frac{\partial u_t}{\partial t}+u_t}\Omega, \quad \omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty),$$

$$(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n = e^{v_\epsilon}\Omega, \quad \omega_\epsilon = \omega_\infty + \epsilon\omega.$$

Choose proper Kähler metrics ω_0 and ω such that $\omega_t = \omega_\epsilon$ for proper sequences of t's and ϵ's. The following t and ϵ will be the correspondent ones from the sequences. And we also take proper Ω such that $\frac{\partial u_t}{\partial t} \leqslant 0$ which we actually used before for the clean convergence of potential flow.

Still by taking quotient, we have

$$\frac{\big(\omega_t + \sqrt{-1}\partial\bar{\partial}v_\epsilon + \sqrt{-1}\partial\bar{\partial}(u_t - v_\epsilon)\big)^n}{(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n} = e^{\frac{\partial u_t}{\partial t}+u_t-v_\epsilon} \leqslant e^{u_t-v_\epsilon}.$$

By maximum principle, we can get $u_t \geqslant v_\epsilon$. Thus we have the one-sided relation for the limits, which together with the global integral equality will tell us that they are actually the same for the regular part. Hence they are the same globally by plurisuharmonicity.

Remark 2.5.1. *In fact, the solution(s) for $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$ we get by these two methods above can both be approximated by sequences of nice decreasing plurisubharmonic functions and that alone can make sure we are getting the same limiting solution. We'll discuss this point of view later in this Chapter by applying the theory about Monge-Ampere operator on unbounded functions.*

Actually there is another kind of perturbation of K_X which is also very natural, namely, $K_X - \epsilon E$ is Kähler for some effective divisor E and $\epsilon \in (0, a)$. So it is also fairly natural to consider the following family of equations:

$$(\omega_\infty - \epsilon E + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega$$

where by abusing of notation, E is used to stand for the curvature form of the line bundle E with some fixed hermitian metric. $\omega_\infty - \epsilon E$ may not be positive as form,

[46] Hopefully the meaning of the lower indices below is self-evident.

but since it is a Kähler class for each $\epsilon \in (0, a)$, we still have a smooth solution u_ϵ for each of these equations. As usual, we'll then derive necessary estimates for all u_ϵ's in order to take limit. The discussion is for each of them, just need to make sure the bounds we get are uniform for all ϵ's.

Still by maximum principle, we get $u_\epsilon \leqslant C$ uniformly for all ϵ's. [47]

Now let's take $\omega_\delta = \omega_\infty + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma'|^2 > 0$ for a nontrivial holomorphic section σ of some proper divisor E' which satisfies $K_X - \delta E'$ being Kähler for some $\delta > 0$. E' doesn't have to be E. $|\cdot|$ is some properly chosen hermitian metric for the line bundle E'. By considering ϵ small enough, we still have $\omega_\delta - \epsilon E$ uniform as (Kähler) metric. Here you still either consider $\sqrt{-1}\partial\bar{\partial}\log|\sigma'|^2$ as the curvature form for this hermitian metric on E' or consider everything out of $\{\sigma' = 0\}$. But we do need to realize that ω_δ is a smooth Kähler metric over X. The equation can now be rewritten as the following

$$(\omega_\delta - \epsilon E + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \delta\log|\sigma'|^2))^n = e^{u_\epsilon}\Omega$$

in $X \setminus \{\sigma' = 0\}$.

By considering the minimal value point of $u_\epsilon - \delta\log|\sigma'|^2$ which clearly exists out of $\{\sigma' = 0\}$ and noticing the uniform positivity of $\omega_\delta - \epsilon E$, we get, at that point, $u_\epsilon \geqslant C_\delta$, and so $u_\epsilon - \delta\log|\sigma'|^2 \geqslant C$ globally. A fixed δ would be enough for our consideration, so we choose to ignore the dependence of the constants on δ.

Now it is quite routine to see the classic Laplacian estimate can be applied for the equation in the above form to give us the bound below:

$$\langle \omega, \omega_\infty - \epsilon E + \sqrt{-1}\partial\bar{\partial}u_\epsilon \rangle \leqslant C|\sigma'|^{-\alpha}$$

where ω is some fixed metric and α is some positive constant.

A little note would be that the inequality used to get this estimate is actually over $X \setminus \{\sigma' = 0\}$ and we need to make sure that when applying maximum principle, the point considered actually exists in this range which is obviously the case just as before.

Then we run the usual machinery to get all the higher derivative estimates locally out of $\{\sigma' = 0\}$ and take a sequence of ϵ tending to 0 to obtain a limit in the local sense. Call this limit u_0 which might depend on a lot of things apriori. And of course $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_0)^n = e^{u_0}\Omega$ out of $\{\sigma' = 0\}$ [48].

[47]Noticing at that point we have $\omega_\infty - \epsilon E \geqslant \omega_\infty - \epsilon E + \sqrt{-1}\partial\bar{\partial}u_\epsilon > 0$, so there is no need to assume the positivity of the background form, $\omega_\infty - \epsilon E$.

[48]Or say the stable base locus set which is trivial to conclude just as before since we can combine the information got by using any σ' for this family of equations.

From all the estimates above, it is easy to see $e^{u_0}\Omega$ has the right integral over X, namely $\int_X {\omega_\infty}^n$, just as in all the previous cases.

In fact by comparing these two perturbations in this subsection:

$$(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega, \qquad (\omega_\infty - \epsilon E + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n = e^{v_\epsilon}\Omega,$$

we can have

$$(\omega_\infty - \epsilon E + \sqrt{-1}\partial\bar{\partial}v_\epsilon + \epsilon(\omega + E) + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - v_\epsilon))^n$$

$$= e^{u_\epsilon - v_\epsilon}(\omega_\infty - \epsilon E + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n.$$

If we use ω positive enough in the first perturbation, which will not affect the limit, to make sure $\omega + E \geqslant 0$, then by maximum principle, we can have $u_\epsilon \geqslant v_\epsilon$ which gives the one-sided relation between the limits. Hence the limits are the same in sight of the integral equality.

But in fact, for the second perturbation, we can also have the convergence as $\epsilon \to 0$ just as for the previous perturbation. The situation here is of course very different. But the argument is still quite easy as follows.

This family of equations can be rewritten in the form:

$$(\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2 + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega$$

where we choose the proper hermitian metric $\|\cdot\|$ so that $E = -\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2$ where the σ is a nontrivial holomorphic section for E in the usual sense. We are now only considering in $X \setminus \{\sigma = 0\}$. As we can see, this can be done for any choice of σ for this E but not for all E' before, so we can't have the result from the following discussion for the domain out of the stable base locus set.

Now for $\epsilon > \lambda > 0$, we have:

$$(\omega_\infty + \lambda\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2 + \sqrt{-1}\partial\bar{\partial}u_\lambda + \sqrt{-1}\partial\bar{\partial}((\epsilon - \lambda)\log\|\sigma\|^2 + u_\epsilon - u_\lambda))^n$$

$$= e^{u_\epsilon - u_\lambda}(\omega_\infty + \lambda\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2 + \sqrt{-1}\partial\bar{\partial}u_\lambda)^n.$$

The maximal value point of $(\epsilon-\lambda)\log\|\sigma\|^2 + u_\epsilon - u_\lambda$ clearly exists in $X\setminus\{\sigma = 0\}$. Let's point out that it is very important to have the same σ in order to draw this conclusion, and so it is difficult to use similar argument to compare the limits of perturbed families with E's not linearly equivalent to each other (i.e., without common σ). Then at that point, we can have $u_\epsilon - u_\lambda \leqslant 0$. That is also just $(\epsilon-\lambda)\log\|\sigma\|^2 + u_\epsilon - u_\lambda \leqslant C(\epsilon-\lambda)$ at that point. Hence it is true globally over X. Actually let's take this constant C to be 0 which can be achieved by rescaling the norm $\|\cdot\|$. So finally we have

$$u_\epsilon + \epsilon\log\|\sigma\|^2 \leqslant u_\lambda + \lambda\log\|\sigma\|^2.$$

Obviously, this would be enough to conclude the smooth convergence of u_ϵ locally out of $\{\sigma=0\}$ [49]. But notice the convergence we can get here for u_ϵ is definitely out of $\{\sigma=0\}$ or say out of the intersection of all σ's for E. So if we want to say anything about the solution globally on X, for example, plurisubharmonic with respect to ω_∞, we still need to use the first kind of perturbation in this subsection.

Also unlike the situation for the first kind of perturbation where the convergence is essentially a decreasing one [50], here we have an increasing convergence of $u_\epsilon + \epsilon\log\|\sigma\|^2$ as $\epsilon \to 0$. Using a little argument in classic pluripotential theory, those functions are plurisubharmonic with respect to ω_∞ on X with the values along $\{\sigma=0\}$ being $-\infty$ which is quite compatible with the expression, and the limiting solution is just $(sup_{\epsilon\in(0,a)}\{u_\epsilon + \epsilon\log\|\sigma\|^2\})^*$ where the upper "$*$" means taking upper semi-continuization since this is the only function plurisubharmonic with respect to ω_∞ which has the desired values out of $\{\sigma=0\}$.

In fact, we can apply maximum principle in a similar fashion as above for the first kind of perturbation when the Kähler class ω used is integral (or rational). Let's sketch it below.

Since the class $[\omega]$ is integral and positive, we can have $\omega = \sqrt{-1}\partial\bar\partial\log|\sigma|^2$ where σ is a nontrivial holomorphic section of the line bundle corresponding to $[\omega]$ and $|\cdot|$ is a proper chosen hermitian metric. Of course one can make sure $|\sigma| \in [0,1]$ by making proper choices. Now over $X \setminus \{\sigma=0\}$, the perturbed equation can be rewritten as:

$$(\omega_\infty + \sqrt{-1}\partial\bar\partial(u_\epsilon - \epsilon\log|\sigma|^2))^n = e^{u_\epsilon}\Omega.$$

Consider $\epsilon > \delta > 0$ and take the quotient of the two equations to get the following

[49]Get first for $u_\epsilon + \epsilon\log\|\sigma\|^2$ using the sequence convergence and monotonicity, then conclude for u_ϵ itself since the other part is that explicit.

[50]This is also the reason why we can get the global information for the limiting solution.

equation:

$$\big(\omega_\infty + \sqrt{-1}\partial\bar\partial(u_\delta - \delta\log|\sigma|^2) + \sqrt{-1}\partial\bar\partial(u_\epsilon - u_\delta - (\epsilon-\delta)\log|\sigma|^2)\big)^n$$

$$= e^{u_\epsilon - u_\delta}\big(\omega_\infty + \sqrt{-1}\partial\bar\partial(u_\delta - \delta\log|\sigma|^2)\big)^n.$$

By maximum principle, considering the minimal value point of $u_\epsilon - u_\delta - (\epsilon-\delta)\log|\sigma|^2$ which clearly exists in $X \setminus \{\sigma = 0\}$, we see at that point, $u_\epsilon - u_\delta \geqslant 0$, and so $u_\epsilon - u_\delta - (\epsilon-\delta)\log|\sigma|^2 \geqslant 0$. Thus we conclude

$$u_\epsilon - \epsilon\log|\sigma|^2 \geqslant u_\delta - \delta\log|\sigma|^2,$$

for $\epsilon > \delta > 0$. Clearly this would provide enough information for the convergence of u_ϵ as $\epsilon \to 0$ by noticing that we can have a lot of sections σ with empty intersection for their 0 locus sets since the class $[\omega]$ is Kähler (ample).

But there is a slightly difference from the second kind of perturbation in this subsection. Basically, we don't have $u_\epsilon - \epsilon\log|\sigma|^2$ above as a global function plurisubharmonic with respect to ω_∞ over X. Here we can't extend the function to $\{\sigma = 0\}$ as before, which is essentially caused by the $-$ sign (i.e., the residue is not positive).

Finally let's use a little remark to end this part.

Remark 2.5.2. *As we can see, the perturbation method is quite robust. One just have to make sure that the forms we used to perturb are small in a uniform way so that they can be dominated by something like $\omega_\infty + \delta\sqrt{-1}\partial\bar\partial\log|\sigma|^2$. Actually an important requirement is that we should guarantee the existence of the solutions for the perturbed equations, in other words, the background classes have to Kähler after being perturbed in order to apply classic results.*

2.5.2 Another Natural Perturbation

We are still considering the degenerate Monge-Ampere equation

$$(\omega_\infty + \sqrt{-1}\partial\bar\partial u)^n = e^u\Omega$$

with $[\omega_\infty] = K_X$ nef. and big for X projective. As before, the discussion can be applied to a general class $[S] + K_X$.

Recall in the previous discussion, a Kähler class $[\omega]$ or some proper divisor E, is used to perturb the left hand side of the equation and make it fit in the classic situation where we have a smooth solution. But using the divisor E before, the following consideration is also quite natural.

Let's rewrite the equation as

$$(\omega_\infty + \epsilon\partial\bar{\partial}\log|\sigma|^2 + \sqrt{-1}\partial\bar{\partial}(u - \epsilon\log|\sigma|^2))^n = e^{u-\epsilon\log|\sigma|^2}|\sigma|^{2\epsilon}\Omega$$

with σ being a nontrivial holomorphic section of line bundle E and $|\cdot|$ being a fixed bundle metric for all $\epsilon > 0$. Strictly speaking, this equation is considered out of $\{\sigma = 0\}$.

Now use the curvature form E instead of $-\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ in the background form and consider $u-\epsilon\log|\sigma|^2$ as the unknown, u_ϵ. We get the following family of equations:

$$(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}|\sigma|^{2\epsilon}\Omega$$

where $\omega_\epsilon = \omega_\infty - \epsilon E$.

The notations are very similar to those for the last section, but clearly we are in a very different situation. Namely, we have also changed the right hand side and there is even a degenerate term $|\sigma|^\epsilon$ there. Now the family does depend on the choice of the section σ and the norm $|\cdot|$ which means we can not combine the properties we might get for all different σ's as before.

But as we can see from how this family of equations comes up, unlike the perturbations in the previous section, it is more like a way of rewriting the degenerate Monge-Ampere equation itself. And so we may expect this perturbation would provide more delicate information about the possible solution.

In fact, these equations have already been considered by Yau a long time ago in [Ya]. The result there tells that we do have a unique solution u_ϵ for each of them.[51] Formally, we can imagine $u_\epsilon = u - \epsilon\log|\sigma|^2$. So it looks very much likely to be true that that we'll have a convergence of these u_ϵ to a solution of the equation started with.

For this family of equations we are considering, there are two tendences of changes as $\epsilon \to 0$ which we'll describe below.

One is that the background form, ω_ϵ, becomes worse with the limit representing something degenerate as a Kähler class;

[51] Notice that we have also fixed the hermitian metric $|\cdot|$. So the background form ω_ϵ may not be positive for all ϵ's. But that won't affect the existence of the solutions for each perturbed equation from the same reasoning as before. More importantly, this will make the perturbation under control just as mentioned in the final remark of the previous subsection.

The other one is that the degenerate term for the right hand side, $|\sigma|^{2\epsilon}$, becomes nicer and finally there would be no degeneration when $\epsilon = 0$.

Let's see that the second tendence is really good for us first. This is indeed just Yau's original argument which is actually uniform for $\epsilon \to 0$ with fixed background metric [52]. In fact this is the main motivation for the discussion in this subsection. Though finally it turns out to be not so necessary for our purpose, we still sketch it a little below.

As usual, we just have to get some uniform estimates for the solutions of the equations

$$(\omega + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n = e^{v_\epsilon}|\sigma|^{2\epsilon}\Omega$$

where ω is now a fixed Kähler metric. From Yau's result, we can solve each of these equations for $\epsilon > 0$ [53] by considering the perturbation $(|\sigma|^2 + \delta)^\epsilon$ by a positive constant δ for the degenerated term on the right hand side and searching for the limit of solutions as $\delta \to 0$.

More precisely, for each fixed positive ϵ, we can get uniform bounds for C^0-norm and Laplacian. From these one can get locally uniform bounds for higher order derivatives out of $\{\sigma = 0\}$. Thus we have a limit v_ϵ which satisfies the original equation. The uniqueness of such a solution can be proved in the way in which we try to prove the uniqueness of our (singular) solution before (i.e., using integration by part). In this case, we have better estimates, so there is no problem to carry through the argument.

Now we just have to check all these estimates can be made uniform for all $\epsilon \in [0, 1]$ in order to see that the limit as $\epsilon \to 0$ of v_ϵ's exists and in fact is just the classic (smooth) solution for the equation with $\epsilon = 0$.

In Yau's argument, there seems to be one step which might break the uniformality of the estimates for all such ϵ's, but in fact that step is quite redundant because the place where it seems to be used can be dodged by noticing there is an unnecessary shifting of power in another step. More details can be found in Appendix.

Now let's come back to the family of equations:

$$(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}|\sigma|^{2\epsilon}\Omega$$

[52] In the case of fixed background form representing a Kähler class, using some smooth function, we can change it to a Kähler metric without essential affecting the feature of the equation itself.

[53] For the case $\epsilon = 0$, the discussion is easier, but it can still be contained in the following discussion.

for $\epsilon \in (0, a)$.

From the sketch above, we know the solution for each equation above does exist. In fact u_ϵ is $C^{1,\alpha}$ for any $\alpha \in [0, 1)$, as we can have bounded C^0-norm and Laplacian, and smooth out of $\{\sigma = 0\}$ where $\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon$ is really a Kähler metric in that range. We denote these metrics by $\tilde{\omega}_\epsilon$ as usual since now they are the metrics which we want to prove the convergence. Also we have that the integration of $\tilde{\omega}_\epsilon^n$ over the regular part is equal to $\int_X {\omega_\epsilon}^n$ by the estimates descrided above.

Consider the maximal value point for $u_\epsilon + \epsilon\log|\sigma|^2$ (for each ϵ) which does exist and should be out of $\{\sigma = 0\}$ (and so is a regular point). Rewrite the equation as:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\epsilon + \epsilon\log|\sigma|^2))^n = e^{u_\epsilon + \epsilon\log|\sigma|^2}\Omega$$

out of $\{\sigma = 0\}$ and we can see at that point,

$$u_\epsilon + \epsilon\log|\sigma|^2 \leqslant \log(max_{x\in X}\{\frac{{\omega_\infty}^n}{\Omega}\}) < C.$$

So $u_\epsilon + \epsilon\log|\sigma|^2 < C$ uniformly for all $\epsilon \in (0, a)$. That is just

$$u_\epsilon < C - \epsilon\log|\sigma|^2 < C - \log|\sigma|^2$$

as $|\sigma| \in [0, C]$. Here the second $<$ just to make the upper bound looks more uniform. And in fact it is still the first $<$ that is used for the Laplacian estimate later. This is the uniform upper bound for u_ϵ locally out of $\{\sigma = 0\}$, a degenerate one just as what we have encountered several times before, but this time it's from above.

Now we use the similar trick to localize the estimates out of $\{\sigma = 0\}$. But notice here we'd better use the same nontrivial holomorphic section σ because our known smooth regular part of u_ϵ is also out of $\{\sigma = 0\}$ and smoothness is important in applying maximum principle.

Set $\omega_{\epsilon,\delta} = \omega_\epsilon + \delta\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2$ for fixed $\delta \in (0, a)$ and some proper hermitian metric $\|\cdot\|$ for the line bundle E such that $\omega_\infty + \delta\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2 > 0$ where the second term is considered to be the corresponding curvature form over X or only considered to be over $X \setminus \{\sigma = 0\}$ when using this expression as usual. We still need that $\omega_{\epsilon,\delta}$ will be uniform as metric for all ϵ sufficiently small [54]. Now one can rewrite

[54]The "suffciently small" here may mean much smaller than $a - \delta$ since we need the form to be positive rather than representing a Kähler class. But clearly this will still be enough since we are considering the limit as $\epsilon \to 0$ anyway.

the equations over $X \setminus \{\sigma = 0\}$ as follows:

$$(\omega_{\epsilon,\delta} + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \delta\log\|\sigma\|^2))^n = e^{u_\epsilon - \delta\log\|\sigma\|^2}|\sigma|^{2(\epsilon+\delta)}\Omega',$$

where Ω' may be a smooth volume form different from Ω where the difference comes from the possible difference between the hermitian metrics $|\cdot|$ and $\|\cdot\|$.

As usual, we consider the minimal value point of $u_\epsilon - \delta\log\|\sigma\|^2$ which clearly exists out of $\{\sigma = 0\}$. At that point, we have $e^{u_\epsilon - \delta\log\|\sigma\|^2}|\sigma|^{2(\epsilon+\delta)}\Omega' \geqslant {\omega_{\epsilon,\delta}}^n$, which gives

$$e^{u_\epsilon - \delta\log\|\sigma\|^2} \geqslant C_\delta|\sigma|^{-2(\epsilon+\delta)} > C_\delta$$

at that point, and so is true over the whole of X. Hence we have $u_\epsilon > -C_\delta + \delta\log\|\sigma\|^2$. This is the uniform lower bound for u_ϵ locally out of $\{\sigma = 0\}$ which is again a degenerate one.

Now using the computation of Laplacian estimate for the following equation

$$(\omega_{\epsilon,\delta} + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \delta\log\|\sigma\|^2))^n = e^{u_\epsilon}|\sigma|^{2\epsilon}\Omega,$$

we have the following inequality over $X \setminus \{\sigma = 0\}$:

$$\begin{aligned} &e^{C_\delta(u_\epsilon - \delta\log\|\sigma\|^2)}\Delta_{\tilde{\omega}_\epsilon}\left(e^{-C_\delta(u_\epsilon - \delta\log\|\sigma\|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle\right) \\ &> -C_\delta - C_\delta\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle + C_\delta\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle^{\frac{n}{n-1}}. \end{aligned}$$

Let's consider the maximal value point of $e^{-C_\delta(u_\epsilon - \delta\log\|\sigma\|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle$. Here we can also see it exists out of $\{\sigma = 0\}$ since we know from Yau's argument that $\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle < C_{\epsilon,\delta}$ over $X \setminus \{\sigma = 0\}$ [55] and u_ϵ bounded for each ϵ. Hence the expression we are considering will go to 0 uniformly when approaching $\{\sigma = 0\}$ for each fixed ϵ. Then at that point, we have: $\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle < C_\delta$, which gives,

$$e^{-C_\delta(u_\epsilon - \delta\log\|\sigma\|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle < C_\delta e^{-C_\delta(u_\epsilon - \delta\log\|\sigma\|^2)}.$$

Using the uniform degenerate lower bound for u_ϵ got above, we conclude

$$e^{-C_\delta(u_\epsilon - \delta\log\|\sigma\|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle < C_\delta$$

[55] This estimate itself is not good enough for us since it clearly depends on ϵ. An interesting point here might be that we use an estimate depending on ϵ to justify an argument which would give an estimate uniform for all such ϵ's.

at that point and hence over X. Furthermore, we arrive at

$$\langle \omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon \rangle < C_\delta e^{C_\delta (u_\epsilon - \delta \log \|\sigma\|^2)} < C_\delta e^{C_\delta (C - (\epsilon+\delta)\log|\sigma|^2)} < C_\delta |\sigma|^{-2C_\delta(\epsilon+\delta)}$$

by the uniform degenerate upper bound for u_ϵ.

Notice here that we are considering the situation when $\epsilon \to 0$, so the Laplacian estimate would essentially be like

$$\langle \omega, \tilde{\omega}_\epsilon \rangle < C_\delta |\sigma|^{-2\delta C_\delta}$$

for any fixed metric ω. Now we have got the uniformality of $\tilde{\omega}_\epsilon$ as metric locally out of $\{\sigma = 0\}$. Thus we can proceed in the standard way to get uniform estimates for higher order derivatives locally out of $\{\sigma = 0\}$.

Now by Ascoli-Arzela's Theorem, we can have a sequence limit, u_0, which satisfies

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial} u_0)^n = e^{u_0}\Omega$$

in $X \setminus \{\sigma = 0\}$ and also has some estimates for it from the controls above. Let's point out that by the above discussion, the limit only exists out of $\{\sigma = 0\}$.

Essentially using $u_\epsilon + \epsilon \log|\sigma|^2 < C$ and by similar argument as before, we have the integral equality

$$\int_{X\setminus\{\sigma=0\}} (\omega_\infty + \sqrt{-1}\partial\bar{\partial} u_0)^n = \int_X e^{u_0}\Omega = \int_X {\omega_\infty}^n.$$

And this integral equality will be used below to prove that this sequence limit is the same as the limiting solutions from other methods. Hence we also get to know the limit here won't depend on the sequence we choose.

In fact, let's consider:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\epsilon + \epsilon \log|\sigma|^2))^n = e^{u_\epsilon + \epsilon \log|\sigma|^2}\Omega$$

over $X \setminus \{\sigma = 0\}$, and

$$(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial} v_\epsilon)^n = e^{v_\epsilon}\Omega$$

where $\omega \geqslant 0$ and represents a Kähler class.

Take quotient to get:

$$\frac{(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}v_\epsilon - \epsilon\omega + \sqrt{-1}\partial\bar{\partial}(u_\epsilon + \epsilon\log|\sigma|^2 - v_\epsilon))^n}{(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n} = e^{u_\epsilon + \epsilon\log|\sigma|^2 - v_\epsilon}.$$

Consider the maximal value point of $u_\epsilon + \epsilon\log|\sigma|^2 - v_\epsilon$ which obviously exists out of $\{\sigma = 0\}$. Clearly at that point,

$$u_\epsilon + \epsilon\log|\sigma|^2 - v_\epsilon \leqslant 0,$$

and so this is for the whole of X. And we know this is true for any ϵ. Now consider any point out of $\{\sigma = 0\}$ and let ϵ goes to 0 in the sequence for the sequence limit of u_ϵ. We see $u_0 \leqslant v_0$ [56] out of $\{\sigma = 0\}$. Then by the integral equality which they both satisfy, we see they are actually the same over $X \setminus \{\sigma = 0\}$. Hence they will be the same if we require plurisubharmonicity over X.

It is also possible to directly see that the sequence limit is independent on the choice of the sequence just by getting some kind of monotonous convergence result.

For $\epsilon > \lambda > 0$ small enough, over $X \setminus \{\sigma = 0\}$, we have:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\epsilon + \epsilon\log|\sigma|^2))^n = e^{u_\epsilon + \epsilon\log|\sigma|^2}\Omega,$$

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\lambda + \lambda\log|\sigma|^2))^n = e^{u_\lambda + \lambda\log|\sigma|^2}\Omega.$$

Take quotient to get:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\lambda + \lambda\log|\sigma|^2) + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - u_\lambda + (\epsilon - \lambda)\log|\sigma|^2))^n$$

$$= e^{u_\epsilon - u_\lambda + (\epsilon-\lambda)\log|\sigma|^2}(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\lambda + \lambda\log|\sigma|^2))^n.$$

Considering the maximal value point of $u_\epsilon - u_\lambda + (\epsilon - \lambda)\log|\sigma|^2$ which should exist out of $\{\sigma = 0\}$, by maximum principle, we have

$$u_\epsilon + \epsilon\log|\sigma|^2 \leqslant u_\lambda + \lambda\log|\sigma|^2$$

over $X \setminus \{\sigma = 0\}$. And this would be enough to conclude the convergence as $\epsilon \to 0$. The convergence is again an increasing one as $\epsilon \to 0$ as in the second perturbation considered in the previous subsection. So the same discussion there gives us the same

[56] They are correspondent sequence limits as $\epsilon \to 0$. Of course for v_ϵ, it's not just a sequence limit as proved before.

form of the limiting solution as $(sup_{\epsilon\in(0,a)}\{u_\epsilon+\epsilon\log|\sigma|^2\})^*$.

It would be interesting to consider the relation of this perturbation with the second perturbation considered in the previous subsection. Now we consider the following two equations over $X\setminus\{\sigma=0\}$ with $\epsilon>\lambda>0$: [57]

$$(\omega_\infty+\lambda\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2+\sqrt{-1}\partial\bar{\partial}u_\lambda)^n=e^{u_\lambda}|\sigma|^{2\lambda}\Omega,$$

$$(\omega_\infty+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2+\sqrt{-1}\partial\bar{\partial}v_\epsilon)^n=e^{v_\epsilon}\Omega.$$

Again we can take the quotient to get:

$$(\omega_\infty+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2+\sqrt{-1}\partial\bar{\partial}v_\epsilon+\sqrt{-1}\partial\bar{\partial}(u_\lambda-v_\epsilon+(\lambda-\epsilon)\log|\sigma|^2))^n$$

$$=e^{u_\lambda-v_\epsilon+\lambda\log|\sigma|^2}(\omega_\infty+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2+\sqrt{-1}\partial\bar{\partial}v_\epsilon)^n.$$

Clearly the minimal value point of $u_\lambda-v_\epsilon+(\lambda-\epsilon)\log|\sigma|^2$ exists out of $\{\sigma=0\}$. Then at that point, $u_\lambda-v_\epsilon+\lambda\log|\sigma|^2\geqslant 0$ which is also

$$u_\lambda-v_\epsilon+\lambda\log|\sigma|^2-\epsilon\log|\sigma|^2\geqslant 0$$

at that point by assuming without loss of generality that $|\sigma|\in[0,1]$. So we have $u_\lambda+\lambda\log|\sigma|^2\geqslant v_\epsilon+\epsilon\log|\sigma|^2$ globally (even makes sense on $\{\sigma=0\}$) for $\epsilon>\lambda>0$. But it is quite easy to see that we don't have the relation in the other direction, i.e., we do not have $v_\epsilon+\epsilon\log|\sigma|^2\geqslant u_\lambda+\lambda\log|\sigma|^2$ for $\lambda>\epsilon>0$, at least from similar consideration.

Combining the result from before, we know both sides increasingly converge to the limiting solution, but in some sense the convergence for the term discussed in this subsection is better than the one before in sight of the inequality above.

Remark 2.5.3. *We found out later that other people had already used this same method to get the (same) singular metric (as in [Su]). Our contribution here would be the uniqueness result of the metrics coming from this method and the further clarification of the convergence.*

[57] Here take both constants positive in order to guarantee the existence, regularity and boundedness for the solutions of the perturbed equations and take different constants to make sure the extremal value point discussed below appears in the regular part.

2.5.3 Big Class

Those perturbation methods discussed above can be applied for nef. and big class $[L]$ (or say $[S]+K_X$). The basic philosophy behind the argument is that we can use Kähler classes to approach $[L]$ and the bigness of this class would provide some degenerate bounds which come from involving some singular-looking terms and applying maximum principle [58].

Now one might want to do the similar thing for merely big class $[L]$. [59] Clearly we still have the perturbations in the same flavor as before. But now the existence of solutions for perturbed equation, which needs the background metrics to be Kähler, would destroy the possiblity to approach the original equation directly. Instead, the approximation has to stop before it gets to the class $[L]$. The limiting class would be like $[L+\epsilon\omega]$ or $[L]-\epsilon E$ for some $\epsilon>0$. Clearly this class and the limiting equation would depend on our choice for the perturbation, i.e., the choice of $[\omega]$ or E.

It's natural to think about how to push the perturbation further towards the class $[L]$. But it's likely to be less natural than the flow method discussed before. The delicacy of the flow approximation should have its advantage in this situation together with its technical difficulties. And we expect some new machinery for further study of this problem.

2.6 Pluripotential-Theoretic Revisit of Uniqueness Result

In this section we give a more complicated way of proving the uniqueness of the solution constructed by either the flow method or the perturbation method using a Kähler metric. This might looks meaningless as we already know the uniqueness and the proof is not hard at all. But the proof here has a strong flavor of comparison principle which can actually fit in a more general picture in pluripotential theory. It'll use the approaching sequence of functions in a more global way. Of course maximum principle is already a global argument, but here we are using the integration instead. No strong derivative is essentially involved and so we need less regularity than before.

[58] It's a lovely fact that the singular terms have the right sign (for ∞) which allows us to run through maximum principle argument before.

[59] The case when $[L]$ is merely nef. is not so justified in this context from the discussion at the beginning, at least when we are considering rational classes.

Suppose we have the following two packages of limiting sequences for $m \to \infty$:

$$u_m \to u_\infty, \omega_m \to \omega_\infty, \omega_m + \sqrt{-1}\partial\bar{\partial}u_m \to \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty, (\omega_m + \sqrt{-1}\partial\bar{\partial}u_m)^n \to e^{u_\infty}\Omega;$$

$$v_m \to v_\infty,\ \phi_m \to \omega_\infty,\ \phi_m + \sqrt{-1}\partial\bar{\partial}v_m \to \omega_\infty + \sqrt{-1}\partial\bar{\partial}v_\infty,\ (\phi_m + \sqrt{-1}\partial\bar{\partial}v_m)^n \to e^{v_\infty}\Omega;$$

where u_m and v_m are smooth, $\omega_m + \sqrt{-1}\partial\bar{\partial}u_m$ and $\phi_m + \sqrt{-1}\partial\bar{\partial}v_m$ are smooth Kähler metrics. Here for each package, the first and third limits are in the sense of local uniform convergence out of $\{\sigma = 0\}$ [60] and with certain uniform estimates locally out of $\{\sigma = 0\}$. Both limits satisfy $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}W)^n = e^W\Omega$ out of $\{\sigma = 0\}$ where W is the function u_∞ or v_∞ which are plurisubharmonic with respect to ω_∞. The last convergence is essentially from the estimates and equations which u_m and v_m satisfy. Finally we require $\phi_m - \omega_m \geqslant 0$ without requiring either of them to be nonnegative.

Clearly if we can prove $u_\infty = v_\infty$ out of $\{\sigma = 0\}$, it'll be enough for proving all the independence of sequence limit on the choice of convergent sequence for any sequences in the flow with any initial metric or sequences in the perturbing family for any perturbing Kähler metric, where probably some auxiliary sequence will be used to compare any chosen pair of sequences.

First there is a baby version for the comparison principle argument in classic smooth case. Suppose $\omega + \sqrt{-1}\partial\bar{\partial}u$ and $\omega + \sqrt{-1}\partial\bar{\partial}v$ are nonnegative where ω is a real smooth closed $(1,1)$-form, u and v are smooth functions. In the following, we prove

$$\int_{\{u>v\}} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leqslant \int_{\{u>v\}} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n.$$

For any $\delta > 0$, we can do the computation below.

$$\begin{aligned}
&\int_{\{u>v+\delta\}} \big((\omega + \sqrt{-1}\partial\bar{\partial}u)^n - (\omega + \sqrt{-1}\partial\bar{\partial}v)^n\big) \\
= &\int_{\{u>v+\delta\}} \sqrt{-1}\partial\bar{\partial}(u - v - \delta)\big((\omega + \sqrt{-1}\partial\bar{\partial}u)^{n-1} + \cdots + (\omega + \sqrt{-1}\partial\bar{\partial}v)^{n-1}\big) \\
= &\int_{\{u>v+\delta\}} d(d^c(u - v - \delta) \wedge \big((\omega + \sqrt{-1}\partial\bar{\partial}u)^{n-1} + \cdots + (\omega + \sqrt{-1}\partial\bar{\partial}v)^{n-1}\big)).
\end{aligned}$$

By Sard's Theorem, we can choose proper sequence of δ going to 0 with the boundary of $\{u > v+\delta\}$ being smooth for each δ. Then by Stokes' Theorem, the last expression

[60] Of course the stable base locus set can be used here instead, but we only require this for simplicity.

is equal to:

$$\int_{\partial\{u>v+\delta\}} d^c(u-v-\delta)\big((\omega+\sqrt{-1}\partial\bar\partial u)^{n-1}+\cdots+(\omega+\sqrt{-1}\partial\bar\partial v)^{n-1}\big),$$

where the orientation of the boundary is determined by the outer normal direction.

In fact the volume form for the boundary is just $\star d\rho$ where ρ is the local defining function for the boundary which is positive outside and negative inside. We also have locally $u-v-\delta = f\cdot\rho$ where f is nonpositive. Suppose

$$d^c(u-v-\delta)\big((\omega+\sqrt{-1}\partial\bar\partial u)^{n-1}+\cdots+(\omega+\sqrt{-1}\partial\bar\partial v)^{n-1}\big) = g\cdot(\star d\rho)$$

locally on the boundary. Notice $d^c(u-v-\delta) = f\cdot d^c\rho$ for the first part and the positivity of the second part. By wedging $d\rho$ on both sides of the equation, it is easy to conclude that $g\leqslant 0$. Thus we have

$$\int_{\{u>v+\delta\}}\big((\omega+\sqrt{-1}\partial\bar\partial u)^{n-1}-(\omega+\sqrt{-1}\partial\bar\partial v)^{n-1}\big)\leqslant 0.$$

Finally by taking $\delta\to 0$, the set we integrate over would enlarge to $\{u>v\}$, so we arrive at the claim above.

Now let's consider the current situation. First we have:

$$\int_{\{u_m>v_m\}}(\omega_m+\sqrt{-1}\partial\bar\partial u_m)^n\leqslant\int_{\{u_m>v_m\}}(\phi_m+\sqrt{-1}\partial\bar\partial u_m)^n\leqslant\int_{\{u_m>v_m\}}(\phi_m+\sqrt{-1}\partial\bar\partial v_m)^n,$$

where the first $\leqslant$ is from the assumption that $\phi_m\geqslant\omega_m$ [61] and the second one is from the result in the smooth case just proved above. Thus for any $\epsilon>0$, we have, for$\delta>0$ sufficiently small and m sufficiently large, that:

$$\int_{\{u_m>v_m+\lambda\}\cap\{|\sigma|>\delta\}}(\omega_m+\sqrt{-1}\partial\bar\partial u_m)^n\leqslant\int_{\{u_m>v_m+\lambda\}\cap\{|\sigma|>\delta\}}(\phi_m+\sqrt{-1}\partial\bar\partial v_m)^n+\epsilon$$

with any constant $\lambda>0$ [62]. Now basically we want to take limit for all the parameters. It needs to be done carefully in sight of all the dependences. We'll do it as follows.

In $\{|\sigma|>\delta\}$, we have uniform convergence of $u_m\to u_\infty$ and $v_m\to v_\infty$ as $m\to\infty$.

[61] So we have $0\leqslant\omega_m+\sqrt{-1}\partial\bar\partial u_m\leqslant\phi_m+\sqrt{-1}\partial\bar\partial u_m$.

[62] The part we remove has very little contribution in the total integral from the uniform upper bound of the volume as included in the assumptions.

Thus we have the relation below for any $\lambda > 0$ and sufficiently large m: [63]

$$\{u_\infty > v_\infty + 2\lambda\} \subset \{u_m > v_m + \lambda\} \subset \{u_\infty > v_\infty\}.$$

Combine these to get:

$$\int_{\{u_\infty > v_\infty + 2\lambda\}\cap\{|\sigma|>\delta\}} (\omega_m + \sqrt{-1}\partial\bar{\partial}u_m)^n \leqslant \int_{\{u_\infty > v_\infty\}\cap\{|\sigma|>\delta\}} (\phi_m + \sqrt{-1}\partial\bar{\partial}v_m)^n + \epsilon$$

which is for any $\epsilon > 0$, $\lambda > 0$, m sufficiently large and $\delta > 0$ sufficiently small. Now let's first take $m \to \infty$ to get:

$$\int_{\{u_\infty > v_\infty + 2\lambda\}\cap\{|\sigma|>\delta\}} e^{u_\infty}\Omega \leqslant \int_{\{u_\infty > v_\infty\}\cap\{|\sigma|>\delta\}} e^{v_\infty}\Omega + \epsilon.$$

Then we let $\lambda \to 0$:

$$\int_{\{u_\infty > v_\infty\}\cap\{|\sigma|>\delta\}} e^{u_\infty}\Omega \leqslant \int_{\{u_\infty > v_\infty\}\cap\{|\sigma|>\delta\}} e^{v_\infty}\Omega + \epsilon.$$

Now take $\delta \to 0$ and get:

$$\int_{\{u_\infty > v_\infty\}} e^{u_\infty}\Omega \leqslant \int_{\{u_\infty > v_\infty\}} e^{v_\infty}\Omega + \epsilon.$$

Actually it is not that necessary to take this limit. We can just replace ϵ by 2ϵ by noticing the part we add back contributes little. Strictly speaking, the integration after taking limit should be over $\{u_\infty > v_\infty\} \cap \{\sigma \neq 0\}$, but clearly it doesn't bring any difference. Finally we let $\epsilon \to 0$ and arrive at:

$$\int_{\{u_\infty > v_\infty\}} e^{u_\infty}\Omega \leqslant \int_{\{u_\infty > v_\infty\}} e^{v_\infty}\Omega.$$

Thus we can see $u_\infty \leqslant v_\infty$ in the regular part and hence for the whole of X by plurisubharmonicity. Notice if we have $\omega_m = \phi_m$, we can also get $u_\infty \geqslant v_\infty$ by symmetry and hence conclude that they are the same. Otherwise, we still have to use the integral equality to draw the conclusion.

For the rest part, let's sketch some results from pluripotential theory related to the discussion above. This can be seen as a warm-up exercise for going into the next

[63] Everything is now restricted to $\{|\sigma| > \delta\}$, i.e., always considering the intersection with $\{|\sigma| > \delta\}$ for each set appearing below.

part of this work, but somehow the discussion below has little relation with what's used there.

Recall that we have not yet seen u_∞, satisfying $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{u_\infty}\Omega$ out of the stable base locus set of K_X, is bounded. Thus though we have it is globally plurisubharmonic, i.e., $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$ is a real positive closed $(1,1)$-current, it's not justified in the usual sense to say $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n$ is a global (Borel) measure over X. But it is quite obvious that $e^{u_\infty}\Omega$ defines a global measure over X. So we are very willing to give the left hand side the similar meaning. And indeed this case has already been considered in pluripotential theory where what people require is a nice approximation of the possibly unbounded plurisubharmonic function.

Though in the place where I learned this theory ([Koj2]), domains in C^n are the main object, the spirit can easily been translated to our situation. And the requirement for the nice approximation can be satisfied by our approximation from flow or perturbation. Basically, we could take the limit of the measures in the sense of measure from the approximation. We can also see the limiting measure will not depend on the nice approximation chosen which of course is of great importance to define a measure for such an unbounded plurisubharmonic function. Obviously the right hand side as a measure is also the limit of the approximation measures. So after making sense of both sides as measure over X, we should be able to see that they are actually the same. Furthermore, we also have comparison principle [64] for those unbounded functions in general. From this we can have the uniqueness of the (possibly unbounded but plurisubharmonic with a nice approximation) solution u for $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$ where the equality is in the sense of measure. At least this uniqueness result looks more general than what's proved before. But essentially we have to require a nice approximation, and at this moment we can only get one by the (continuity) methods discussed before.

We sketch the related argument as follows. Basically it's quoted from [Koj2] where it is contributed to [Ce]. Local picture (of a domain V in $\mathbb{C}^n$) is considered in those works. We'll deal with the closed manifold case instead. Up to now, what we have achieved is less than that of the local case, but it is still enough for the application. Anyway, the argument below is essentially theirs after some modification. [65]

Suppose $u \in PSH_{\omega_\infty}(X)$ is the (pointwise) limit of a decreasing sequence of functions, $u_j \in PSH_{\omega_j}(X) \cap C^0(X)$ such that $u_j \leqslant 0$ and $\int_X (-u_j)^p(\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n <$

[64] The integral inequality for u_∞ and v_∞ proved before which is of course just a special case. We'll discuss it more later.

[65] Professor Kolodziej informed me that Zeriahi and others had also treated the closed manifold situation for this theory.

C for some $p \geqslant 1$ and any j. Here assume $\omega_j \to \omega_\infty$ as $j \to \infty$ in a nice (linear for example) way and they are all semi-positive [66].

Then we want to prove that $(\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n$ is weakly convergent to a (Borel) measure $d\mu$ which is independent on the choice of u_j as above. So we can define $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = d\mu$. Clearly, the global intergral of the measure would be $\int_X {\omega_\infty}^n$ by the weak convergence. In the following, we justify this definition.

Let ϕ be a smooth funtion on X (i.e., a test function). Define $v_{[k} := max\{v, -k\}$ for any $k > 0$ [67]. Of course $u_j = u_{j[k}$ on $\{u_j \geqslant -k\}$ which is a closed set and $(\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n = (\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n$ on $\{u_j > -k\}$ which is open [68]. Now we can have the following computation:

$$
\begin{aligned}
&|\int_X \phi\big((\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n - (\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n\big)| \\
\leqslant &\int_{\{u_j \leqslant -k\}} \phi\big((\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n + (\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n\big) \\
= &\, k^{-p} \int_{\{u_j \leqslant -k\}} k^p \phi\big((\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n + (\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n\big) \\
\leqslant &\, C_\phi k^{-p}\big(\int_X (-u_j)^p (\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n + \int_X (-u_{j[k)})^p (\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n\big).
\end{aligned}
$$

In the last expression, the first term in the bracket is uniformly controlled by assumption. If we can also do that for the second term, then we arrive at

$$
|\int_X \phi\big((\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n - (\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n\big)| \leqslant C_\phi k^{-p}.
$$

Let's first show how this is going to give us the unique limit. We know the measures $(\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j[k)})^n$ converge weakly to the measure $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_{[k)})^n$ as $j \to \infty$ since the potentials are (uniformly) bounded after truncation. [69] Then it's easy to see the Cauchy property for the sequence of integrals $\int_X \phi(\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n$. Thus we can have the weak convergence of $(\omega_j + \sqrt{-1}\partial\bar{\partial}u_j)^n$ to some (positive Borel) measure $d\mu$. We also want this measure $d\mu$ to be independent on the choice of $\{u_j\}$.

[66]This semi-positivity seems to be involved for all the argument using pluripotential theory. At many places, we can trivially reduce the general case to it. However we really seems to need this assumption here. So this discussion is for a more restrictive situation in comparison to the earlier part of this section, but we have less assumption for u_j's.

[67]In the case when the background form is semi-positive, this would preserve the plurisubharmonicity of the functions since the constant function is plurisubharmonic itself.

[68]This seems to be the only place where the continuity of the approximation functions is needed.

[69]This result will be discussed later. The nice convergence of background forms ω_j makes it legal to use the classic results for domains in $\mathbb{C}^n$ by chooosing proper local potentials.

This is also quite easy from the inequality above. Taking $j \to \infty$ in the inequality, by the weak convergences already got, we conclude that

$$|\int_X \phi\big(d\mu - (\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_{[k]})^n\big)| \leqslant C_\phi k^{-p}$$

which obviously guarantees the uniqueness of such a measure $d\mu$.

Now let's justify that inequality. As mentioned before, it'll be done if we can bound the terms containing $u_{j[k}$ uniformly. Of course an obvious idea would be to use the corresponding term with u_j to control these terms. More precisely, we'll prove the following. Notice u_j and $u_{j[k}$ are bounded functions.

Claim: For any $0 \geqslant v \geqslant u$ with $v, u \in PSH_\omega(X) \cap L^\infty(X)$ with $\omega \geqslant 0$, with $p \geqslant 1$ we have

$$\int_X (-v)^p(\omega + \sqrt{-1}\partial\bar{\partial}v)^n \leqslant C \int_X (-u)^p(\omega + \sqrt{-1}\partial\bar{\partial}u)^n$$

for some universal positive constant C.

Proof. [70] Set $I_m = \int_X (-u)^p {\omega_u}^m \wedge {\omega_v}^{n-m}$ for $m = 0, \cdots, n$ where the natural notations, $\omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u$ and also ω_v, are used. Rewrite I_m as following when $m < n$:

$$I_m = \int_X (-u)^p {\omega_u}^m \wedge {\omega_v}^{n-m-1} \wedge \omega + \int_X (-u)^p {\omega_u}^m \wedge {\omega_v}^{n-m-1} \wedge \sqrt{-1}\partial\bar{\partial}v.$$

Now let's deal with the first term by the following computation:

$$\begin{aligned}
&\int_X (-u)^p {\omega_u}^m \wedge {\omega_v}^{n-m-1} \wedge \omega \\
&= \int_X (-u)^p {\omega_u}^{m+1} \wedge {\omega_v}^{n-m-1} - \int_X (-u)^p (\sqrt{-1}\partial\bar{\partial}u) \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
&= I_{m+1} - \int_X \sqrt{-1}\partial\big((-u)^p \bar{\partial}u \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1}\big) \\
&\quad + \int_X \sqrt{-1}\partial(-u)^p \wedge \bar{\partial}u \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
&= I_{m+1} - p\int_X (-u)^{p-1}\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
&\leqslant I_{m+1}.
\end{aligned}$$

[70]Since these functions are not smooth, many expressions below are not understood in the classic sense. But the meaning is very natural from pluripotential theory using the boundedness of the functions. Some terms below are indeed defined by the equation used. It would be very tedious to make everything down to the ground at this moment. Basically, let's keep in mind that approximation argument can be used to justify those bad terms, and so we can actually treat them as if they are smooth. More discussions about these terms can be found in Appendix.

For the second term, we have the following:

$$
\begin{aligned}
&\int_X (-u)^p(\sqrt{-1}\partial\bar{\partial}v) \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
=& \int_X v\big(\sqrt{-1}\partial\bar{\partial}(-u)^p\big) \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
=& \, p(p-1)\int_X v(-u)^{p-2}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
& - p\int_X v(-u)^{p-1}(\sqrt{-1}\partial\bar{\partial}u) \wedge {\omega_u}^m \wedge {\omega_v}^{n-m-1} \\
\leqslant& \, p\int_X (-v)(-u)^{p-1}{\omega_u}^{m+1} \wedge {\omega_v}^{n-m-1} \\
\leqslant& \, pI_{m+1}.
\end{aligned}
$$

In the above steps, the facts $\sqrt{-1}\partial u \wedge \bar{\partial} u \geqslant 0$, $\omega \geqslant 0$ and $-u \geqslant -v \geqslant 0$ are used. And when $p = 1$, it's actually easier. Thus we have $I_m \leqslant (p+1)I_{m+1}$ which gives $I_m \leqslant C_m I_n$. Hence we have $\int_X (-v)^p {\omega_v}^n \leqslant \int_X (-u)^p {\omega_v}^n = I_0 \leqslant C\int_X (-u)^p {\omega_u}^n$ which is exactly our goal.

□

In fact, in the computation above, we can switch the role of v and u. The computation would be more like the original computation for domains in $\mathbb{C}^n$. Let's do it below. Set $J_m = \int_X (-v)^p {\omega_v}^m \wedge {\omega_u}^{n-m}$. Then for $m = 0, \cdots, n-1$, we can have:

$$
\begin{aligned}
& J_m - J_{m+1} \\
=& \int_X (-v)^p\big(\sqrt{-1}\partial\bar{\partial}(u-v)\big){\omega_v}^m \wedge {\omega_u}^{n-m-1} \\
=& \int_X (u-v)\big(\sqrt{-1}\partial\bar{\partial}(-v)^p\big){\omega_v}^m \wedge {\omega_u}^{n-m-1} \\
=& \, p(p-1)\int_X (u-v)(-v)^{p-2}\big(\sqrt{-1}\partial v \wedge \bar{\partial} v\big) \wedge {\omega_v}^m \wedge {\omega_u}^{n-m-1} \\
& - p\int_X (u-v)(-v)^{p-1}\big(\sqrt{-1}\partial\bar{\partial}v\big) \wedge {\omega_v}^m \wedge {\omega_u}^{n-m-1} \\
\leqslant& \, p\int_X (v-u)(-v)^{p-1}\big(\sqrt{-1}\partial\bar{\partial}v\big) \wedge {\omega_v}^m \wedge {\omega_u}^{n-m-1} \\
\leqslant& \, p\int_X (v-u)(-v)^{p-1}{\omega_v}^{m+1} \wedge {\omega_u}^{n-m-1} \\
\leqslant& \, p\int_X (-u)(-v)^{p-1}{\omega_v}^{m+1} \wedge {\omega_u}^{n-m-1} \\
\leqslant& \, p{I_{n-m-1}}^{\frac{1}{p}}{J_{m+1}}^{\frac{p-1}{p}},
\end{aligned}
$$

where the last step is by Hölder inequality for the measure ${\omega_v}^{m+1} \wedge {\omega_u}^{n-m-1}$. Thus we arrive at:

$$J_m \leqslant p({I_{n-m-1}}^{\frac{1}{p}} + {J_{m+1}}^{\frac{1}{p}}){J_{m+1}}^{\frac{p-1}{p}} \leqslant 2p{I_{n-m-1}}^{\frac{1}{p}}{J_{m+1}}^{\frac{p-1}{p}}.$$

Combining with the result above, we can see all the terms I_m and J_m are bounded by I_n.

Remark 2.6.1. *This control is not as good as the original one for the domain in $\mathbb{C}^n$. For example, it won't give the convexity of the set of functions which have this kind of approximation. More specifically, we don't know the summation of two such functions still has such an approximation unless these two functions are comparable.*

In order to see that comparison principle still works for the measure defined like this, by the quasicontinuity of these functions, we only need to see the measure ${\omega_{u_j}}^n$ for any small relative capacity set is uniformly small for all j where $\omega_{u_j} = \omega_j + \sqrt{-1}\partial\bar{\partial}u_j$. This convention has been used before, but here the background forms are changing. The relative capacity is defined locally using the classic definition for domains in $\mathbb{C}^n$. [71] In fact it is easy to be seen as follows.

For some $\epsilon > 0$, suppose O is a subset of X with $Cap(O) \leqslant \epsilon$. Then for any $C_0 > 0$,

$$\begin{aligned}\int_O {\omega_{u_j}}^n &= \int_{O\cap\{u_j \geqslant -C_0\}} {\omega_{u_j}}^n + \int_{O\cap\{u_j < -C_0\}} {\omega_{u_j}}^n \\ &\leqslant {C_0}^n\epsilon + \int_X {C_0}^{-p}(-u_j)^p {\omega_{u_j}}^n \\ &\leqslant {C_0}^n\epsilon + C\cdot {C_0}^{-p}.\end{aligned}$$

By taking C_0 large enough and ϵ small enough, we can guarantee $\int_O {\omega_{u_j}}^n$ to be uniformly small.

Another issue for justifying comparison principle for two of these functions (with the same $\omega_\infty \geqslant 0$ of course) is about the changing background forms for the approximation. Basically, we'll need the forms to be the same for each pair of approximation functions or at least have the one-sided relation in the favorable direction. However

[71]One can also define it globally using some background form. It'll be equivalent to the local definition if the background metric is positive. Since our background forms ω_j are not uniformly positive, the globally defined ones won't be uniform. But this is not needed for us here. Indeed, we only need the uniform control of globally defined capacity by the locally defined one which is rather obvious.

this would not be a problem if the ω_j's are like $\omega_\infty + \delta\omega$ for $\omega > 0$ since we can take proper subsequences of the approximations. There would be no crue left after taking the limit. And we can treat the sets $\{v > u\}$ and $\{u > v\}$ respectively where u and v are the functions to compare. In fact for our application which is to prove the uniqueness of such a (possibly unbounded) solution for the equation

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

where Ω is a smooth nondegenerate (nowhere 0) volume form. It is enough to see $\{v > u\}$ is empty since $u \geqslant v$ together with the global integral equality would tell $u = v$.

Remark 2.6.2. *Here the main difficulty is coming from the possible unboundedness (and also discontinuity) of the limiting functions which would give some trouble in making sense of the limiting distributions as measure and applying the usual comparison principle. But as mentioned before, it might be the case that the limit is bounded (even with certain Hölder continuity). This problem will be heavily considered later.*

The discussion in this section says we can actually apply comparison principle for (apriori) unbounded functions like that. Indeed we can see that continuous (or even just bounded) functions which are plurisubharmonic with respect to some $\omega_\infty \geqslant 0$ are also considered in the above discussion [72], which is clearly consistent with the classic discuss for them.

Thus if by any means, we have a continuous (or even just bounded) solution for this equation, then we know it is also the solution got before by continuity methods. This actually can be used to identify the orbifold Kähler-Einstein metric and the (singular) metric got before for any minimal surface of general type as mentioned in [TiZh].

[72]For a continuous function, the approximation could consist of just the function itself which clearly has the proper uniform bound for the global integrals even if one uses some more positive background form $\omega_\infty + \omega$ in sight of the boundedness of the function itself. In fact, the current discussion works for bounded functions. The corresponding approximation result will be discussed later in this work. The background form can be more positive than ω_∞ but converges to it nicely.

Chapter 3

Kolodziej's Argument and Direct Application

This chapter is devoted to sketch the argument in [Koj1] and [Koj2] by S. Kolodziej. The argument is quite original and fairly different from classic PDE methods for Monge-Ampere equation. More importantly, it provides important information about the solution in some situation where other methods fail to help (at least as we see it at this moment). Let's first introduce the classic situation considered in his works.

Over a closed Kähler mainfold X, the equation considered is:

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = F\omega^n,$$

with ω being a Kähler metric and nonnegative $F \in L^p(X)$ for some $p > 1$ satisfying $\int_X F\omega^n = \int_X \omega^n$. [1] Basically, the existence of a (weak) solution in $PSH_\omega(X) \cap L^\infty(X)$, which is also continous and unique, is proved.

Remark 3.0.3. *Kolodziej's original argument applies for F in more general class of functions. But the above restricted version will be enough for our main interest. And at certain places, it'll allow us to simplify the argument to make it easier to generalize the argument to our situation. The affect will be emphasized when it occurs.*

3.1 Classic Results in Pluripotential Theory

In this section, we'll introduce some of the necessary notions and related results in classic pluripotential theory used in Kolodziej's argument. The discussion in this

[1] Whenever we use the notation $L^p(X)$, a smooth nondegenerated volume form is assumed while the choice clearly doesn't have any essential affect.

section is supposed to be very concise. But sometimes we may also go into details, especially when it is fairly related our generalization later.

It seems necessary to begin with the definition of plurisubharmonic functions. We use $PSH(V)$ to stand for the class of plurisubharmonic functions over V which is an open set in $\mathbb{C}^n$. Instead of writing down one definition (as in [Le] for example), it might be better to just illustrate what are those functions.

Basically, they are functions which will be subharmonic when restricted to any complex direction, in other words, $\sqrt{-1}\partial\bar{\partial}u$ will be a positive $(1,1)$-current. The meaning is quite clear in smooth case. Generally speaking, it's in the sense of distribution. In the following, we list some basic features of these functions [2]:

i) They are L^1_{loc} functions, but they are not equivalent classes of functions, i.e., the value for each point is decided. They can take the value $-\infty$, but not $+\infty$;

ii) They have the same mean value property as subharmonic functions. In fact, since plurisubharmonic functions are subharmonic (i.e., with positive distributional Laplacian), so this is just inherited from subharmonic functions. Moreover, we have maximum principle for them just as for subharmonic functions;

iii) They are upper-semicontinuous and Borel measurable. Furthermore, they are also essentially upper-semicontinuous, i.e., for $f \in PSH(V)$, $\forall x \in V$, we have $lim_{i\to\infty} esssup_{U_i} f = f(x)$ where $\{U_i\}$ is a basis of neighbourhoods for x. Thus the values for all points in X are decided by values almost everywhere, namely, one can ignore values over any measure 0 set; [3]

ix) Their restrictions out of small open sets are continuous. The "small" here means with relative capacity as small as one wants where relative capacity is an important notion in this business which will be discussed in great details later;

x) Their convolutions are smooth plurisubharmonic functions and the sequences of functions coming from convolution will decrease to themselves pointwisely. [4]

The last one is frequently used to prove anything local about plurisubharmonic functions because the smooth ones are usually easy to deal with directly and then

[2]Some of them come directly from definition, but some are not so trivial results by themselves.

[3]This is a better restriction for the value at each point then the usual upper-semicontinuity. It can be used to understand the first part of property i) and is indeed quite useful for us later. The measure 0 set is usually a subvariety for our application.

[4]Of course, the convolution is the classic one with some compactly supported and rotationally symmetric smooth function, ρ, with normalized total integral. The sequence is from rescaling this function ρ by proper constants for the value and variable which maintains the total integral. The set, where the convolution can be defined, is relatively compact in V but exhausts the set for a proper sequence of rescaling constants.

by taking a limit for the result with respect to the convolutions, we can hopefully get the result for the original function. Moreover, combining x) with ix) and Dini's Theorem, we can see the convergence is really not so bad.

A trivial observation would be that from definition, we can define plurisubharmonic functions over a domain with a (holomorphic) complex structure exactly in the same manner as before. All the local properties above will remain unaffected. But now we have to worry about the existence of convolution since there might be no global coordinates. And in fact, this is the main difficulty arising when the domain V is no longer in $\mathbb{C}^n$ and would be a huge cloud over the simple-mindedly straightforward generalization of Kolodziej's orginal argument.

In the following, we put together some results in classic pluripotential theory. There is no logic order guarding the list of results. We still mainly consider domains in $\mathbb{C}^n$ for these results.

Remark 3.1.1. *The domain V is usually assumed to be connected. It'll make our picture simpler. But it is easy to see that the discussion below still applies to the case when there are multiple (even countably many) connected components for V. Basically we can just consider each component separately and "sum up" the results.*

There could be some exceptions since some results are hard to take sum, for example, boundedness of the functions would be for each components. So sometimes we'll need a uniform bound (of the sizes, for example) for all components in order to get the result for the whole domain V, which should be easy to see from the context.

The case when V is a connected and unbounded open set in $\mathbb{C}^n$ [5] will also be of some interest. And we do not require V to be bounded for everything below unless explicitly stated.

It's easy to see that a plurisubharmonic function over $\mathbb{C}^n$ bounded from above will have to be trivial, i.e., a finite constant by considering the extension on each complex direction to the point ∞ [6]. But that's not the case for a general unbounded domain V. For example, when $n = 1$, the upper half plane is biholomorphic to the unit disk, so it can have a lot of nontrivial plurisubharmonic functions which are not bounded from above. Anyway, this is not so related to our main consideration here, but still of quite some interest by itself.

a) Comparison Principle

[5] The meaning of "unbounded" clearly needs some ambient space which is $\mathbb{C}^n$ for now and can be other ones with similar properties.

[6] Similar argument will be used later in another context.

Proposition 3.1.2. *For a bounded domain V in $\mathbb{C}^n$, $\forall u, v \in PSH(V) \cap L^\infty(V)$, if $\forall p \in \partial V$, $\underline{\lim}_{x \to p}(u - v)(x) \geqslant 0$, then*

$$\int_{\{u<v\}} (\sqrt{-1}\partial\bar{\partial}v)^n \leqslant \int_{\{u<v\}} (\sqrt{-1}\partial\bar{\partial}u)^n.$$

Illustration of the proof:

First, let's point out that under the assumption of this proposition, $(\sqrt{-1}\partial\bar{\partial}u)^n$ can be defined as a Borel measure. [7] So the meaning of the integration in the conclusion is clear. Notice that the integrals on both sides may be $+\infty$, but $\leqslant$ would still be true in natural sense.

The case when everything is smooth and the set integrated over is relative compact in V is quite clear from Stokes' Theorem (as V is bounded). "Everything" means the functions and the boundary of the set $\{u < v\}$.

For the general case as assumed in the proposition, we can use smooth approximation to do the job. Here since we are in $\mathbb{C}^n$, convolution will provide a nice approximation sequence of functions. And during the process, we also need to make sure the situation for "small" sets (as appeared in property ix) of plurisubharmonic functions) is under control, but the boundedness of the functions can provide enough help.

There are quite some technical details for a rigorous proof (see in [BeTa]). For example, the convolution will only be defined for sightly smaller sets which are relatively compact in V, but since the comparison relation near the boundary is also available from the assumptions, we'll be fine by using some more approximation arguments.

Remark 3.1.3. *Comparison principle may be the most important tool for classic pluripotential theory. The result is quite natural if one takes the magnitude of positivity of $(\sqrt{-1}\partial\bar{\partial}u)^n$ as some kind of "convexity" of the plurisubharmonic function u.*

Anyway, the justification and application of comparison principle in generalized situation will be one of our main topics later.

b) Relative Capacity and Relative Extremal Function

• Relative Capacity

Consider any compact subset K in V and define the relative capacity of K with

[7] There would be more discussion about this later. It actually is a big issue and we just feel it might be too early to bring it up at this time since the picture here is quite natural if one only considers smooth functions.

respect to V as follows:

$$Cap(K, V) = sup\{\int_K (\sqrt{-1}\partial\bar{\partial}u)^n | u \in PSH(V), \ -1 \leqslant u \leqslant 0\}.$$

Then for any subset E of V, one defines the corresponding capacity as:

$$Cap(E, V) := sup\{Cap(K, V) | K \ compact, K \subset E\}.$$

It might look favorable if we can use the definition in case of compact sets for any subset. And it's easy to see that will depend on how well we can use compact sets to exhaust any set. In fact, we can do that for Borel sets, and so the definition for compact sets can be used for Borel sets (or say these two definitions are equivalent for Borel sets). But for general Lebesgue measurable sets, the exhaustion will be up to a (Lebesgue) measure 0 set and the integration of $(\sqrt{-1}\partial\bar{\partial}u)^n$ for some $u \in PSH(V) \cap L^\infty(V)$ over a measure 0 set might be positive as we can see later by example.

Anyway the definition here (for any subset) makes it only necessary to only consider compact sets in most cases which is actually very convenient for us.

• Relative Extremal Function:

For E any subset of V, we define relative extremal function of E with respect to V as follows:

$$u_{E,V} = u_E := \big(sup\{u | u \in PSH(V), \ u \leqslant 0 \ on \ V, u \leqslant -1 \ on \ E\}\big)^*.$$

Here the "*sup*" is taken pointwisely and upper semi-continuization (the upper "$*$") is also used to make sure the function we finally got is still plurisubharmonic (see [De1] for example).

Notice that upper semi-continuization is used in the definition, and so it is not true that $u_E = -1$ on E, which is obviously the case without "$*$" in the definition. This makes the actual values of u_E on $E \setminus \mathring{E}$ hard to describe. But there is a general result for this situation as follows.

Lemma 3.1.4. *For a family of plurisubharmonic functions which are locally uniformly bounded from above, the pointwise supremum function, u, which may not be plurisubharmonic, will have its upper semi-continuization, u^*, plurisubharmonic. Moreover the set $\{u^* > u\}$ is pluripolar.*

We don't want to get into too much detail about this and just want to mention that "pluripolar" means the set would have (Lebesgue) measure, relative capacity and outer relative capacity 0 which essentially allows us to ignore this set when doing integration. This handy fact actually shows up in a lot of places, for example, the proof of a very classic fact about relative capacity in [AlTa] whose generalization is quite important for the proof of our essential estimate later which can be seen as the punchline for our argument.

For many purposes as will become clear, it would be great to see that it is equivalent to use the function class "$PSH(V) \cap C^0(\bar{V})$" instead of "$PSH(V)$" in the definitions. And in fact this is the case for the definition of relative extremal functions for compact sets when V is hyperconvex whose definition will be given explicitly soon. Basically it means there is a nice exhaustion function defined in a neighbourhood of $\bar{V}$. The argument is essentially by approximation argument which is under frequent use for this business and so we sketch it below. [8]

Essentially, one uses convolution to get a continuous (smooth in fact) approximation of any element, u, in $PSH(V)$ which has proper values as in the definition of u_E, and then Dini's Theorem can be used to describe the approximation more carefully. The condition on V is used to extend any plurisuharmonic function to a neighbourhood of $\bar{V}$, which would make the convolutions defined on $\bar{V}$, as follows.

First we can assume u is valued in $[-1, 0]$ over V and equal to -1 over E since it is justified to take supremum only over functions like $max\{u, -1\}$. It is even enough to consider functions like $max\{u, h\}$, where h is the defining function for hyperconvexity of V [9], in the original definition of relative extremal function, u_E.

Now consider $max\{u, h + \epsilon\}$ for $\epsilon > 0$. These functions can obviously be extended by $h + \epsilon$ to a neighbourhood of $\bar{V}$ where h is defined, which then could have convolutions defined over V. And as $\epsilon \to 0$, they decreases to $max\{u, h\}$. By diagonalizing argument, we can have a sequence of smooth plurisubharmonic functions defined on $\bar{V}$ and decreasing to $max\{u, h\}$. They valued in $[-1, \delta]$ where δ decreases to 0 along the sequence.

The compactness assumption of E is for concern of applying Dini's Theorem. Notice that $u = 0$ on $\bar{V}$ and $u = -1$ on E which of course means it is continuous over $\bar{V} \cup E$. So by Dini's Theorem, the values of the approximation functions are controlled well over $\bar{V} \cup E$. Now we can apply an elementary property of plurisubharmonic func-

[8] It would be a slightly different issue for relative capacity which will be discussed in details later.

[9] By taking multiple if necessary, we can make sure the h here has very negative values (say < -2) over the set E.

tions, maximum principle, which is mentioned before, to see that the approximation functions take almost the proper values over V as required in the definition of u_E. Then it's routine to see the relatively extremal function defined using only continuous plurisubharmonic function, which will be smaller than the original one, u_E apriori, is actually equal to u_E.

Of course just from the definitions, we can get some elementary properties for relative capacities and relative extremal functions, for instance, the monotonicity when the corresponding sets (V and E) become larger or smaller. It's quite obvious and so we even omit the statements.

Also when $V \subset \mathbb{C}^n$ is bounded, it's easy to see by definition that

$$\lambda(K) \leqslant C \cdot Cap(K, V)$$

where λ is the standard Lesbegue measure from the standard Euclidean potential, K is a compact subset and C is a positive constant depending on the size of V. In fact by the discussion before, we can see that it is also the case for any subset E since the affect of ignoring measure 0 sets contributes in the favorable direction. This simple-minded observation, in some sense, can be taken as a guide for us. Stronger form of it would play an essential role later in our generalization.

Finally, let's point out a nice relation between these two notions which is at least beautiful in its own way.

Proposition 3.1.5. *Assume V hyperconvex (with meaning as before), then for any compact set K in V, we have:*

$$Cap(K, V) = \int_V (\sqrt{-1}\partial\bar{\partial}u_K)^n$$

and in fact the measure $(\sqrt{-1}\partial\bar{\partial}u_K)^n$ is supported on ∂K.

Idea of the proof:

It is actually natural and important to realize that the support of the measure $(\sqrt{-1}\partial\bar{\partial}u_K)^n$ is ∂K at first. For the interior of K, easy to see the measure vanishes there since u_K is identically equal to -1 by definition. For the complement of K, we can use the idea of pluriharmonic lifting which is exactly in the same spirit as Perron method for constructing harmonic functions, i.e., locally (in a ball contained in the complement, for example) lifting the continuous plurisubharmonic function started with to be the pluriharmonic one with the same boundary data. Here we'll need hyperconvexity for V as continuous functions are used instead of general ones which is justified before.

Remark 3.1.6. *For a while, it looks like we can weaken the meaning of hyperconvex here to require the defining function only defined over $\bar{V}$ by using an exhaustion sequence of open sets $\{V_\epsilon\}$ for V coming from the defining function and getting the convergence of u_{K,V_ϵ} to $u_{K,V}$ which would give the result for V from those for V_ϵ's. Notice obviously that V_ϵ's are hyperconvex as described before.*

The problem is in the attempt to get the convergence mentioned above. Of course the (decreasing) limit of $\{u_{K,V_\epsilon}\}$ is a nonpositive plurisubharmonic function defined on V which is no less than $u_{K,V}$ since each of them is no less than $u_{K,V}$. But on the other hand, we don't have that the limit is $\leqslant -1$ on K since u_{K,V_ϵ} may not be smaller than or equal to -1 there because in the definition of relative extremal function, upper semi-continuization is applied for the "sup" [10]. *Thus it is not clear (at least by definition) that the limit will be (smaller than or) equal to $u_{K,V}$.*

Then we basically just need to compare the integration of $(\sqrt{-1}\partial\bar{\partial}u_K)^n$ on K with that of any other function which we can try in the definition of $Cap(K,V)$ and see it is not smaller. After proving that, since we can also try u_K itself in the definition of $Cap(K,V)$, it is done. Here a good point about compact sets is that in the definition of relative capacity, we use $\int_K$ in the definition of relative capacity and ∂K is contained in K which is the support of $(\sqrt{-1}\partial\bar{\partial}u_K)^n$.

From the hyperconvexity of V, we know that u_K is definitely close to 0 near ∂V. Now there is just a little technicality left to cook up the proper function to compare with u_K by Stokes' Theorem.

Remark 3.1.7. *In fact, after getting this equality for K compact, we can get for some other sets by using approximation argument. For example, we can get for any open set from the following simple argument.*

A bounded open set, U, can be exhausted by a sequence of closed sets, $\{K_\epsilon\}$ and a sequence of open sets, $\{U_\epsilon\}$, contained in the closed sets, i.e. $U_\epsilon \subset K_\epsilon$, respectively by considering the distance to the boundary.

Then one can see the decreasing convergence of the relatively extremal functions u_{K_ϵ} to u_U [11]*, thus we have the convergence of currents from the result in c) below.*

[10] Of course the limit "$\leqslant -1$" on $\mathring{K}$, but K is compact, so $\mathring{K} \neq K$.

[11] It is easy to see the limit is greater or equal to u_U, and we get the other direction by using the definition of u_U, noticing the exhaustion by open sets makes sure the limit is -1 on U.

Now we see $(\sqrt{-1}\partial\bar{\partial}u_{_U})^n$ *supported on* $\bar{U}$ *(*∂U *in fact). Finally the result is from the earlier definition of* $Cap(U,V)$ *using "sup" by noticing it'll just be the increasing limit of* $Cap(K_\epsilon, V)$*.*

Actually we can also define something called "outer capacity" for any set by taking infimum of the relative capacities of open sets containing the set being considered. The similar equality as above would hold for any general relative compact subset in V. We sketch the argument as follows. First one gets for compact sets where this new capacity turns out to be the same as the old one. Then it's basically left to use the definition to deduce for general sets relatively compact in V (see [Koj2] for more details).

We would like to point out that for our argument, usually this relation above between relative capacity and relative extremal function is not that needed. We just use it to make the picture more clear in some cases. But the result itself is fairly nice and illuminating. And more importantly, we have a natural observation from the above proof of this relation (for the original relative extremal capacity) as follows.

Recall in the first part of the proof, continuous plurisuharmonic functions are so easy to work with. But because we use general plurisubharmonic functions in the definition of $u_{_E}$, hyperconvexity has to be imposed on the domain V and E has to be compact in order to justify the use of continuous plurisubharmonic functions instead of general ones.

It really seems like that we are giving ourselves unnecessary trouble by defining $u_{_E}$ like that. In fact let's try the following definition instead:

$$u^c_{_{E,V}} = u^c_{_E} := \big(sup\{u | u \in PSH(V) \cap C^0(\bar{V}), u \leqslant 0 \text{ } on \text{ } V, u \leqslant -1 \text{ } on \text{ } E\}\big)^*.$$

Now we see $(\sqrt{-1}\partial\bar{\partial}u^c_{_E})^n$ is supported on ∂E. The proof is just like before, but now we can use continuous plurisubharmonic functions by definition. And we have this for any relatively compact subset E in V. [12]

But for the whole result before, we still need V to be hyperconvex and E to be compact since we want to compare the integrations by using the function $u^c_{_E}$ and other properly modified functions. Notice we do not change the definition of relative capacity accordingly.

In fact, this observation would make our life much easier later when more general

[12]The relative compactness is from the consideration that $(\sqrt{-1}\partial\bar{\partial}u)^n$ is a current on V anyway, so we want to have something really inside V which is important for us to apply Stokes' Theorem at certain stage. In fact if E is just any subset of V, we still have that the support is on $\partial E \cap V$ since that part of the proof still goes through.

domain V is considered. And because this new definition is equivalent to the one before in many classic cases, it will also give us no trouble in applying the classic results.

c) Weak Convergence

Proposition 3.1.8. *i) Let* $\{u_k^j\}_{j=1}^{\infty}$ *be a uniformly bounded sequence of plurisubharmonic functions in* V *for* $k = 1, \cdots, m$*, and* $u_k^j \to u_k \in PSH(V) \cap L^{\infty}(V)$ *with respect to capacity for each* k*. Then we have:*

$$(\sqrt{-1}\partial\bar{\partial}u_1^j) \wedge \cdots \wedge (\sqrt{-1}\partial\bar{\partial}u_m^j) \to (\sqrt{-1}\partial\bar{\partial}u_1) \wedge \cdots (\sqrt{-1}\partial\bar{\partial}u_m)$$

in the sense of distribution (i.e., weakly). And more generally, we have

$$(\sqrt{-1}\partial\bar{\partial}u_1^j) \wedge \cdots \wedge (\sqrt{-1}\partial\bar{\partial}u_m^j) \wedge T \to (\sqrt{-1}\partial\bar{\partial}u_1) \wedge \cdots (\sqrt{-1}\partial\bar{\partial}u_m) \wedge T$$

in the sense of distribution where T *is any closed positive* (l, l)*-current;*

ii) If an everywhere decreasing (increasing) sequence $\{u_j\}_{j=1}^{\infty}$ *with* $u_j \in PSH(V) \cap L^{\infty}(V)$ *decreases (or increases) to some* $u \in PSH(V) \cap L^{\infty}(V)$ *almost everywhere, then the convergence is actually with respect to capacity. Moreover, using the setting in i), if the convergences are monotonous (decreasing or increasing), we also have*

$$u_1^j(\sqrt{-1}\partial\bar{\partial}u_2^j) \wedge \cdots \wedge (\sqrt{-1}\partial\bar{\partial}u_m^j) \wedge T \to u_1(\sqrt{-1}\partial\bar{\partial}u_2) \wedge \cdots (\sqrt{-1}\partial\bar{\partial}u_m) \wedge T$$

weakly where T *is as above.*

Here $u_j \to u$ as $j \to \infty$ over V with respect to (relative) capacity means for any compact set K in V and $\epsilon > 0$, we have

$$lim_{j\to\infty}Cap(K \cap \{|u_j - u| > \epsilon\}, V) = 0.$$

Remark 3.1.9. *The sequence in ii) is obviously (globally) uniformly bounded, so we could use the convergence results in i). And in fact, since the conclusion in i) is rather local and the decreasing (increasing) convergence is also a local property, we can assume the functions are in* $L_{loc}^{\infty}(V)$ *instead of* $L^{\infty}(V)$ *for ii).*

Also notice that for ii) we only need the convergence to be almost everywhere, [13] *but for the decreasing case, this would imply the convergence everywhere by a fundamental property of plurisubharmonic functions, and for the increasing case,* u *would be the upper semi-continuization of the pointwise limiting function.*

[13] The monotonicity of the sequence is required everywhere.

Another trivial point from the properties of plurisubharmonic functions would be that any function in $PSH(V) \cap L^\infty(V)$ would be bounded everywhere by the L^∞-norm simply by semi-continuity.

Finally, let's point out that since all the currents involved are positive [14], so the convergence in the sense of current would imply the convergence in the weak topology of measure, i.e., we can use compactly supported continuous functions instead of smooth ones to test the convergence.

We do not require the continuity of all these functions and that's somehow the interesting thing about this result. In fact, if we require the continuity for all of them, then uniform convergence of the potentials can also imply the convergence of the currents in the weak sense (as in [BeTa] for example). In the following, we say something about the proof.

First, notice the convergence with respect to capacity is defined in the same spirit as convergence with respect to measure. But it is more like a local property as there is a compact set K involved.

The detail of the proof is more or less technical in a standard way. We have to be careful about the meaning of all the currents, as well as all the previous places where terms like $(\sqrt{-1}\partial\bar{\partial}u)^n$ appeared. We feel it might be a good place to talk about this a little bit.

Generally speaking, there is no wedge product of currents. Even for positive currents, we still need to be very careful about taking wedge product. Everything starts with the following definition

$$\sqrt{-1}\partial\bar{\partial}u \wedge T := \sqrt{-1}\partial\bar{\partial}(uT)$$

for $u \in PSH(V) \cap L^\infty_{loc}(V)$ and T a positive current. The right hand side actually makes sense because uT is also a current from the fact that T is positive and u is a locally bounded Borel measurable function. And of course the operator $\partial\bar{\partial}$ acts on it in the sense of current (distribution). If we further assume that T is closed, then it can also be seen that the right hand side is still a positive current by using approximation from convolution and general result about convergence of measures. Thus we can (repeatedly) use this explanation to make sense of all the currents involved above. More details can be found in Appendix.

In sight of the complication of just making sense of these currents, it can be imagined that people should be very careful in dealing with them rigorously. A lot

[14] In fact, nonnegative might be a better way to say it.

of formulae used in computation which are so obvious in the smooth case need to be justified. Basically, we just need to make sure that the expression in each step actually makes sense. The situation is not that bad indeed since all the definitions should be natural in the smooth case and we can use the smooth case as a guide. Local approximation (from convolution), together with classic results in measure theory, is usually all we need to justify them.

Anyway, the results are classic and well known. It would be rather tedious to give all the details at this moment. We might talk more about some computation related when they are used somewhere else.

d) CLN Inequalities

They are a set of very classic inequalities which were first introduced in [CLeNi]. We only recall the following two inequalities which are most useful for us.

Proposition 3.1.10. *For any open set U such that $U \subset\subset V$ (i.e., relatively compact in V), we have positive constants $C = C(U, V)$, such that for any $u_j \in PSH(V) \cap L^\infty(V)$ for $j = 1, \cdots, n$ and a compact set K in U, the following inequality holds:*

$$i) \quad \int_K (\sqrt{-1}\partial\bar{\partial}u_1) \wedge \cdots \wedge (\sqrt{-1}\partial\bar{\partial}u_n) \leqslant C\|u_1\|_{L^1(V)}\|u_2\|_{L^\infty(V)} \cdots \|u_n\|_{L^\infty(V)}.$$

Moreover if $u_0 \in PSH(V) \cap L^\infty(V)$ and $u_0 \leqslant 0$, we also have:

$$ii) \quad \int_K |u_0|(\sqrt{-1}\partial\bar{\partial}u_1) \wedge \cdots (\sqrt{-1}\partial\bar{\partial}u_n) \leqslant C\|u_0\|_{L^1(V)}\|u_1\|_{L^\infty(V)} \cdots \|u_n\|_{L^\infty(V)}.$$

The proof is essentially by formal integration by part. The idea is to use a sequence of (fixed) smooth plurisubharmonic functions to take away the $\partial\bar{\partial}$ before u_j and the auxiliary terms would contribute to the constant $C(U, V)$.

In the computation, it would be convenient to normalize the u_j's, whose L^∞-norms appear on the right hand side of the inequalities, to be valued in $[-1, 0]$. This can be easily justified by noticing that the L^∞-norm will at most be doubled if we add some constant to the original function to make it nonpositive, and rescaling by positive constants clearly won't affect the result. This would be fairly enough to picture the argument for i).

For ii), there is yet some subtlety about u_0 which plays a role different from others, and we are requiring it to be nonpositive. In fact, if we can formally integrate by part and switch the role of u_0 and u_1, the result would follow from the computation for i). But that can only be carried out if "boundary values" of u_0 and u_1 are the same since then we can use

$$u_0(\sqrt{-1}\partial\bar{\partial}u_1) - u_1(\sqrt{-1}\partial\bar{\partial}u_0) = u_0\sqrt{-1}\partial\bar{\partial}(u_1 - u_0) + (u_0 - u_1)(\sqrt{-1}\partial\bar{\partial}u_0),$$

and integration by part can now be justified by using compactly supported smooth functions to approximate $u_0 - u_1$ which vanishes on the boundary. Here something called "Localization Principle" naturally kicks in and let's discuss it below.

• Localization Principle: in order to prove the weak convergence or local estimates for a family of locally uniformly bounded plurisubharmonic functions, there is no loss of generality to assume that the functions are defined in a ball and equal over some neighbourhood of the boundary.

It is quite easy to prove this principle itself since we can use the rescaling of standard exhaustion function for a Euclidean ball to modify the family of functions being considered. Then the positive currents correspondent to the functions will be the same for inside part which would be enough for proving the statement for weak convergence and local estimates, while the boundary value would be just the value for the exhaustion function.

Let's emphasize that the above CLN inequalities are fairly local in spirit which make it ready to have more global form. Of course we can still have some information in case of $u_0 \leqslant C$ for some positive C instead of $u_0 \leqslant 0$ simply by noticing $u_0 = u_0 - C + C$ and combining i) and ii). This trivial observation is actually used to derive the following result which is what we are actually going to apply later.

Claim: Suppose U open and relatively compact in V. Then for any compact set K in U, $u \in PSH(V) \cap L^\infty(V)$ and $u \leqslant C_0$ for some $C_0 > 0$, there exists positive constant $C(U, V, C_0)$ such that:

$$Cap(K \cap \{u < -j\}, V) \leqslant \frac{C\|u\|_{L^1(V)} + C}{j}.$$

Proof. For any $v \in PSH(V)$ and valued in $[-1, 0]$, consider any compact set, $K' \subset$

$K \cap \{u < -j\}$, we have:

$$\begin{aligned}\int_{K'} (\sqrt{-1}\partial\bar{\partial}v)^n &\leqslant \frac{1}{j}\int_K |u|(\sqrt{-1}\partial\bar{\partial}v)^n \\ &\leqslant \frac{1}{j}\int_K (|u - C_0| + C_0)(\sqrt{-1}\partial\bar{\partial}v)^n \\ &\leqslant \frac{C\|u - C_0\|_{L^1(V)} + C}{j} \\ &\leqslant \frac{C\|u\|_{L^1(V)} + C}{j}.\end{aligned}$$

From the definition of capacity, this would give the inequality above.

□

Remark 3.1.11. *The notions and results listed above could by no means give a complete and accurate picture about classic pluripotential theory. We just feel it might be too distracting to introduce them in the next section which describes Kolodziej's orginal arguments if we do not want to leave too many black boxes. We still need a few other notions and results along the way. But they are more or less involved in the arguments themselves.*

3.2 Kolodziej's Original Argument

In this section, let's get into the original arguments in [Koj1] and [Koj2]. We do this for two good reasons. First, a large part of the arguments can be directly used by us and so we can put them here for later reference. Second, by putting some of them in our own language, it could give a more illuminating way for understanding and generalizing them. For later convenience, we shall separate the arguments into six parts in the following discussion.

• Part (1): Bound Capacity by Measure

For V an open bounded set inside $\mathbb{C}^n$ which is hyperconvex, i.e., $\exists h \in C^0(V') \cap PSH(V')$ where V' an open neighbourhood of $\bar{V}$ such that $\rho = 0$ on ∂V, sets $\{\rho < -s\}$ are relatively compact in V for $s > 0$ [15] and $V = \{\rho < 0\}$.

Now consider $u \in PSH(V) \cap L^\infty(V)$, $v \in PSH(V) \cap C^0(\bar{V})$. Assume that $U(s) = \{u < v + s\}$ is relatively compact (maybe empty) in V for $s \in [S, S + D]$

[15] "Relatively compact" means the corresponding closure in $\mathbb{C}^n$ is contained in V. A more intuitive description would be "really inside". These sets give the nice exhaustion of this domain V mentioned before.

with $D \geqslant 0$ of course. Then we claim the following which is the goal for Part (1): for $t \in [0, S + D - s]$,

$$t^n Cap(U(s), V) \leqslant \int_{U(s+t)} (\sqrt{-1}\partial\bar{\partial}u)^n.$$

Proof. The inequality is trivial for $t = 0$ and also for empty set case, so we assume $t > 0$ and $U(s)$ is nonempty below.

Consider any compact set $K \subset U(s)$. Define the function $w := \frac{u-s-t}{t}$ and set $W := \{w < u_K + \frac{v}{t}\}$ where u_K is the relative extremal function of K with respect to V. Then we can see $K \subset W \subset U(s+t)$ as follows:

On K ($\subset U(s)$), $w = \frac{u-s-t}{t} < \frac{v-t}{t} = -1 + \frac{v}{t} \leqslant u_K + \frac{v}{t}$, so $K \subset W$.

On W, $\frac{u-s-t}{t} < u_K + \frac{v}{t} \leqslant \frac{v}{t}$, so $u < v + s + t$, thus $W \subset U(s+t)$.

Basically we just use $-1 \leqslant u_K \leqslant 0$ above. Now we have the following computation:

$$Cap(K, V) = \int_K (\sqrt{-1}\partial\bar{\partial}u_K)^n \leqslant \int_K \left(\sqrt{-1}\partial\bar{\partial}(u_K + \frac{v}{t})\right)^n \leqslant \int_W \left(\sqrt{-1}\partial\bar{\partial}(u_K + \frac{v}{t})\right)^n$$
$$\leqslant \int_W (\sqrt{-1}\partial\bar{\partial}w)^n = t^{-n} \int_W (\sqrt{-1}\partial\bar{\partial}u)^n \leqslant t^{-n} \int_{U(s+t)} (\sqrt{-1}\partial\bar{\partial}u)^n.$$

Let's give a little explanation for the steps. The first step is from the relation between relative capacity and relative extremal function discussed before. The rest are basically from the relation between the sets proved above and comparison principle (over V), where the boundary condition for comparison principle is justified by the assumption about the relative compactness of $U(s+t)$ in V. [16]

Since the estimate can be done for any compact subset of $U(s)$, we get it for $U(s)$ itself just by the definition of relative capacity for general sets.

□

For this result, we can have u and v defined over a set which contains V. There is no trouble if we have to restrict them to V (in order to satisfy the assumptions). We mention this to emphasize the locality of the above discussion.

Remark 3.2.1. *When proving the important chain of sets above, $K \subset W \subset U(s+t)$, we only used the fact that u_K is valued in $[-1, 0]$. So we may just use any $\phi \in PSH(V)$ valued in $[-1, 0]$ instead of u_K (also in the definition of W) in the whole computation above. And finally by taking supremum over all such ϕ's, we can still get the same estimate for $Cap(K, V)$. So there is in fact no need to use the relation $Cap(K, V) =$*

[16] Notice that the second step is rather trivial from the plurisubharmonicity (and boundedness) of v. Though that's the only place where we need v to be plurisubharmonic other than the application of comparison principle, it is in fact quite crucial for the argument.

$\int_K(\sqrt{-1}\partial\bar{\partial}u_{_K})^n$. And so the hyperconvexity of V is not that essential here. It's easy to see the continuity of v is also not needed. We only need v to be bounded and plurisubharmonic.

• Part (2): A Fundamental Arithmetic Result

We keep the set-up in Part (1) and further assume that for any (Borel) subset E of V, we have:

$$\int_E(\sqrt{-1}\partial\bar{\partial}u)^n \leqslant A\cdot Cap(E,V)Q\big((Cap(E,V))^{-\frac{1}{n}}\big)^{-1}$$

for some positive constant A, where $Q(r)$ is an increasing function for positive variable r with positive value. Moreover, we also require the set $U(s)$ defined before to be nonempty for $s\in[S,S+D]$.

For the case when $Cap(E,V)=0$, the inequality above can be trivially understood. [17] This requirement is bascially for the case when $Cap(E,V)>0$. From now on, we call this condition as "Condition (A)".

At least for the situation we are mainly interested in, in the sense of Borel measure, $(\sqrt{-1}\partial\bar{\partial}u)^n = f\cdot d\lambda$ where $d\lambda$ is the standard Euclidean measure and (nonnegative) $f\in L^p(V)$ for some $p>1$ (or even just say f is Lebesgue integrable). So Condition (A) above is equivalent to,

$$\int_E f\cdot d\lambda \leqslant A\cdot Cap(E,V)Q\big((Cap(E,V))^{-\frac{1}{n}}\big)^{-1}.$$

Thus this condition is also equivalent to require only for compact subsets of V instead since we can use compact sets to approximate all (Lebesgue measurable) subsets up to some measure 0 set and the inequality is OK for taking such an exhausting limit since the left hand side is precisely the limit and the right hand side is no less than the limit [18]. So for our concern, it's enough to require Condition (A) for any compact set E. [19]

The main claim for this part is as follows: under the set-up in Part (1) with Condition (A) and $U(s)$ nonempty for $s\in[S,S+D]$, we have

$$D\leqslant\kappa(Cap(U(S+D),V))$$

[17] In this case, the left hand side is of course 0 from essentially the boundedness of u and the definition of $Cap(E,V)$.

[18] The monotonicity of function Q is used here.

[19] Actually, in the original form of this condition, we are only considering E to be any Borel subset since the measure $(\sqrt{-1}\partial\bar{\partial}u)^n$ is only Borel, so the exhaustion argument used above should work in general. Hence we always only need to take care of compact sets.

for the function

$$\kappa(r) = C_n A^{\frac{1}{n}} \Big(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1} (Q(y))^{-\frac{1}{n}} dy + \big(Q(r^{-\frac{1}{n}}) \big)^{-\frac{1}{n}} \Big),$$

where C_n is a positive constant only depending on the complex dimension n.

Idea of proof: the argument is a little technical but quite elementary in spirit. Since the detail has already appeared in [Koj1], we'll only sketch the idea below.

The inequality proved in Part (1) looks like "$Cap \leqslant measure$".

Condition (A) above gives the other direction "$measure \leqslant Cap$".

So we can combine them to get some information about the length of the interval coming from t in the inequality proved in Part (1). The assumption about nonemptiness of the set $U(s)$ is needed because we have to divide both sides by $Cap(U(\cdot), V)$, which has to be nonzero, in order to get something explicit for t. Then we can sum all these small t's up to get control for D.

We need the trivial fact that for a bounded domain in $\mathbb{C}^n$, nonemptiness, nonzero (Lebesgue) measure and nonzero (relative) capacity are equivalent for sets like $U(s)$. [20]

Of course we'd better use a clever way to carry out all computation just in sight of the rather complicate final expression of functon κ. And it has been done in great details very carefully in [Koj1]. We emphasize that except for the little fact above, all those involved are fairly elementary analysis.

Finally, let's point out that in the detailed argument, we do not have a positive lower bound for the t's summed up, so it is important that the inequality from Part (1) holds (uniformly) for all small enough $t > 0$.

Remark 3.2.2. *It's actually not necessary for the V in Condition (A) to be the same one as the V in Part (1). If we can use a larger V' for the "Cap" in Condition (A) which would make the condition even stronger than the correspondent one using V, then the conclusion will be also using "Cap" with respect to V' instead of V. The reason is that by noticing $Cap(K, V') \leqslant Cap(K, V)$, the result of Part (1) can be trivially translated to the statement using V'.*

Moreover, from the little sketch of the argument above, we see that in this case, we still only need Condition (A) for the sets $U(s)$ in V. Also there is no need for V' to be hyperconvex which is needed for V as in the original argument for Part (1).

[20] From the properties listed at the beginning for plurisubharmonic functions, $U(s)$ having measure 0 would imply $U(s)$ empty, and (relative) capacity controls measure as mentioned before. In fact, this is true for more general domains which might not be in $\mathbb{C}^n$ and the simple argument we just used can be adjusted to that case quite easily.

[21] *But we still emphasize this because it shows that there could be a lot of flexibility about the domain considered. This is pointed out because it's easier to get a (uniform) control for $Cap(U, V)$ when U's are "uniformly" inside V (by CLN inequalities for example), i.e., all contained in a set relatively compact in V, which is useful in Part (4) below.*

• Part (3): Condition (A)

In this part, we want to show that $(\sqrt{-1}\partial\bar{\partial}u)^n = f \cdot d\lambda$ for nonnegative $f \in L^p(V)$ with some $p > 1$ will be enough to justify the condition (A) introduced in Part (2) for some proper function $Q(r)$, which is essentially like $(1+r)^m$ for some positive m [22].

Recall the condition we want to justify is the following:

$$\int_K f \cdot d\lambda \leqslant A \cdot Cap(K, V)\big(Q((Cap(K, V))^{-\frac{1}{n}})\big)^{-1}$$

for any compact subset K of V (with positive capacity). A should be a positive constant which depends only on the L^p-norm of f and the domain V.

Let's first notice that it will follow from:

$$\int_V |g|^n Q(|g|) f d\lambda \leqslant A,$$

where $g = (Cap(K, V))^{-\frac{1}{n}} u_{K}$. This is simply because

$$\begin{aligned} A &\geqslant \int_V |g|^n Q(|g|) f d\lambda \\ &\geqslant \int_K |g|^n Q(|g|) f d\lambda \\ &\geqslant (Cap(K, V))^{-1} Q\big((Cap(K, V))^{-\frac{1}{n}}\big) \int_K f d\lambda, \end{aligned}$$

which is just what we are heading for. Here we have used $u_{K} = -1$ *a.e. on* K since the upper semi-continuization only changes values in a pluripolar set which has relative capacity and so Lebesgue measure 0 from the facts mentioned before.

Now we only have to get a good upper control of $\int_V |g|^n Q(|g|) f d\lambda$. Using the condition $f \in L^p(V)$ for some $p > 1$, by Hölder inequality, we have an upper control

[21] Of course, the remark at the very end of discussion for Part (1) says the hyperconvexity of V is also not that essential.

[22] Essentially, we need that the function $\kappa(r)$, which is decided by the function Q, goes to 0 as r goes to 0. The reason will become clear when we combine all the results to draw the conclusion.

of the whole integral by the L^p-norm of f and $\int_V |v|^l d\lambda$ for some large l's. Here we've already taken the explicit form of $Q(r)$, $(1+r)^m$, into account. The L^p-norm of f is under control by our assumption. The other part should be easily taken care of once we have some kind of uniform bound of the measure of small (very negative) value part of g, noticing $g \leqslant 0$. The following will be dedicated to get this bound.

Obviously the claim below will be enough for our goal.

Claim: For any bounded hyperconvex domain U in $\mathbb{C}^n$, consider $u \in PSH(U) \cap L^\infty(U)$ with the limit to ∂U existing at each boundary point and being nonnegative, and $\int_U (\sqrt{-1}\partial\bar{\partial}u)^n \leqslant 1$. Set $U_s = \{u < -s\}$ for any $s > 0$.

Then we have $\lambda(U_s) \leqslant C_\alpha \cdot e^{-2\pi\alpha s}$ with $0 < \alpha < 2$ [23] and C_α being a positive constant not depending on u with those properties.

Essentially we want to apply this claim to the function g above. And in order to justify the conditions in the claim for g, we need V to be hyperconvex. [24].

In order to prove the above claim, there is a useful subclaim that we'll prove first. We need a few more notions before getting into that.

e) [25] Lelong Class

The following class of functions is what is called Lelong class:

$$\mathcal{L} := \{u \in PSH(\mathbb{C}^n) | u(z) - \log(1+|z|) < C_u\},$$

where C_u stands for a constant which might well depend on the specific function u and $|z| = \sqrt{|z^1|^2 + \cdots + |z^n|^2}$ where $\{z^1, \cdots, z^n\}$ is the Euclidean coordinate system for $\mathbb{C}^n$.

Basic examples of elements: $\log|z|$, $\log(1+|z|)$ and some simple modification of them.

Basic property for such functions: log-growth control as explained below:

For $u \in \mathcal{L}$ and any complex line I in $\mathbb{C}^n$ which may or may not passing through the origin, consider the function $u - \log|w|$ on I, where w is the induced complex coordinate for I. Clearly $u - \log|w|$ is (pluri)subharmonic and bounded from above in $I \backslash \{w = 0\}$ since $\log|w|$ is harmonic for complex dimension 1 and $|w|$ is still essentially just the distance to the origin of $\mathbb{C}^n$ for points on I. Thus we can extend the function $u - \log|w|$ subharmonically to the ∞ on I, i.e., $u - \log|w|$ is now subharmonic on the

[23] This inequality is trivial for $\alpha \leqslant 0$.

[24] We can see this is also not necessary later from another observation.

[25] We are continuing the introduction of pluripotential theory from before and so starts with "e" here.

punctured Riemann sphere extended from I without origin $w = 0$.

Now by applying maximum principle for balls centered at infinity, we get:

$$sup_{\partial B_r}(u - \log|z|) \leqslant sup_{\partial B_s}(u - \log|z|)$$

for $0 < s < r$, and so

$$sup_{\partial B_r} u - sup_{\partial B_s} u \leqslant \log\big(\frac{r}{s}\big)$$

for $0 < s < r$.

This is a much better description of the growth of u than before and is called "log-growth control".

Remark 3.2.3. *We have used the balls in $\mathbb{C}^n$ for above inequalities which is OK since we can have for each complex line through the origin respectively and the results can be combined to get for $\mathbb{C}^n$. The "sup" can of course be replaced by "max" by upper semi-continuity of plurisubharmonic functions.*

But we should emphasize here that the argument (using $u - \log|w|$) can not be carried through for $\mathbb{C}^n$ for $n > 1$ directly because $-\log|z|$ is not plurisubharmonic on $\mathbb{C}^n \setminus \{0\}$. This would cause big trouble in some later consideration for generalizing the argument in this part.

f) Global Extremal Function

For a bounded set $E \in \mathbb{C}^n$, define

$$L_E = (sup\{u | u \in \mathcal{L}, u \leqslant 0 \text{ on } E\})^*.$$

The "*sup*" is taken pointwisely and "$*$" means taking upper semi-continuization just as in the definition of relatively extremal function before.

There are cases when L_E is in fact $+\infty$ everywhere. Those sets are in fact pluripolar (see [Koj2] for example) and will not cause any trouble for our consideration as basically they are of Lebesgue measure 0. Other than this case, we actually have $L_E \in \mathcal{L}$. The upper control is basically coming from the log-growth control for elements in $\mathcal{L}$ discussed before. In fact there is also a trivial lower control of L_E by some log-growth function which comes directly from the boundeness of E and the definition of L_E by noticing those examples of elements in $\mathcal{L}$ listed before.

There would be some trivial value comparison results for these functions just as for relatively extremal functions. Similarly, we also have to be very careful about the affect of the upper semi-continuization which is necessary for the plurisubharmonicity but makes the function more subtle to understand.

The same argument as for relative extremal function $u_{_E}$ will give us that the (Borel) measure $(\sqrt{-1}\partial\bar{\partial}L_E)^n$ will be supported on ∂E for E compact. The compactness of E is required since we want to be able to use continuous functons only for the definition of L_E just as for $u_{_E}$. [26]

There is another property about the functions like L_E, i.e., those plurisubharmonic functions on $\mathbb{C}^n$ with log-bounds (from both sides) which gives some very strict restriction about them. In fact, by comparison principle, it is easy to see $\int_{\mathbb{C}^n}(\sqrt{-1}\partial\bar{\partial}u)^n$ has to be the same for any such function. Thus we can get the precise value by computing $\int_{\mathbb{C}^n}\left(\sqrt{-1}\partial\bar{\partial}\frac{1}{2}\log(1+|z|^2)\right)^n$ which is equal to the volume of Fubini-Study metric over $\mathbb{CP}^n$ (up to some conventional constant).

g) Global Capacity

For a compact set K in $\mathbb{C}^n$, we define global capacity of K as follows:

$$T_R(K) := e^{-sup_{|z|\leqslant R}L_K(z)}$$

for some fixed $R > 0$.

Directly from the above definition and the result about the global integral of $(\sqrt{-1}\partial\bar{\partial}L_E)^n$ from f), we can have control of $T_R(K)$ by $Cap(K, B_R)$ for compact K in B_R which will be used later.

The control for the other direction is also available for more restricted K (i.e., closer to the center). The proof is slightly more involved and makes use of the geometry of Euclidean space in a more subtle way. Since it is not that useful for us, we'll ignore it at this moment.

Now we state the subclaim mentioned before.

Subclaim: For any $0 < \alpha < 2$, there exists $C_{\alpha,n}$ such that in $\mathbb{C}^n$, for all $u \in \alpha\mathcal{L}$, $B = B_0(1)$, we have:

$$\int_B e^{sup_B u - u} d\lambda \leqslant C_{\alpha,n}.$$

Proof. Clearly we can assume $0 = sup_B u \leqslant u(a)$ for some $a \in \bar{B} \subset B_0(2)$. Set $E_k = \{z \in B_0(2) | \log(k-1) < u \leqslant \log k\}$, $F_k = \cup_{j=k}^{\infty} E_j$.

We can see $v(z) := \frac{1}{\alpha}(u(z) + \log(k-1))$ belongs to $\mathcal{L}$ and is nonpositive over F_k. Noticing $v(a) = \frac{1}{\alpha}\log(k-1)$, from the definition of the global capacity T, we know

[26] Here since the "background" domain is $\mathbb{C}^n$ now, the convolution is globally OK and there is no need to extend the functions, so it is slightly less involved to justify the restriction to continuous functions in this case.

for any complex line I passing through a:

$$T_2(I \cap F_k) \leqslant (k-1)^{-\frac{1}{\alpha}}$$

where the T_2 corresponds to $B_0(2)$ as in the definition.

Now from the classical result about the relation between standard measure and the global capacity in $\mathbb{C}^1$ (see [Tsm] for example), we have:

$$\lambda_1(I \cap F_k) \leqslant C\big(T_2(I \cap F_k)\big)^2 \leqslant C(k-1)^{-\frac{1}{\alpha}}.$$

Then easy computation will provide: $\lambda(F_k) \leqslant C(k-1)^{-\frac{2}{\alpha}}$. Finally a trivial argument gives $\int_{B_0(2)} e^{-u} d\lambda \leqslant C_{\alpha,n}$. The condition on α appears naturally for the convergence of the infinite summation involved.

□

Finally we can prove the claim stated before.

Proof. Without loss of generality, assume $U \subset B(1)$ where $B(1)$ is of course the unit ball in $\mathbb{C}^n$. Then it's easy to see $\lambda(U_s) \leqslant \int_B e^{-\alpha L_{U_s}} d\lambda$ simply by noticing $L_{U_s} = 0$ almost everywhere in U_s. Here the choice of α is quite flexible and we can take it to be 1. [27] From the subclaim, we have

$$\lambda(U_s) \leqslant C \cdot e^{-\alpha \cdot sup_{B(1)} L_{U_s}} = C\big(T_1(U_s)\big)^{\alpha}.$$

Also notice the inequalities

$$T_1(U_s) \leqslant e^{-2\pi (Cap(U_s,B(1))^{-\frac{1}{n}})} \leqslant e^{-2\pi (Cap(U_s,U))^{-\frac{1}{n}}}$$

where the first $\leqslant$ is the control of global capacity by relative capacity mentioned in g) and the second $\leqslant$ is a rather trivial relation. Combine all of them to arrive at:

$$\lambda(U_s) \leqslant C \cdot e^{-2\pi\alpha (Cap(U_s,U))^{-\frac{1}{n}}} \cdots\cdots (\star).$$

Now for any $t > 1$ and compact set K in U_s, by comparison principle, we have:

$$Cap(K,U) = \int_K (\sqrt{-1}\partial\bar{\partial} u_K)^n = \int_{\{\frac{tu}{s} < u_K\}} (\sqrt{-1}\partial\bar{\partial} u_K)^n$$
$$\leqslant \frac{t^n}{s^n} \int_U (\sqrt{-1}\partial\bar{\partial} u)^n \leqslant \frac{t^n}{s^n}.$$

[27] It just needs to be some positive constant strictly less than 2 in order to apply the subclaim.

Finally we can conclude $Cap(U_s, U) \leqslant s^{-n}$ by letting $t \to 1$ which gives us the claim in sight of $(\star)$.

$\square$

Until now we have proved the main result for this part which is basically the justification of Condition (A) for $L^{p>1}$ measure.

Remark 3.2.4. *The argument above (quoted from [Koj2]) works for more general class of functions than $L^{p>1}$ which is in fact what is originally proved in Kolodziej's works. But the special case of L^p is fairly sufficient for our consideration. And in fact it's also for this case that we can simplify and generalize the argument a little bit to study the degenerated Monge-Ampere equation.*

Indeed, we'll observe later that it is the inequality $(\star)$ for any compact set K of U instead of U_s that is essential for our purpose. And from the argument above, it's quite clear that no hyperconvexity is required for U in order for this to be true. We only need U to be bounded in $\mathbb{C}^n$.

• Part (4): Bounded Solution

Recall that the equation we are considering is:

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = F\omega^n$$

over X where $F \in L^p$ is nonnegative for some $p > 1$ with the proper integral over X and ω is a Kähler metric on X.

There are two strongly related goals for this part. One is to get an apriori L^∞ bound for a bounded weak (i.e., in $PSH_\omega(X) \cap L^\infty(X)$) solution for this equation (after proper normalization). The other one is to get such a solution (or say to prove existence for bounded solution). The essential argument is the same for both.

For the second one, we need an approximation sequence of equations whose solutions exist by classic results. The approximation solutions would have uniformly bounded L^∞-norm by the apriori estimate which would allow us to take the (weak) limit to get a bounded solution for the original equation. Let's take care of this first.

In this case, the main obstacle to get existence of a solution for the original equation is from the general right hand side which may not be a smooth nondegenarated volume form. So to begin with, we construct a sequence of smooth and positive functions $\{F_j\}$ converging to F in L^p and satisfying $\int_X F_j\omega^n = \int_X \omega^n$ as described below.

The positivity is easy to deal with by adding small positive constants to F. Then the smoothness can be achieved by using partition of unity and convolution. The positivity is preserved under this action. Thus we have constructed a sequence of smooth positive functions converges to F in L^p-norm. Finally rescaling by a proper sequence of constants, which clearly converges to 1, will give us the desired F_j's above.

By the classic result of Yau's (see [Ya]), there is a unique sequence of smooth functions $\{u_j\}$ satisfying

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_j)^n = F_j\omega^n$$

with $max_X u_j = 0$.

If we can have a uniform L^∞ bound for u_j's, it's quite easy to take limit to get a bounded solution for the original equation. The detials have been carried out carefully in [Koj1] and so we might just illustrate it a little bit later. In fact, the limit would also have supremum (maximum) 0 by a simple argument using Hartogs' Lemma.

Now we want to prove that u_j's are uniformly bounded in L^∞. Notice the $L^{p>1}$-norms for F_j's are uniformly controlled. So it's enough for us to get a uniform apriori L^∞ bound for bounded solutions for equations with uniformly bounded $L^{p>1}$ measure on the right hand side. Let's denote the solution by u and the measure by $Fd\lambda$ in the following.

Remark 3.2.5. *If we only want to get a bounded solution, then it's enough to consider the approximation solutions which are smooth. And as we'll see below, the life is much easiler if one only considers smooth solutions. But of course the apriori estimate is interesting in its only way and at least makes the result more complete.*

First, using Green's function and noticing u is nonpositive with maximum value 0, we can see that the L^1-norm can be controlled uniformly. The argument is standard for the classic case when the solution is smooth (C^2 would be enough). So that'll directly work for approximation solution u_j. We need to use approximation argument for solutions with less regularity. For a continuous solution u, Richberg's method of approximation provides a sequence of smooth functions $\{u_k\}$ uniformly converges to u as $k \to \infty$ with $(1 + \frac{1}{n})\omega + \sqrt{-1}\partial\bar{\partial}u_n \geqslant 0$. This would give us the desired L^1-norm control for u. For a merely bounded solution, other approximation methods will be used. When X is projective, there is an approximation which gives a decreasing sequence of smooth functions plurisubharmonic with respect to some metric, which may be more positive than ω, converging to u. For a general closed manifold X, a more recent result appearing in [BlKol] also gives an approximation of u similar to that. [28] Classic results in pluripotential theory says the convergence would actually be in L^1 space (as in [Ho] for example). [29] Anyway, we get the L^1-norm bound needed.

Then let's observe that since X is a closed complex manifold, it's easy to use local coordinate balls to reduce the picture to finitely many domains in $\mathbb{C}^n$ where the notions introduced before can be directly used. By the uniform bound of L^1-norm and CLN inequalities, we can see that the capacity of $\{u < -s\}$ locally in each coordinate ball will be uniformly controlled by $\frac{C}{s}$. Strictly speaking, we are considering $\{u < -s\} \cap B_r$ inside B_s where B_r and B_s are balls with $0 < r < s$ in the (same) coordinate patch (say $r = 1$, $s = 2$). In each local coordinate balls, we have the local potential, ϕ, for the background metric ω [30]. Since ω is a Kähler metric, we have no trouble to make the potential convex in a standard way, i.e., the central part is strictly smaller than the very outer part by some positive number with some uniform gap in between. Anyway, we have now switched the picture of a manifold X to finite Euclidean balls with the background information uniform for all of them.

Clearly we only need to get a uniform lower bound of u. Let's consider the "minimal" value point of it. There is no trouble to consider this point for a continuous u. It might seem to be a problem for a merely bounded function. But in fact, we can just consider the point whose value is very close (up to some small constant $\delta > 0$ which can be as small as we want) to be the infimum of u. The argument can go through once we choose δ to be controlled by another uniform constant which will appear below.

This point must fall in the central part of one of the coordinate (unit) balls. Since we have mentioned above about the uniform controls of the background information, there should be no trouble to ignore the fact that we don't know exactly which ball it falls in. Now we want to apply the results from the previous parts.

Considering in B_1, the function "u" in Step (1) would be $u+\phi$ here and "v" would be the minimal value for ϕ of the outer part, c, which we can choose to be 0. It's easy to see $\{u+\phi < c+s\}$ will be relatively compact in the coordinate unit ball for s small enough and won't be empty for proper choice of s [31]. We need the point chosen

[28] We'll talk about these approximations in greater details later in discussion about comparison principle.

[29] The above way to see uniform L^1-norm bound might look too complicated especially for bounded functions. I believe classic pluripotential theory might have more elementary ways to see this.

[30] This terminology of background metric is quite illusive and will be used repeatedly during the process.

[31] For example, take $s = inf_X u$. It doesn't matter that this value is not uniform apriori. We only need the gap "D" of the interval $[S, S + D]$ to be uniform.

to have the "minimal" value of u to achieve the relative compactness here. But we also see that it's still OK if the constant δ before is chosen to be dominated by the uniform gap of the local potential ϕ.

Clearly there would be some room for the constant s from the strict convexity of ϕ. So we some uniform room, which is essentially just the difference between the values of ϕ for central and outer parts of the ball, to move s around while still having $\{u + \phi < c + s\}$ nonempty and relatively compact in B_1. That room will be our D.

Remark 3.2.6. *The uniform room is coming from the uniform convexity of local potential of ω, i.e., from the positivity of ω. This would be the main difference for the degenerate situation that we are interested in and the current situation.*

Condition (A) for Part (2) is justified since L^p-norm for some $p > 1$ is uniformly bounded for F. Hence the constant A in the condition is uniformly controlled as from Part (3). Now from the conclusion of Part (2), we have for $Q(r) = (1 + r)^m$,

$$D \leqslant \kappa(Cap(U(S + D), V)).$$

The D is uniformly bounded from below as mentioned above, and so $Cap(U(S+D), V)$ must be uniformly bounded from below. But we've already seen the upper control of $Cap(U(S + D), V)$ as $-\frac{C}{S+D}$. Here we use B_2 as V, but $U(S + D)$ is still contained in B_1 by definition. [32] So we see S has to be uniformly bounded from below (say $S > -C$), which means $U(-C)$ would have to be empty. This indicates uniform lower bound for u. Hence we have got the apriori L^∞ bound for the solution.

Remark 3.2.7. *Actually, a more global argument has already been carried out in [Koj2] for this result which looks more concise and simple as one doesn't have to go through the previous point-pick construction. We shall use the essential computation of it later. But the local argument is still interesting in the sense that it tells you what is really needed for the argument in a more explicit way, and so we can locate the difficulties more easily when trying to go through similar argument for more general situation.*

As mentioned before, after getting the uniform L^∞ bound for the approximation solutions, a quite routine argument using some general results about plurisubharmonic functions allows us to take a limit of them in a weak sense (say pointwise almost

[32] This freedom of choosing domains has been discussed before.

everywhere or in L^1-norm) [33] and the upper semi-continuation of the limit, u, will be bounded and plurisubharmonic with respect to ω (but may not be continuous apriori). Of course we also know u solves

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = F\omega^n$$

where both sides are (Borel) measures over X.

We've mentioned that Hartogs' lemma will tell us that the maximum of the solution u thus got over X is still 0, just like the approximation solutions. This argument just uses the convergence of u_j to u. But we can have another argument as follows. Actually, we can have the limiting function u as a limit of a decreasing sequence of functions $\{v_j\}$ plurisubharmonic with respect to ω and with maximal value 0. [34], and then we can see the maximum value of u is also 0 which simply uses the fundamental fact that if a decreasing sequence of closed sets has empty set as the limit, then already one of them would have to be empty since $\{u \geqslant C\}$ is closed for u upper semi-continuous and any constant C.

The cohomological condition $\int_X F\omega^n = \int_X \omega^n$ is naturally involved when one tries to deduce from the inequality $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n \geqslant F\omega^n$, which is got by taking limits, the equality $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = F\omega^n$ by noticing the integrals of both sides are the same over X.

Remark 3.2.8. *In fact, the argument up to now can already be used to provide L^∞ (or say C^0) estimate for the classic problems considered in Yau's original paper ([Ya]). This method is very different from the classic one featuring maximum principle and can be used to improve some of the classic results a little bit.*

In fact, if one tracks down all the steps more carefully, the argument can actually give L^∞-norm bound for the normalized (with 0 as the maximum value) solution by L^p-norm of F in a more explicit way as follows:

$$\|u\|_{L^\infty} \leqslant C \cdot \|F\|^n_{L^p}$$

where constant $C > 0$ depends only on X, ω and $p > 1$.

The argument is pretty elementary and fairly universal, i.e., it can be directly

[33] Let's point out that there is an expression of this limit using the whole sequence which is just the upper semi-continuization of the upper limit for the sequence. So the choice of a subsequence involved will not affect the limit. It's just that sometimes it would be more convenient to use a convergent sequence.

[34] Just take $v_j = (sup_{k \geqslant j} u_k)^*$ which clearly has maximal value 0 and decreases to u pointwisely.

applied to our generalized situation and gives the relation claimed in Theorem 1.3.2. Let's give the details below. There are some relations used below which will be more obvious from later treatment. They are also true for the current situation.

The constant A in Condition (A) would now be chosen as $\|F\|_{L^p}$ with function $Q(r)$ be chosen as $C_m \cdot (1+r)^m$ for some $m > 0,\ C_m > 0$. The lower index m of C indicates the dependence on m. This choice is slightly different from before because we want the affect of $\|F\|_{L^p}$ to be more explicit for the current purpose. The constant C_m in $Q(r)$ contains the other constant which is essentially from the upper bound of Lebesgue measure by relative capacity.

Now we can consider $\kappa(r)$ from the pretty complicated definition and get

$$\kappa(r) \leqslant C_m \cdot A^{\frac{1}{n}}\Big(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1-\frac{m}{n}}dy + r^{\frac{m}{n^2}}\Big) \leqslant C_m \cdot A^{\frac{1}{n}} r^{\frac{m}{n^2}}.$$

It seems to me that it's still most convenient to get the estimate by following the logic of contradiction argument used before. Suppose the sublevel set of u, $U(s)$, is relatively in V and nonempty for any $s \in [T-D, T]$ with some $T < 0$ where the D is the uniform gap got before by the local construction [35], then we need to have

$$D \leqslant \kappa\big(Cap(U(T), V)\big) \leqslant C_m \cdot A^{\frac{1}{n}} Cap(U(T), V)^{\frac{m}{n^2}} \leqslant C_m \cdot A^{\frac{1}{n}}(-T)^{-\frac{m}{n^2}}$$

where in the last step , we have used the same CLN inequality as before.

Now we have $T \geqslant -C_m \cdot A^{\frac{n}{m}}$. This is an explicit expression about how negative the $S+D$ used before can be. From the way in which this picture is chosen [36], we have the following L^∞ estimate for any $m > 0$ [37]

$$\|u\|_{L^\infty} \leqslant C_m \cdot \|F\|_{L^p}^{\frac{n}{m}}.$$

The one claimed above is for $m = 1$. It might seem better to have bigger m in sight of the lower L^1 bound of F. If one considers "rescaling" relation between u and F, $m = n^2$ might be a good choice.

Remark 3.2.9. *This above discussion makes use of the local argument. As mentioned*

[35] Let's emphasize that this D only depends on the background form ω over X. Of course, we can choose it to be smaller than 1 for the convenience of later application where D is more favorable to be like that. It's not that essentially anyway. This T is just the $S+D$ before.

[36] The function $v = 0$ and the minimal value of u would be larger than $T - C$ for a positive constant C which depends only on the background form ω over X.

[37] We need the obvious lower L^1 bound of F from the cohomology condition, i.e., the global integral over X is known to be a positive constant.

before, there is a corresponding global argument for boundedness result and we can also use it to do the same thing as above. Actually for this argument, we can consider an interval $[T-\frac{1}{2}, T]$ where the values of u can stretch over which means the sublevel sets are not empty. [38] *Then similarly we can get a lower bound of T and hence for u. It seems to be simpler than the previous argument. Usually, we can argue more brutally and concisely using global argument. But in my opinion, local argument gives better description about what's really happening. Furthermore, local argument is necessary for the continuity result in the case that we are mainly interested in at least for now.*

There might be another way to get this bound as follows.

If we can choose $S+D$ such that

$$C_m \cdot A^{\frac{1}{n}}\big(Cap(U(S+D),V)\big)^{\frac{m}{n^2}} = \frac{1}{2} \cdots\cdots (W),$$

then we have $\frac{1}{2} \leqslant C_m \cdot A^{\frac{1}{n}}(-S-D)^{-\frac{m}{n^2}}$ which gives $-S-D \leqslant C_m \cdot A^{\frac{n}{m}} = C_m \cdot \|F\|_{L^p}^{\frac{n}{m}}$. Combining with $D \leqslant \kappa\big(Cap(U(S+D),V)\big) \leqslant \frac{1}{2}$, we have the same L^∞ estimate for any $m > 0$ as above.

Here we do need the equality in (W) [39], but I am not so sure about whether we can actually do it. The trouble comes from the picture of u when relative capacity of $U(T)$ might jump to "0" which seems possible to me. This seems to be an interesting question.

- Part (5): Continuity of Bounded Solution

For many reasons, it would be good to know the solution from above is continuous. For example, there are a lot of choices involved in getting the solution, but we would like to see the solution is independent on all the choices, i.e., we want some kind of uniqueness of the solution. It's quite natural to see a proof for such a uniqueness result would involve comparison principle. For the case of a closed manifold, continuity of the functions we want to compare is usually required in classic version of comparison principle. [40] And in classic pluripotential theory, it is also more favorable if the function is continuous.

[38] Clearly if the values of u can't stretch over an interval of length $\frac{1}{2}$, we are done since the maximum is 0.

[39] What we need is upper and lower bounds of the left hand side by constants. In the previous discussion, we immediately apply the CLN inequality. Maybe now we have chosen a wrong time to fix the number.

[40] We'll see later that for projective or even closed Kähler manifolds, this is not so necessary. We can compare any two bounded plurisubharmonic functions.

Now we start to prove the continuity of the solution u got before. The essential step of the argument, as in [Koj1], is mainly contained in the proof of the uniform L^∞ bound above. Let's point out that this argument actually works for any bounded solution u for the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = F\omega^n$$

where $\omega + \sqrt{-1}\partial\bar{\partial}u \geqslant 0$ in the sense of current and both sides are in the sense of (Borel) measure. We do not need that the solution is from the approximation method.

All the following is quoted from [Koj1]. Suppose a bounded solution u is not continuous (over X). Since it's already upper semi-continuous from plurisubharmonicity, we can assume $d := sup_X\{u - u_*\} > 0$ where the lower "$*$" indicates taking lower semi-continuization. Provided the boundedness of u, we know $d < \infty$.

By noticing that $-u_*$ is upper semi-continuous, we can safely use "max" instead of "sup" in the definition of d and in fact the set $Y := \{u - u_* = d\}$ is closed and nonempty by assumption. Moreover we can have $x_0 \in Y$ such that $u(x_0) = inf_Y u$ as follows.

Suppose $u(x_j) \to inf_Y u$ as $j \to \infty$ with $x_j \in Y$. Since Y is closed, we can assume $x_j \to x_0$. Just need $u(x_0)$ has the right value which does not trivially follow from the upper semi-continuity. But we can get this from the simple argument below.

We already have as $j \to \infty$,

$$u(x_j) \to inf_Y u, \quad - u_*(x_j) = d - u(x_j) \to d - inf_Y u.$$

From upper semi-continuity, we know

$$u(x_0) \geqslant inf_Y u, \quad - u_*(x_0) \geqslant d - inf_Y u.$$

Now we can take the summation to conclude $u(x_0) - u_*(x_0) \geqslant d$. It has to be equal from the definition of d, and so the two $\geqslant$'s before the summation would have to be $=$'s. That's just what we want.

In the following, we want to draw a contradiction from the existence of such a point x_0.

Consider a coordinate chart centered at the point x_0. Assume it's a unit ball, $B(0,1)$ where x_0 is the origin 0. Then as before, we can take the local potential for

ω, ϕ, which is minimal at the center x_0 and there is a uniform gap between the values of ϕ for the very outer part of the ball and x_0, saying $inf_S\phi - \phi(0) > b > 0$ where $S = \bar{B}(0,r) \setminus B(0,\frac{r}{2})$ with $r \in (0,1)$ being fixed and b is a positive constant which does not depend on the location of x_0 on X [41]. Here of course ϕ is also uniformly bounded no mattter where x_0 is. Since we know that u is bounded over X, it is justified to assume $v := u + \phi > 0$ on $B(0,1)$ and $A := u(x_0) + \phi(0) > d$ by using some proper choice. Of course we have already implicitly identified the coordinate neighbourhood on X with the unit ball in $\mathbb{C}^n$.

Using convolution in $B(0,1)$, we can get a sequence of smooth plurisubharmonic functions v_j decreasing to v in any relatively compact sets in $B(0,1)$. In fact we just need to have the decreasing approximation for a smaller ball in the following discussion.

Now we claim that considering in $B' = B(0,r)$, for some $a_0 > 0$ and $t > 1$, the sets $W(j,c) := \{w + c < v_j\}$ where $w := tv + d - a_0$ are nonempty and relatively compact in B' for c belonging to a uniform interval which does not depend on j for j sufficiently large. Here the constant a_0 and t might well depend on the situation. But this would still be enough for us to apply the results before to draw a contradiction at the end.

Proof of the claim: We consider the family of sets

$$E(a) := \{v - v_* \geqslant d - a\} \cap \bar{B}' = \{u - u_* \geqslant d - a\} \cap \bar{B}'$$

for $a \in [0,d]$ where we've used the smoothness of the local potential ϕ for ω.

We have $0 \in E(a)$, $E(0) = Y \cap \bar{B}'$, and $E(d) = \bar{B}'$. Furthermore, they are all closed by upper semi-continuity of $u - u_*$ and of course $E(a)$ shrinks to $E(0)$ as $a \to 0$.

Define $c(a) := u(0) - inf_{E(a)}u$. We can see $lim_{a\to 0}c(a) = 0$ below.

First, since $c(a) \geqslant 0$ which is trivial from above, we have $\underline{\lim}_{a\to 0}\, c(a) \geqslant 0$.

Now suppose $\overline{\lim}_{a\to 0}\, c(a) > 0$. We have a positive constant γ and a sequence $\{a_k\}$ going to 0 such that $u(0) - inf_{E(a_k)}u > 2\gamma$. Moreover, we can take a sequence $\{x_k\}$ such that $x_k \in E(a_k)$ and $u(0) - u(x_k) > \gamma$. Then take a subsequence if necessary, we may assume $x_k \to x \in E(0) \subset Y$.

From the definition of x_0 (i.e., the origin of the ball), $u(x) \geqslant u(x_0)$. And using the upper semi-continuity of u and $-u_*$ together with the inequality for x_k's above,

[41] Actually we don't need the uniformality for all points on X here. This argument for continuity is more local in flavor than that for boundedness.

we get:

$$\overline{\lim_{k\to\infty}} u(x_k) \leqslant u(0) - \gamma \leqslant u(x) - \gamma, \quad \overline{\lim_{k\to\infty}} (-u_*(x_k)) \leqslant -u_*(x).$$

Sum them up to get:

$$u(x) - u_*(x) - \gamma \geqslant \overline{\lim_{k\to\infty}} (u(x_k) - u_*(x_k)) \geqslant \overline{\lim_{k\to\infty}} (d - a_k) = d$$

which contradicts the choice of d.

Thus we have $\overline{\lim}_{a\to 0} c(a) \leqslant 0$ and so $lim_{a\to 0} c(a) = 0$.

Now let's fix a positive constant a_0 such that

$$0 < a_0 < min(\frac{b}{3}, d), \quad 0 \leqslant c(a) < \frac{b}{3}$$

for $a \leqslant a_0$. Then we can choose $t > 1$ such that

$$(t-1)(A-d) < a_0 < (t-1)(A-d+\frac{2b}{3}).$$

A special version of Hartogs' Lemma as follows would be useful for us.

Lemma 3.2.10. *With all the notations from before, if $v - tv_* < C$ on a compact set $K \subset \bar{B}'$, then we have $v_j < tv + C$ on K for j sufficiently large.*

Proof. For any $x \in K$, we have that there is a neighbourhood V of it and a constant $C' < C$ such that $tv > sup_{\bar{V}} v - C'$ over $\bar{V}$ as follows.

Suppose it is not true. Then for any constant $C' < C$ and a sequence of shrinking (to x) neighbourhoods $\{V_k\}$, there exist $\{x_k\}$ such that $x_k \in \bar{V_k}$ and $t \cdot v(x_k) \leqslant sup_{\bar{V_k}} v - C'$. Taking limit as $k \to \infty$, we have $t \cdot v_*(x) \leqslant \underline{\lim}_{k\to\infty} t \cdot v(x_k) \leqslant v(x) - C'$.

Thus $v(x) - tv_*(x) \geqslant C'$ which clearly contradicts the assumption of the lemma at x since this would be for any $C' < C$.

In the mean time, from Hartogs' Lemma, we have $v_j \leqslant sup_{\bar{V}} v + (C - C')$ for j sufficiently large and on a smaller neighbourhood of x. [42]

Combining these two inequalities, we would have the result for a neighbourhood of x and the compactness of K can now be used to conclude the proof for this version of Hartogs' Lemma.

□

[42] In fact, we can have this without using the original form of Hartogs' Lemma since our v_j's are from convolution of v.

Now let's come back to our picture. Considering $y \in S \cap E(a_0)$ and recalling the definition for all the constants, we have:

$$v_*(y) = \phi(y) + u_*(y) \geqslant \phi(0) + b + u(y) - d \geqslant \phi(0) + b + u(0) - c(a_0) \geqslant A - d + \frac{2b}{3}.$$

We can also get $(t-1)v_*(y) > a_0$ from the definition of constants t and a_0, which would obviously imply the following:

$$v(y) \leqslant v_*(y) + d < tu_*(y) + d - a_0.$$

By a simple contradiction argument using the semi-continuity for both sides and the closedness of $S \cap E(a_0)$, we see this inequality actually holds for a neighbourhood of $S \cap E(a_0)$.

By the special version of Hartogs' Lemma discussed above, we see $v_j < tv + d - a_0$ on a neighbourhood V of $S \cap E(a_0)$ for j sufficiently large.

Then consider the part $S \setminus V$. We have $v - v_* < d - a_0$ over it since there is no intersection with $E(a_0)$. Thus we can use Hartogs' Lemma again, noticing $t > 1$ and $v > 0$, to get:

$$v_j < v + d - a_0 < tv + d - a_0$$

over $S \setminus V$ for j large enough.

Hence we have $v_j < tv + d - a_0$ on S for j sufficiently big. Thus if we set $w = tv + d - a_0$ as stated before, it's easy to see $W(j, c)$ will be relatively compact in B' for $c \geqslant 0$ and j sufficiently large.

Also notice that we have:

$$tv_*(0) + d - a_0 - v(0) = t(A - d) + d - A - a_0 = (t-1)(A-d) - a_0 < -a_1$$

for some positive constant a_1 from the choice of t.

That is just $tv_*(0) + d - a_0 < v(0) - a_1 \leqslant v_j(0) - a_1$. From the definition of v_* and the smoothness (even just continuity) of u_j, we can see below that for $c \leqslant a_1$, there are points as near to 0 as possible which will belong to $W(j, c)$ for any j and this would imply the nonemptiness of those $W(j, c)$'s.

More precisely, if $t \cdot v(x) + d - a_0 \geqslant v_j(x) - c$ for x in some small neighbourhood of 0, then we see for any $\delta > 0$,

$$tv(x) + d - a_0 \geqslant v_j(0) - c - \delta$$

in some small neighbourhood of 0 depending on δ from the continuity of (each fixed) v_j. Thus $t\cdot v_*(0)+d-a_0 \geqslant v_j(0)-c-\delta$ for any $\delta > 0$, and so $tv_*(0)+d-a_0 \geqslant v_j(0)-c$. This would provide a contradiction if $c \leqslant a_1$.

So far we have seen that for all j's sufficiently large, $W(j,c)$ would be nonempty and relatively compact in B' for $c \in [0, a_1]$.

Remember the equation v (similarly for w after a rescaling) would satisfy in $B(0,1)$:

$$(\sqrt{-1}\partial\bar{\partial}v)^n = f d\lambda$$

with nonnegative function $f \in L^p$ for some $p > 1$ and $d\lambda$ being the standard Lebesgue measure. This would justify Condition (A) in Part (2) from the discussion of Part (3) just as before.

Thus we are completely in the set-up for using the previous results and can arrive at:

$$a_1 \leqslant \kappa\big(Cap(W(j,0), B(0,s))\big)$$

for all j's large enough where $0 < r < s < 1$. This would give a lower bound for $Cap\big(W(j,0), B(0,s)\big)$ for all large j's. But notice that

$$W(j,0) = \{w < v_j\} = \{tv + d - a_0 < v_j\} \subset \{v + d - a_0 < v_j\}$$

and the fact mentioned before that v_j would converge to v with respect to capacity. This would give a contradiction for j large enough since we have $d - a_0 > 0$ from the choice of a_0 and $W(j,0)$'s are in B'.

So such a situation can not appear. Hence u has to be continuous.

Remark 3.2.11. *In fact there is another way of proving the continuity of the solution got by approximation argument. The essential computation has also been carried out in [Koj2] in a local version. But it is easy to observe that the argument would work perfectly in case of a closed manifold. We'll give some details later. Unlike the proof above, this proof works only for any bounded solution got by approximation, i.e., the method we actually use to derive such a solution since basically we are proving the convergence is actually uniform there.*

Though it is at least not apriori trivial to see that any bounded solution can be approximated like that, since the first argument works for any bounded solution, using the stability result in the part coming up, we see that it is indeed the case.

- Part (6): Stability and Uniqueness of Continuous Solution

For this part, we are going to use some global version of the notions, which are defined previously for domains in $\mathbb{C}^n$, for the closed manifold X. And of course, global version of the results from previous parts will also be used. Since such generalization of the notions and results are quite natural and in fact will be put into details later, we'll just use them without explicitly defining or stating them here.

Just notice that for now we only use continuous functions basically for the consideration of comparison principle following the argument in [Koj2] [43] and some other results. The subtlety will be put out along the way and probably more related details will appear for further consideration.

Specifically, in this part, all the plurisubharmonic functions with respect to the Kähler metric ω (or say ω-PSH) are continuous by definition [44]. It makes sense to only study these functions for uniqueness result of bounded solution for the current Monge-Ampere equation as we have already proved in Part (5) that bounded solution for this equation is actually continuous.

The main difference from the local discussion before, where there is no background metric ω appearing explicitly [45], would be that convex linear combination is now used in order to preserve the background metric. But actually we can easily see this is a rather superficial point and at many places, there is no need to follow this rule that seriously.

Basically, all the following argument is directly quoted from [Koj2].

Claim: Let $\phi, \psi \in PSH_\omega(X)$ (i.e., being ω-PSH functions on X) and satisfy $0 \leqslant \phi \leqslant C$, then for $s > C+1$, we have

$$Cap_\omega(\{\psi + 2s < \phi\}) \leqslant \left(\frac{C+1}{s}\right)^n \int_{\{\psi+s<\phi\}} (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n.$$

Proof. Define $E(s) := \{\psi + s < \phi\}$. Take any $\rho \in PSH_\omega(X)$ valued in $[-1, 0]$. Set $V = \{\psi < \frac{s}{C+1}\rho + (1 - \frac{s}{C+1})\phi - s\}$.

Since $-s \leqslant \frac{s}{C+1}\rho - \frac{s}{C+1}\phi \leqslant 0$, we can easily deduce the following chain relation of sets:

$$E(2s) \subset V \subset E(s).$$

Now quite similar to what is done in Part (1), we can have the following compu-

[43] Richberg's method of approximation is well known to be able to justify comparison principle in this case. However, it turns out that we can do much better than this as we'll see later.

[44] This is not the case before!

[45] Local potential is used, so we only need to study plurisubharmonic functions.

tation:

$$
\begin{aligned}
(\frac{s}{C+1})^n \int_{E(2s)} (\omega + \sqrt{-1}\partial\bar{\partial}\rho)^n &\leqslant \int_V (\frac{s}{C+1}\omega_\rho + (1 - \frac{s}{C+1})\omega_\phi)^n \\
&\leqslant \int_V {\omega_\psi}^n \\
&\leqslant \int_{E(s)} {\omega_\psi}^n
\end{aligned}
$$

by the relation of sets above and applying comparison principle for the two functions appearing in the definition of the set V. Here we have used the notation $\omega_\rho = \omega + \sqrt{-1}\partial\bar{\partial}\rho$, and similar for the others. This natural simplification of the notation has appeared in Chapter 2 and will also be frequently used for less regular case here.

Finally we can conclude the result from the definition of Cap_ω just as for the local version of relative capacity.

□

Now let's state the following version of stability result which is slightly weaker than what's achieved in [Koj2] because it can be proved for more general class of functions (measures) than L^p class. But it quite suffices for our concern.

Theorem 3.2.12. *In the same set-up as usual, for any nonnegative L^p-functions f and g with $p > 1$ which have the proper total integral over X, i.e., $\int_X f\omega^n = \int_X g\omega^n = \int_X \omega^n$, suppose that ϕ and ψ in $PSH_\omega(X)$ satisfy ${\omega_\phi}^n = f\omega^n$ and ${\omega_\psi}^n = g\omega^n$ respectively and are normalized to have $max_X\{\phi - \psi\} = max_X\{\psi - \phi\}$ by adding constants.*

If $\|f - g\|_{L^1} \leqslant \gamma(t)t^{n+3}$ for $\gamma(t) = C\kappa^{-1}(t)$ with some proper nonnegative constant C depending only on the L^p-norms of f and g [46] , where $\kappa^{-1}(t)$ the inverse function of essentially the κ function appearing in Part (2), then we can conclude that

$$\|\phi - \psi\|_{L^\infty} \leqslant Ct$$

for $t < t_0$ where $t_0 > 0$ depends on γ and C depends on the L^p-norms of f and g.

Proof. Suppose $\|f\|_{L^p}, \|g\|_{L^p} \leqslant A$. We'll be careful about the fact that the constants in the argument will only depend on A and the function γ.

[46] The dependence on the manifold X and Kähler metric ω should be clear.

For simplicity, let's normalize to have $\int_X \omega^n = 1$. And in fact, we can also assume $max_X\{\phi - \psi\} = max_X\{\psi - \phi\} > 0$ since the case for $\leqslant 0$ is trivial [47].

Without loss of generality, assume $\int_{\{\psi<\phi\}}(f+g)\omega^n \leqslant 1$ since $\int_X f\omega^n = \int_X g\omega^n = 1$.

Then by adding the same constant to ϕ and ψ which obviously affects nothing, we can assume $0 \leqslant \phi \leqslant a$ where "a" is a positive constant only depending on A from the boundedness result before.

Of course it is OK here to take a larger "a", which we'll actually do below, as long as the dependence on A is clear enough, or say finally we can still fix it to be some positive constant only depend on A.

As $lim_{t\to 0}\gamma(t) = 0$ by definition and the property of the function κ, we can fix $0 < t_0 < 1$ sufficiently small such that $\gamma(t_0){t_0}^{n+3} < \frac{1}{3}$, which will also hold for $0 < t < t_0$ since γ is obviously decreasing.

Fix such a t for now and set $E_k = \{\psi < \phi - kat\}$ where the "a" is from above but we still have not made the choice yet.

Clearly we have:

$$\int_{E_0} g\omega^n = \frac{1}{2}\int_{E_0}\big((f+g)+(g-f)\big)\omega^n \leqslant \frac{1}{2}(1+\frac{1}{3}) = \frac{2}{3}.$$

Now we construct a function g_1 which is equal to $\frac{3g}{2}$ over E_0 and some other nonnegative constant power of g for the complement. By the above estimate, it is easy to see that one can choose a proper constant (in [0,1]) such that g_1 is still nonnegative with L^p-norm bounded by $\frac{3A}{2}$, and more importantly it has the proper total integral over X.

So we can find a continuous solution $\rho \in PSH_\omega(X)$ as before by the approximation method such that

$${\omega_\rho}^n = g_1\omega^n, \quad max_X\rho = 0$$

with lower bound of ρ only depend on A [48]. By enlarging "a" if necessary which clearly won't affect the set E_0, we can assume the lower bound of ρ is $-a$. Now we can finally fix our constant "a" which clearly only depends on A in an explicit way.

[47] In this case, we can have $\phi - \psi \leqslant 0$ and $\psi - \phi \leqslant 0$, which says $\phi = \psi$. In other words, we have the compatible direction.

[48] Here besides directly using L^p-norm, one can also use the fact that $\int_S g_1\omega^n \leqslant \frac{3}{2}\int_S g\omega^n$ for any set S to justify Condition (A) in Part (2). Essentially, we still have to use the result in Part (3), and the control would of course be the same. Notice we've used the existence of continuous solution at this point for the solution ρ.

By noticing that $-2at \leqslant -t\phi + t\rho \leqslant 0$, it is easy to see

$$E_2 \subset E := \{\psi < (1-t)\phi + t\rho\} \subset E_0.$$

Let's denote the set $\{f < (1-t^2)g\}$ by G. Then over $E_0 \setminus G$, we have:

$$\big((1-t^2)^{-\frac{1}{n}}\omega_\phi\big)^n \geqslant g\omega^n, \quad \big((\frac{3}{2})^{-\frac{1}{n}}\omega_\rho\big)^n = g\omega^n.$$

Hence we can conclude that over $E_0 \setminus G$, as measure,

$$(\frac{3}{2})^{-\frac{n-k}{n}}(1-t^2)^{-\frac{k}{n}}{\omega_\phi}^k \wedge {\omega_\rho}^{n-k} \geqslant g\omega^n.$$

Remark 3.2.13. *This is a rather trivial result in smooth case which is just a direct application of algebraic-geometric mean value inequality. Then by approximation argument, it should also hold in our case here. For the conclusion above, there is no need to restrict us to the set $E_0 \setminus G$. We can consider over X and use $g\chi_{E_0 \setminus G}\omega^n$ for the right hand side. Actually the rigorous approximation argument is local and uses quite some results about Dirichlet problem for Monge-Ampere equation. The continuity of the functions are very involved in the proof which seems to be the main obstacle to go through the whole argument in this part for merely bounded solutions [49]. We'll give some details later when feeling necessary.*

Let's set $q = (\frac{3}{2})^{\frac{1}{n}} > 1$, and rewrite the above inequality as:

$$\omega_\phi^k \wedge {\omega_\rho}^{n-k} \geqslant q^{n-k}(1-t^2)^{\frac{k}{n}} g\omega^n$$

over $E_0 \setminus G$. Now the following computation is quite obvious: [50]

$$\begin{aligned} {\omega_{t\rho+(1-t)\phi}}^n &\geqslant \big((1-t)(1-t^2)^{\frac{1}{n}} + qt\big)^n g\omega^n \\ &\geqslant \big((1-t)(1-t^2) + qt\big)^n g\omega^n \\ &\geqslant \big(1 + t(q-1) - t^2\big) g\omega^n \\ &\geqslant \big(1 + \frac{t}{2}(q-1)\big) g\omega^n. \end{aligned} \tag{3.1}$$

[49] It might seems unnecessary to consider merely bounded solutions in the current case as they are in fact continuous. But it might still be interesting for more general cases.

[50] t below can be taken to be sufficiently small.

From the definition of G and assumption of the theorem, we also have:

$$t^2 \int_G g\omega^n \leqslant \int_G (g-f)\omega^n \leqslant \gamma(t)t^{n+3}$$

which is just:

$$\int_G g\omega^n \leqslant \gamma(t)t^{n+1}. \tag{3.2}$$

Hence we can have the following inequalities:

$$\begin{aligned}
\big(1+\frac{t}{2}(q-1)\big)\int_{E\setminus G} g\omega^n &\leqslant \int_E {\omega_{t\rho+(1-t)\phi}}^n && (\textit{the measure inequality } (3.1))\\
&\leqslant \int_E {\omega_\psi}^n && (\textit{comparison principle})\\
&\leqslant \int_{E\setminus G} g\omega^n + \gamma(t)t^{n+1} && (\textit{the integration inequality } (3.2)).
\end{aligned}$$

and arrive at:

$$\frac{q-1}{2}\int_{E\setminus G} g\omega^n \leqslant \gamma(t)t^n.$$

Therefore by noticing $E_2 \subset E$, we get:

$$\frac{q-1}{2}(\int_{E_2} g\omega^n - \gamma(t)t^{n+1}) \leqslant \frac{q-1}{2}(\int_{E_2} g\omega^n - \int_G g\omega^n) \leqslant \frac{q-1}{2}\int_{E\setminus G} g\omega^n \leqslant \gamma(t)t^n,$$

and so we have

$$\int_{E_2} g\omega^n \leqslant (t+\frac{2}{q-1})\gamma(t)t^n \leqslant \frac{3}{q-1}\gamma(t)t^n$$

for t small enough.

The claim proved before tells us:

$$Cap_\omega(E_4) \leqslant (\frac{a+1}{2at})^n \int_{E_2} g\omega^n.$$

Combining this with the previous inequality, we have:

$$Cap_\omega(E_4) \leqslant (\frac{a+1}{2a})^n \frac{3}{q-1}\gamma(t).$$

Thus if $E' := \{\psi < \phi - (4a+2)t\}$ is nonempty, by the argument for boundedness result before, we should have:

$$2t \leqslant \kappa(Cap_\omega(E_4)) \leqslant \kappa((\frac{a+1}{2a})^n \frac{3}{q-1}\gamma(t)) = t.$$

Clearly this is a contradiction for $t > 0$. Here as mentioned at the beginning, we have used a slightly different version of the result from what's quoted before, which considers the global case when we are not in a domain in $\mathbb{C}^n$ and have a background metric. We'll say more about this later.

Anyway, we have from above that $\psi \geqslant \phi - (4a + 2)t$.

Hence $max_X(\psi - \phi) = max_X(\phi - \psi) \leqslant (4a + 2)t$, which will give the desired conclusion.

□

Now from this stability result, it is easy to get uniqueness result for continuous plurisubharmonic solutions which are normalized to have maximum 0 over X for Monge-Ampere equation with the general right hand side since they would be the same up to a constant [51], but the constant would have to be 0 by the normalization.

One can easily see the proof can be simplified a little if we only care about the uniqueness result. But this result above actually gives much better description of the variation of the solution under the perturbation of the right hand side of the equation (i.e, the measure).

Remark 3.2.14. *For our main consideration, as we'll see later, the discussion in this section would be of little use since we have an extra term e^u on the right hand side of the equation mainly interested in and the argument above can't be carried through directly.*

In the mean time, the favorable sign of u here might make comparison principle alone sufficient for proving uniqueness result, and so it's only left to justify the application of comparison principle in all kinds of situation. In fact there would be some other ways to get uniqueness result for this modified equation in various cases. More discussion can be found in Appendix.

However, the discussion in this part is quite logically satisfying as a generalization of classic uniqueness result where the right hand side is smooth.

3.3 Application to Modified Equation

In this section, we apply or adjust the argument quoted above for the modified equation below which is more like the equation discussed in the previous chapter:

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^u F \omega^n$$

[51] Consider the case $f = g$ here. In fact the constant C in the definition of γ can be 0 in this case.

where ω is a Kähler metric over X and F is a nonnegative $L^p(X)$ function over X for some $p > 1$ with $\int_X C \cdot F\omega^n = \int_X \omega^n$ for some positive constant C. In other words, we are in exactly the same situation as before with just an extra term e^u on the right hand side.

In classic consideration about Monge-Ampere equation, this term e^u makes our life much easier [52]. For the current discussion, it at least won't give us too much extra trouble as we'll see below.

Let's first prove the existence of bounded solution for this equation. We still try to solve it by using approximation (of F) in the same spirit as before. In other words, we want an apriori estimate for (smooth) approximation solution.

Considering the following family of equations:

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_j)^n = e^{u_j} F_j \omega^n$$

where $\{F_j\}$ is a sequence of positive smooth functions over X which converges to F in L^p-norm as constructed before.

Classic result guarantees the existence and uniqueness of smooth u_j's. As before, we should first see u_j's are uniformly bounded. Being motivated by the classic argument for this kind of equations, let's consider another family of equations:

$$(\omega + \sqrt{-1}\partial\bar{\partial}v_j)^n = C_j \cdot F_j \omega^n$$

where C_j's are positive constants chosen to satisfy $\int_X C_j \cdot F_j\omega^n = \int_X \omega^n$. It's easy to see that as $j \to \infty$, $C_j \to C$, and moreover $C_j \cdot F_j \to C \cdot F$ in L^p-norm. Clearly C_j's are uniformly bounded.

From the results before, we have a uniform bound for all v_j's if we require $max_X v_j = 0$. Thus by taking a positive constant C large enough, for $\widetilde{v}_j := v_j + C$, we have

$$(\omega + \sqrt{-1}\partial\bar{\partial}\widetilde{v}_j)^n = C_j \cdot F_j\omega^n \leqslant e^{\widetilde{v}_j} F_j \omega^n.$$

Comparing this with the equation for u_j, we arrive at

$$(\omega + \sqrt{-1}\partial\bar{\partial}\widetilde{v}_j)^n \leqslant e^{\widetilde{v}_j - u_j}(\omega + \sqrt{-1}\partial\bar{\partial}u_j)$$

[52] We should have got a little taste of this fact with the discussion in the previous chapter, though we are already in degenerate case there.

which can be rewritten as

$$\big(\omega+\sqrt{-1}\partial\bar{\partial}u_j+\sqrt{-1}\partial\bar{\partial}(\widetilde{v}_j-u_j)\big)^n\leqslant e^{\widetilde{v}_j-u_j}(\omega+\sqrt{-1}\partial\bar{\partial}u_j)^n.$$

By maximum principle, considering the minimal value point of $\widetilde{v}_j-u_j$, it is easy to see $\widetilde{v}_j\geqslant u_j$.

Thus we have got a uniform upper bound for all u_j's. Similar argument will provide the uniform lower bound. Just as the discussion in the previous chapter, the sign of u in the term e^u is very important to carry through this argument. Smoothness is important for maximum principle argument used above, and so we do not claim the apriori estimate for bounded solution here. [53] Actually, it can still be achieved by going around this point using the continuity and uniqueness result which will be discussed shortly. Basically, we want to say that any bounded (and then continuous in this case) solution appears like that.

Now we want to get a solution for the original equation by taking limit in some proper sense and then taking the upper semi-continuation of the limit. There is a little difference in justifying that the limit would be a solution, i.e., make both sides of the equation be the same measure, which can be clarified as follows.

In fact we only have to justify the function u got from the above procedure will satisfy:

$$\int_X e^uF\omega^n=\int_X\omega^n=\int_X(\omega+\sqrt{-1}\partial\bar{\partial}u)^n,$$

which then guarantees the measure inequality $(\omega+\sqrt{-1}\partial\bar{\partial}u)^n\geqslant e^uF\omega^n$ got from the limiting argument [54] an equality. And this can be easily done below.

Obviously we have $\int_X e^{u_j}F_j\omega^n=\int_X\omega^n$. The following inequality is also trivial:

$$\int_X|e^{u_j}F_j-e^uF|\omega^n\leqslant\int_X|e^{u_j}F_j-e^{u_j}F|\omega^n+\int_X|e^{u_j}F-e^uF|\omega^n.$$

The control for the first term on the right would come from the uniform bound of u_j's and the convergence of F_j to F. And we can use Dominated Convergence Theorem, which is justified by the uniform boundedness of u_j's got before, to control the second term.

[53] The L^∞-norm of u is involved in the L^p-norm of the measure on the right hand side. It's even in the exponent which makes the explicit bound derived before hardly be of any help.

[54] Use lower limit for u_j on the right hand side to get u there. The limit would be pointwisely almost everywhere, and so different kinds of limits will actually give the same limit.

Hence the sequence $\{e^{u_j}F_j\}$ converges to e^uF in L^1-norm [55]. Finally, we can conclude that

$$\int_X e^u F\omega^n = \int_X \omega^n.$$

It is easy to observe that the continuity of the above solution u follows from exactly the same argument in Part (5) before. This is also the case for any bounded solution for this equation. Actually, we can directly apply the result there since the measure on the right hand side is clearly $L^{p>1}$ provided the solution u being bounded.

Then we can see the uniqueness of continuous solutions as follows. It should be pointed out that the full strength of the stability result as in Part (6) in the previous section is useful here because the obvious argument using comparison principle and the favorable sign of u on the right hand side of the equation can not quite get us there in sight of that our F may have 0 locus with nonzero measure. We have to be more careful than that. More precisely, by comparison principle, if we have two continuous solutions u and v for the equation, then F has to be almost everywhere 0 in the set $\{u \neq v\}$. Thus the right hand side of the equation will have to be the same L^p-measure after plugging in u or v. Now by applying the stability result in the previous section, we conclude that u and v will be up to a constant. Hence they should actually be the same simply by considering the integral of the right hand side of the equation over X.

Remark 3.3.1. *As mentioned before, continuous and uniqueness results actually justify the apriori L^∞ bound for bounded solution. But this method would not be so satisfying logically.*

Besides, it's also very natural to ask about similar stability result for the equation discussed in this section. As mentioned before, the original argument in the previous section would not do the job directly. Intuitively, if one directly apply the stability result there, then since the solution is also involved in the measure on the right hand side of the equation, the assumption looks even stronger than the result. [56] *But a more delicate argument is likely to do the trick.*

[55] We can actually have the convergence in L^p-norm by the above argument.

[56] This is fairly similar to the concern before about apriori L^∞ estimate for bounded solution of this equation.

3.4 Direct Application of Kolodziej's Results

In this section, we want to directly apply the previous results in this chapter to the case considered in the previous chapter. But first, we'll give some idea about how the set-up of Theorem 1.3.2 stated in Introduction comes into being.

Recall our main interest there is the case when the class $[\omega]$ is no longer Kähler, i.e., for the equation below

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

over a projective manifold X where Ω is a smooth volume form and ω_∞ is not a Kähler metric but with $[\omega_\infty]$ integral, nef. and big.

Let's further assume $[\omega_\infty]$ is semi-ample. Thus we can have a map $P : X \to \mathbb{CP}^N$ for some positive integer N using the holomorphic sections of the holomorphic line bundle correspondent to $k[\omega_\infty]$ with some large enough positive integer k. We also know the (possible singular) image will have the same dimension as X since we have $[\omega_\infty]^n > 0$ from the nef. and big assumption.

Remark 3.4.1. *Classic results in algebraic geometry (as in [Ka2]) tell us this would be the case if we are talking about nef. and big canonical bundle, K_X. But it is not true that general nef.and big line bundle would be semi-ample. This semi-ample assumption is quite essential for our argument in this whole business.*

Moreover, instead of the general semi-ample line bundle, we'll frequently talking about the canonical bundle, K_X, where there is some other terminology related, but it should be quite clear that our argument works perfectly for general line bundle in the corresponding situation.

Now we have $[\omega_\infty] = \frac{1}{k} \cdot P^\star[\omega_{FS}]$ [57] where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^N$. Of course our main interest is for P not being an embedding. But we'll still require it to be birational to its image at least for now [58] Basically one just considers the blow-down picture at this moment.

By a simple transform of the original equation, we can use $\frac{1}{k} \cdot P^\star\omega_{FS}$ instead of ω_∞ in the expression without changing any essential character. Thus we basically arrives at the set-up in Theorem 1.3.2 in Introduction.

[57] Usually, $*$ is used to stand for pullback or pushforward action, but in this section we use $\star$ instead since upper semi-continuization also appears below which we have used $*$ to denote it.

[58] We'll see later that it's indeed the case if we take the interger k large enough.

In this section, we consider the case when the image, $P(X)$, is actually a complex submanifold of $\mathbb{CP}^N$ [59]. And we also assume that P is a birational morphism from X to $P(X)$, i.e., P is biholomorphic after removing some (strictly lower dimensional) subvarieties from X and $P(X)$ respectively.

In this case, we can consider the original equation over X as an equation over $P(X)$ as follows:

$$(\frac{1}{k}\omega_{_{FS}} + \sqrt{-1}\partial\bar{\partial}v)^n = e^v\Omega',$$

where Ω' is a measure over $P(X)$ with its pullback over X by P to be the (smooth) measure Ω, i.e., $P^\star\Omega' = \Omega$ in the sense of measure.

For the well-definedness of Ω' with any general smooth volume form Ω over X, we have used the birationality of the map P. In fact, Ω' is essentially just the pullback of Ω by the inverse of P out of some subvariety in $P(X)$. In other words, $\Omega' = P_\star\Omega$ where $P_\star\Omega$ is considered as the pushforward of the measure Ω over X.

It is quite easy to see that Ω' will be L^p for some $p > 1$ with respect to a usual smooth measure over $P(X)$ which corresponds to a smooth volume form on $P(X)$. The idea is that actually one can see the singularities (or poles) are just along the part of $P(X)$ where X has some subvarieties crushed onto each point by the map P, moreover the orders of the poles basically just correspond to the dimensions of the corresponding subvarieties which are clearly strictly smaller than the dimension of X (and also $P(X)$). Using this picture, as we can see that $P_\star\Omega$ is clearly L^1 over $P(X)$, by the explicit form of (possible) singularities as $\frac{1}{r^\alpha}$ where r is like the distance to the subvariety, we can conclude it to be actually L^p for some $p > 1$ since it is an open condition for r here.

The equation we want to solve now is exactly in the situation discussed in the previous section. So we'll have a continuous solution v of it which is plurisubharmonic with respect to $\omega = \frac{1}{k}\omega_{_{FS}}$. And both sides of the equation are measures over $P(X)$.

Now we can pull back everything to X by P. The following equation over X is what we get:

$$P^\star\big((\frac{1}{k}\omega_{_{FS}} + \sqrt{-1}\partial\bar{\partial}v)^n\big) = e^{P^\star v}P^\star P_\star\Omega.$$

Notice obviously that $P^\star v$ is a continuous plurisubharmonic function with respect to $P^\star(\frac{1}{k}\omega_{_{FS}}) = \omega_\infty$, and so $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}P^\star v)^n$ is a (Borel) measure over X.

At the first sight, it looks like that by a local argument using approximation (from

[59]Of course with complex dimension n from discussion before.

convolution) and the classic weak convergence result, we can deduce:

$$P^\star\big((\frac{1}{k}\omega_{_{FS}}+\sqrt{-1}\partial\bar{\partial}v)^n\big)=(\omega_\infty+\sqrt{-1}\partial\bar{\partial}P^\star v)^n.$$

The natural idea is as follows. Both sides are obviously the same when v is smooth, so it would be true for the smooth approximation functions and then taking the limit seems to give us the conclusion. Here for the left hand side, we are using the natural-looking fact that the pullback of a convergent sequence of measures (as measure, i.e., the integration over any Borel set converges) is still a convergent sequence of measures with the corresponding limit. We are not saying

$$P^\star(\frac{1}{k}\omega_{_{FS}}+\sqrt{-1}\partial\bar{\partial}v)=\omega_\infty+\sqrt{-1}\partial\bar{\partial}P^\star v$$

(for complex dimension greater than 1) and in fact the left hand side above does not even make sense in general.

But notice there is yet some other problem for the above argument. More precisely, the convergence of measures coming from the monotonous convergence of the potentials is in the sense of distribution and will not give the convergence in the sense of measure as used above (though it gives the convergence in the weak sense of measure, i.e., one can use compactly supported continuous functions to test the convergence) generally speaking. We can use the following simple example to illustrate this point.

Consider in the Euclidean ball $B_2\subset\mathbb{C}^n$. We have seen before that the (Borel) measure $(\sqrt{-1}\partial\bar{\partial}u_{\bar{B}_1})^n$ is supported on ∂B_1, where $u_{\bar{B}_1}$ is the relative extremal function of $\bar{B}_1$ with respect to B_2. Now consider the convolution of the function $u_{\bar{B}_1}$, u_ϵ. At this moment, it should be easy to see $(\sqrt{-1}\partial\bar{\partial}u_\epsilon)^n$ converges to $(\sqrt{-1}\partial\bar{\partial}u_{\bar{B}_1})^n$ in the sense of current (distribution) as $\epsilon\to 0$ [60]. Since the measures correspondent to u_ϵ's are smooth, the integrals of them over ∂B_1 should be 0. But the integral of the limiting distribution (measure) over ∂B_1 is equal to $Cap(\bar{B}_1,B_2)$ from the results introduced before which is clearly positive. Hence there is no convergence in the sense of measure like that in general case.

So it is not justified to use the convergence of the pullback measures on X. But if a positive top degree distribution is $=$ or $\geqslant$ to another one in the sense of distribution, then this relation also holds in the sense of measure by definition. This is of course a trivial but handy fact for us.

[60] We can say this over B_2 as $\epsilon\to 0$. The convolution which is defined in a slightly smaller set but enlarging to $\bar{B}_2$.

As we see it, the result itself considering the pullback equation on X is so natural, and so it should be true. In fact a more careful global argument will justify it as follows.

For convenience, we set $\omega = \frac{1}{k}\omega_{FS}$. Let's take a look at how we get v first.

Consider the equation on $P(X)$: $(\omega + \sqrt{-1}\partial\bar{\partial}v)^n = e^v P_\star\Omega$.

Take a sequence of functions $\{F_j\}$ such that

$$F_j \in C^\infty(P(X)), \quad F_j > 0, \quad F_j \to \frac{P_\star\Omega}{\omega^n} \quad in \quad L^p(P(X)) \quad as \quad j \to \infty.$$

We start with solving the approximation equations,

$$(\omega + \sqrt{-1}\partial\bar{\partial}v_j)^n = e^{v_j}F_j\omega^n.$$

And the solution v for the original equation is basically the pointwise limit of v_j almost everywhere as $j \to \infty$ after taking a subsequence if necessary, which is in fact the everywhere pointwise limit for the first convergence below from the property of plurisubharmonic functions. Let's set

$$U_j = (max_{k\geqslant j}v_k)^*, \quad W_j = min_{k\geqslant j}v_k, \quad E_j = min_{k\geqslant j}F_k.$$

It's easy to see from definition that

$$U_j \to v \;\; decreasingly, \;\; W_j \to v \;\; increasingly, \;\; E_j \to \frac{P_\star\Omega}{\omega^n} \;\; increasingly$$

almost everywhere on $P(X)$ as $j \to \infty$.

Furthermore, it is OK to pull back the approximation equations to X as follows since they all contain only smooth objects:

$$\big(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(P^\star v_j)\big)^n = e^{P^\star v_j}P^\star F_j{\omega_\infty}^n.$$

And we also have the correspondent convergences:

$$P^\star U_j \to P^\star v \; decreasingly, \; P^\star W_j \to P^\star v \; increasingly, \; P^\star E_j \to \frac{\Omega}{{\omega_\infty}^n} \; increasingly$$

almost everywhere over X and in fact the convergence is everywhere for the first one as mentioned before. It might be a good place to point out the trivial fact that the pullback of a plurisubharmonic function by a holomorphic map is still plurisubharmonic (see [Le], here the situation of pullback being $-\infty$ clearly won't happen.).

It is quite easy to see:

$$\left(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(P^\star U_j)\right)^n \geqslant e^{P^\star W_j} P^\star E_j {\omega_\infty}^n$$

by applying the classic fact about the measure from the function $max\{u, v\}$ with u and v plurisubharmonic, and the weak convergence of current coming from the monotonous convergence of potential which is stated before. And of course, the inequality for two positive currents will also give the inequality in the sense of measure.

Now by taking $j \to \infty$, we arrive at

$$\left(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(P^\star v)\right)^n \geqslant e^{P^\star v}\Omega.$$

Here we've used the monotonicity of the (increasing) convergences to draw the convergence for the right hand side. Finally, one uses

$$\int_X e^{P^\star v}\Omega = \int_{P(X)} e^v P_\star \Omega = [\omega]^n = [\omega_\infty]^n = \int_X \left(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(P^\star v)\right)^n$$

to get $=$ from $\geqslant$. The first and third $=$'s above comes from the birationality of P and the rest two make use of the fact that v is bounded. Of course, $[\omega]^n$ and $[\omega_\infty]^n$, which are topological pairings, also make sense as integration of smooth forms over closed manifolds.

Anyway, now we can conclude that

$$\left(\omega_\infty + \sqrt{-1}\partial\bar{\partial}(P^\star v)\right)^n = e^{P^\star v} P^\star P_\star \Omega.$$

The birationality of P and boundedness of $P^\star v$ will justify that the right hand side is the same as $e^{P^\star v}\Omega$ in the sense of measure. So we can say $u = P^\star v$ is a (continuous) solution for the original equation (over X) being considered.

For this case, we can have the uniqueness of such (continuous) solution from the simple argument using comparison principle since Ω is a smooth (nondegenerate) volume form over X.

Remark 3.4.2. *We only know the solution is continuous by the discussion in this section. Actually we can have more properties of it.* [61] *In fact, we can see that this solution is actually the same one got from flow or perturbation methods used in the*

[61] We have already assumed that $P(X)$ is smooth and P is birational to the image up to now.

previous chapter. One just has to prove that solution got there is continuous (or even just bounded [62]*) in order to apply classic versions of comparison principle and go through the simple argument above. Thus we can combine the properties got from all these methods.*

Actually this also the reason why we assume that Ω is a smooth volume form which can clearly be weakened only for the argument in this section. Of course, then the uniqueness argument won't be so trivial as above.

The discussion in this section up to now is in the direction of pullback. Strictly speaking, we have only treated the existence of continuous solution for the original equation over X using it.

Actually the other direction can also be treated in this case. More precisely, start with any bounded solution for the original equation over X. We can push forward it to the smooth image, $P(X)$ [63]. Using classic extension results for plurisubharmonic functions, we can easily see the pushforward function would be a bounded solution for the corresponding equation over $P(X)$ considered before. And clearly these two operations, pullback and pushforward (of the solutions), are indeed inverse to each other. So we can get everything for the equation over X in this degenerate case just as in Kolodziej's case.

Furthermore, it is quite straightforward to justify that the argument quoted before from Kolodziej's works can be well adjusted to the case when X is an orbifold. Of course then all the things involved in the equation would be required to be compatible with the orbifold structure.

Basically, instead of considering a domain in $\mathbb{C}^n$, now we need to consider the quotient of it by a finite group. It might seem to be quite different, but in fact all the essential discussion would still be done in the domain contained in $\mathbb{C}^n$, i.e., using the orbifold coordinate chart, and all the notions from pluripotential theory are still for this domain.

With this observation, we can see that the previous results discussed in this section will still hold when the image, $P(X)$, is a Kähler orbifold with the restriction of ω_{FS} being an orbifold Kähler metric and the map P also being compatible with the orbifold structure [64].

[62] Actually, we do not need any more informationm than we already have now to see they are the same in sight of the discussion for unbounded functions at the end of the previous chapter.

[63] Birationality is important to justify this.

[64] In order to preserve nice pullback property. In fact, later we'll also consider the case of oribifold image when the orignal argument in [Koj1] is generalized. At that time, we'll need much less

This ends our attempt to directly apply the results from Kolodziej's works. Later on, we'll try to generalize the original arguments to prove more general and interesting results, i.e., Theorem 1.3.2 in Introduction.

compatibility with the orbifold structure, for example, the map P doesn't have to be a map between orbifolds.

Chapter 4

Generalized Results and Strategy to Prove

Recall the (degenerate Monge-Ampere) equation we want to study is the following:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

over a projective [1] manifold X, where $[\omega_\infty]$ is big and semi-ample, (not ample though, i.e., $\omega_\infty \geqslant 0$ with $\{{\omega_\infty}^n = 0\}$ nonempty and not X itself) and Ω is a smooth volume form for X.

In this chapter, we want to see which parts of Kolodziej's original argument quoted in the previous chapter need to be modified in order to be carried out similarly for our case above. The main goal is still to find a bounded (and even continuous) solution of this equation.

4.1 Generalization

This equation above has e^u on the right hand side whose affect has been discussed before. In other words, for L^∞ estimate (of smooth approximation solution), little effort is needed to translate the result for a proper equation without e^u to that for this equation. So instead of the equation above, let's consider the following equation

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = \Omega$$

[1] The projectivity is required to be compatible with the terminologies in algebraic geometry. We can consider general closed Kähler manifolds using a more general picture as used in Theorem 1.3.2 which will be explained later.

with the smooth volume form Ω satisfying $\int_X \Omega = \int_X {\omega_\infty}^n > 0$, which is more like a generalization of the main equation considered by Yau a long time ago in [Ya]. This equation is also the main object in Kolodziej's argument quoted in the previous chapter.

Just as in Part (4) of Kolodziej's argument, the solution is supposed to be obtained by taking the limit of solutions for a family of approximation equations. Of course we want the solvability of the approximation equations to be classic (well known). This strategy is natural and I haven't found any other essentially different way.

Unlike the situation before where the difficulty is from the general right hand side of the equation, our trouble now comes from the semi-positivity of ω_∞.

Remark 4.1.1. *Actually, for most consideration using pluripotential theory, there is no need for Ω to be a smooth (nondegenerated) volume form. We merely need it to be a nonnegative L^p volume form just as in the original argument quoted before. And for the approximation equations, one only needs to combine the approximation of Ω by smooth volume forms with the additional approximation which will be introduced below. Indeed, by directly applying Kolodziej's results, we don't have to make the measure on the right hand side of the equation to be regular in order to have a (unique) continuous solution for the following approximation equations.*

We have already got a natural family of approximation equation as used in the discussion about one of the perturbation methods in Chapter 2 as follows [2]:

$$(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = C_\epsilon\Omega$$

where $\omega > 0$ and $\int_X C_\epsilon\Omega = \int_X(\omega_\infty+\epsilon\omega)^n$ for $\epsilon \in (0,1]$. Obviously, we have $C_\epsilon \in (1, C]$.

From the classic (or Kolodziej's when Ω is not so regular) results, we have no trouble to find a unique (continuous) solution for each of these equations after requiring the normalization $sup_X u_\epsilon = 0$ (or $max_X u_\epsilon = 0$).

Just as before, we only have to prove that u_ϵ's are uniformly bounded (from below) in order to get a bounded solution for the equation

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = \Omega$$

[2]This strategy also works when using flow method or other perturbation methods. One just notices the background metrics can be made in the form $\alpha\omega_\infty + \omega$ where α is a positive constant and ω is a changing metric, so the generalized argument below for boundedness (and other properties) still works.

by taking a limit of (a sequence of) u_ϵ's as $\epsilon \to 0$. [3] There are also some other properties we want to prove for the possible solution got like that. But as one can easily observe in the quoted argument in the previous chapter, the boundedness argument is really the heart of the whole program. We shall see later that it is almost the same case for our case. For readers' convenience, let's restate the main theorem (Theorem 1.3.2) that we prove in this part of the thesis by pluripotential-theoretic argument below.

Theorem 4.1.2. *Let X be a closed Kähler manifold with $dim_{\mathbb{C}}X = n \geqslant 2$. Suppose we have a holomorphic map $P : X \to \mathbb{CP}^N$ with the image $P(X)$ of the same dimension as X. Let ω_M [4] be any Kähler form over some neighbourhood of $P(X)$ in $\mathbb{CP}^N$. For the following equation of Monge-Ampere type:*

$$(P^*\omega_M + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega, \tag{4.1}$$

where Ω is a fixed smooth volume form over X and f is a nonnegative function in $L^p(X)$ for some $p > 1$ with the proper total integral over X (i.e., $\int_X f\Omega = \int_X (P^*\omega_M)^n$), we have the following:

(1) (Apriori estimate) Suppose u is a weak solution in $PSH_{P^*\omega_M}(X) \cap L^\infty(X)$ of the equation with the normalization $sup_X u = 0$, then there is a constant C such that $\|u\|_{L^\infty} \leq C\|f\|^n_{L^p}$ where C only depends on P, ω_M and p;

(2) (Existence of bounded solution) There would always be such a bounded solution for the equation;

(3) (Continuity and uniqueness of bounded solution) If P is locally birational, then any bounded solution is actually the unique continuous solution.

The understanding of the above statement should not be a problem now after all the previous discussion in pluripotential theory. Let's emphasize that we require an extra assumption for (3) which is the local birationality of the map P. The meaning of it should be self-evident at least intuitively and we have explained it in Introduction. It would be better to explain this in the actual argument. We also know that this extra assumption is not that restrictive and will be satisfied in most interesting geometric pictures.

[3] In the original equation interested in, there is the term e^u on the right hand side of the equation. We can still see that the limit, which would officially be the upper semi-continuization of the upper limit, indeed satisfies the equation. The crucial idea has already appeared in the previous chapter where by taking subsequence if necessary, the limit would also be pointwisely almost everywhere as in [Ho].

[4] The "*M*" is the initial of "*model*". ω_M can be understood as the model metric of original degenerated metric interested in.

Remark 4.1.3. *Let's point out that the integral* $\int_X (P^*\omega_M)^n$ *is actually positive. This is very easy to see by Sard's Theorem since the dimension of* $P(X)$ *is the same as that of* X.

For projective X*, the bigness and semi-ampleness of* $[\omega_\infty]$ *is to provide us with a map* P *as in the theorem. We require the image to be in* $\mathbb{CP}^N$ *in order to control the image* $P(X)$ *a little bit.*

To connect with the equation considered at the beginning, we put $\omega_\infty = P^*\omega_M$ *which is semi-positive.*

In the following several sections, we discuss the idea of the proof for this theorem, focusing on the differences from Kolodziej's case and the arising difficulties. Brief description about the way to treat them will also be provided.

4.2 Apriori L^∞ Estimate

In this section, we consider (1) in Theorem 4.1.2 above. Let's consider the apriori L^∞ bound for those approximation solutions first.

We can't just use the original argument of Kolodziej's since $\omega_\infty + \epsilon\omega$ is no longer uniformly positive for $\epsilon \in (0, 1]$ which means if we consider the local potentials in coordinate balls, they will be no longer uniformly convex. And then we can not have a uniform "D" for the interval "$[S, S + D]$" considered there which is of course very crucial for the (uniformality of) argument. This actually says that the picture of a coordinate ball is too local as we see it now.

A closer look will tell us that this will only cause trouble when the minimum of u_ϵ occurs close to the set $\{{\omega_\infty}^n = 0\}$. An important observation is that if one chooses a domain which has those degenerated directions (of ω_∞ as metric) going around inside, we may still have the uniform convex potential for the domain, i.e., the values for the very outer part are greater than those of the central part by a uniform positive constant.

More precisely, assume each component of $\{{\omega_\infty}^n = 0\}$ is mapped to a point by the map P. If we take a ball around that point in $\mathbb{CP}^N$, then the preimage of that ball in X would be a neighbourhood, V, of the component we start with. As $\omega_\infty = \frac{1}{k}P^*\omega_{FS}$, we have the (global) potential of ω_∞ in V be convex in the sense above. This domain V can no longer sit in $\mathbb{C}^n$ since it contains a closed subvariety.

Furthermore, we can see the domain V is also hyperconvex in the usual sense which is quite important at several places for the original argument in [Koj1] as we've seen before. Of course there are a lot of other pullbacks on V of plurisubharmonic functions in that ball in $\mathbb{CP}^N$. Though we've observed before that apparently hyperconvexity may not be that essential at those places, it still give us the confidence to treat this general domain V just as a nice domain in $\mathbb{C}^n$.

It seems the treatment above will cause another problem since the perturbing metric ω may (and should) not have a global potential in the domain V. But we can deal with this by considering plurisubharmonic functions in V with respect to $\omega_\infty + \epsilon\omega$ for each $\epsilon \in (0, 1]$. In fact we can also include the case when $\epsilon = 0$ in the discussion where they essentially coincides with the usual plurisubharmonic functions since ω_∞ has a global potential in V. The idea is that our argument would actualy be uniform for all such ϵ's once we get for the worst one, $\epsilon = 0$ (or say ω_∞ which is degenerate as metric).

From all the previous consideration, we see that in order to go through similar argument for uniform boundedness of the approximation solutions in our situation, it's necessary (and sufficient) to have all those results in Parts (1) and (3) for the domain V which is no long inside $\mathbb{C}^n$ together with the CLN inequality which is used to control $Cap\big(U(S + D), V\big)$. Part (2) will be the same since it only contains algebraic computation using the result from Part (1) [5].

We can easily have the version of CLN inequality over the above domain V from the locality of the result itself. [6] We just need to get the uniform bound of L^1-norm for u_ϵ's. As before, that should just be an application of Green's formula using the fact that $sup_X u_\epsilon = 0$. Let's provide some details since at first sight the changing background metric $\omega_\infty + \epsilon\omega$ might look troublesome.

For the L^1-norm, we choose to use the smooth volume form ω^n, and in fact we use the Green's function for ω. For fixed ϵ, suppose $u_\epsilon(x) = 0$ [7] and $C > G_\omega$ where G_ω is the Green's function for the metric ω. Since $\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon > 0$, we have

$$\Delta_\omega u_\epsilon = \langle \omega, \sqrt{-1}\partial\bar{\partial}u_\epsilon \rangle > -n\epsilon - \langle \omega, \omega_\infty \rangle \geqslant -C$$

where C is uniformly chosen for $\epsilon \in (0, 1]$. Basically, this tells that there should be no worry for the changing background metric. Then we have the following standard

[5] We'll see later there is a little difference here from the slightly different result for Step (1) with background metrics. But it won't affect the main argument.

[6] We can just divide V further into domains in $\mathbb{C}^n$. The inequality for V can be got by taking summation. The background metric and the local potential are changing in a very controllable way.

[7] u_ϵ should achieve its maximal value only by upper semi-continuity.

computation:

$$\begin{aligned} 0 = u_\epsilon(x) &= \int_X u_\epsilon \omega^n + \int_{y\in X} G_\omega(x,y)\Delta_\omega u_\epsilon \cdot \omega^n \\ &= \int_X u_\epsilon \omega^n + \int_{y\in X} \big(G_\omega(x,y) - C\big)\Delta_\omega u_\epsilon \cdot \omega^n \\ &\leqslant \int_X u_\epsilon \omega^n - C\int_{y\in X} \big(G_\omega(x,y) - C\big)\omega^n \\ &\leqslant \int_X u_\epsilon \omega^n + C, \end{aligned}$$

which gives the uniform L^1 bound for u_ϵ's by noticing that they are all nonpositive. [8]

Part (3) looks like a difficult one to be generalized since the geometry for $\mathbb{C}^n$ is fairly involved in the original discussion. It'll be our main concern later.

4.3 Existence of Bounded Solution

Once we get all the necessary preparations above done, Part (4) can be directly carried through for our case and provides the uniform boundedness of the approximation solutions. Hence we can get a bounded solution for the degenerated Monge-Ampere equation by taking a limit just as in Kolodziej's where the argument is very local.

Remark 4.3.1. *Indeed, u_ϵ is essentially decreasing as $\epsilon \to 0$ [9] which will make the limit easier to take. More details about this and some uniqueness results for bounded solutions will appear later.*

4.4 Continuity of Bounded Solution and Stability

In Part (5) of the original argument, we need a local smooth (continuous) approximation of the bounded solution. At that time, we have no trouble since we have convolution in $\mathbb{C}^n$. But now we also need to consider the domain V where convolution is not available. This makes the whole argument more involved.

[8] The subtlety about less regular function arises similarly as before and can be treated in the same way.

[9] We have this by maximum principle argument when the volume for Ω is good. Here we are saying that it's also the case when Ω is less regular.

Remark 4.4.1. *There are quite some classic and recent results about constructing a decreasing sequence of smooth approximation functions for a (bounded) plurisubharmonic function. We might get into details of some of them later. But it seems to me that the positivity of the background form is very crucial for these results. It's slightly different from the situation about comparison principle which will be discussed later.*

The first attempt is to restrict us to the solution got from above since we have some kind of approximation for it. Indeed, for the equation

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega,$$

we have, from before, a smooth sequence of u_ϵ decreasing to it globally on X with the background metric linearly converging to ω_∞. Now for simplicity, we are considering the case when Ω is a smooth volume form. In fact this is essentially also the case in general as mentioned above. This looks enough for us at the first sight provided the uniform L^∞ bound is available.

But there is a rather serious problem here. Recall in the argument of Part (5), finally, we need to use the results from Parts (1) to (3) for the sets similar to "$\{u_\infty - s < u_\epsilon\}$". At this moment, they are plurisubharmonic with respect to different background metrics, ω_∞ and $\omega_\infty + \epsilon\omega$. More importantly, u_ϵ is not plurisubharmonic with respect to ω_∞. Of course we might want to use $\omega_\infty + \epsilon\omega$ for both, but then we don't have

$$(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = F\Omega$$

with some nonnegative $F \in L^p$ for some $p > 1$, i.e., the correspondent Condition (A) is not quite justified. In fact, if we can justify this condition, then the same result as in [Koj1] would have given the continuity of this solution. However we can trivially see $F \in L^1$ as follows.

The global integration is OK from the boundedness of u_∞ which means the left hand side above makes sense as a (Borel) measure. And clearly we can have F positive and smooth out of $\{\sigma = 0\}$ for the regularity of the solution u by defining it from the equation itself. Moreover, the integral of it over this range is finite and obviously equal to that of the left hand side of the equation. Finally, since the set $\{\sigma = 0\}$ is pluripolar (and measure 0), it won't contribute for the total measure of $(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\infty)^n$. Here the boundedness of u_∞ is used to justify this. We know also any L^1 function would have the property that values over a Lebesgue measure 0 set are not important. Hence with a proper chosen (*a.e.*) positive L^1 function F, both sides are (Borel) measures and they are equal as Borel measure. But L^1 is not enough for us.

Be careful that we do need the regularity out of $\{\sigma = 0\}$ in order for the measure to be L^1. In general, boundedness of potential would give finite total integral over X, but the measure might have nonzero contribution from Lebeague measure 0 sets as illustrated also by the example before.

Remark 4.4.2. *It looks like we are facing the same problem about the degenerate estimates as in Chapter 2. In fact, if we want to use the degenerate control for Laplacian there, the assumption for the power is less restrictive (better) for $F \in L^p$ for some $p > 1$ than for the solution to be in some proper Sobolev space which would also gives the continuity.* [10] *But our argument here requires the boundedness of the potential and being in the proper Sobolev space, which is more restrictive, actually gives Hölder continuity of the solution u which is a stronger result.*

However, if we can have the singularities of F along $\{\sigma = 0\}$ be like (not just controlled by) $|\sigma|^{-\alpha}$, then $F \in L^1$ would imply $F \in L^p$ for some $p > 1$. This observation is used before in direct application of Kolodziej's results for some special degenerate situations.

There seems to be another problem for the equation

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = \Omega$$

with the volume form Ω having the correct total volume over X. Obviously we can get a bounded solution by going through the above stategy. But apriori, it is hard to get the monotonicity of the approximation even when Ω is smooth and we can use maximum principle argument for the approximation solutions. This is where the final remark for Part (5) about essential decreasing of the approximation will help us a little. More details will be provided later.

Remark 4.4.3. *There is a subtle difference of the discussion above and the original one in Part (5). Namely, at that place, we can prove that any bounded solution is actually continuous, i.e., we do not care how the solution comes up. But the discussion above can only apply to the solution got by approximation. Of course, if we can prove*

[10] The estimate related here is the following Laplacian estimate:

$$\langle \omega, \omega_\infty + \sqrt{-1}\partial\bar{\partial}u \rangle \leqslant C|\sigma|^{-\alpha}.$$

For the argument above, we basically need $|\sigma|^{-(n-1)\alpha}$ to be integrable. For u being in some proper Sobolev space, $W^{2,q}$, one would need $|\sigma|^{-n\alpha}$ integrable where $2n$ is the real dimension.

the uniqueness for the boundedness (which is actually done later in this work for the main case being considered), then this would not be a problem. This attempt above doesn't seem to very successful anyway.

Actually, we seems to be in the wrong track to prove the continuity result. Professor Kolodziej pointed out a classic extension result in [FoNar] which says a weakly plurisubharmonic function over a (singular) variety can be extended locally to the ambient space $\mathbb{C}^N$. [11] In the current situation, if we can push forward the solution which is plurisubharmonic on X to $P(X)$ by the map P, then we might be able to get a local continuous approximation using convolution locally on $\mathbb{CP}^N$. Notice that local continuous approximation is just what we need to go throught Kolodziej's orginal argument for continuity. In order to push forward any (bounded) plurisubharmonic function, we require the local birationality [12] of the map P which is really not that restrictive.

Once we have the continuity result, it would be easy to see that the further discussion about stability works without any modification. But since there seems to be a little more assumption about the continuity, we also want to consider the stability result for solutions which are merely bounded. [13] So in the following, we ignore the continuity result from above.

Though the argument for stability for continuous solution in Part (6) seems to be OK by itself [14], it might be completely meaningless since we do not have the continuity of the solution got before.

We can actually see that there is only one obstacle to carry through the stability argument for bounded solution in this case which is just the fact used there that is trivial for smooth case and proved for continuous case by approximation argument using results about Dirichlet problem. But for bounded case, it's not justified for now or even may not be true. A little more effort using approximation can give us the uniqueness (or stability) result for all the solutions we can get (by approximation). But that's still not for any bounded solution.

[11] With Professor H. Rossi's help, I am able to understand the argument there.

[12] Basically, we do not want any serious branching for the map P.

[13] In fact, the continuity result took me quite a long time to prove. During the process, I also thought about the stability result in case we might not have continuity of solution got by approximation. There seems to be something interesting and I would like to point it out.

[14] It looks like that we need the existence of a continuous solution for the original argument in [Koj2], but later we'll see that it can be easily dodged by approximation argument.

Rather differently, for the equation

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega,$$

we can have the uniqueness result for bounded solutions once we can justify the application of comparison principle. Somehow, we can do this by introducing a fairly involved approximation. Here Ω needs to be nonzero almost everywhere.

4.5 Comments about Local and Global Arguments

As in [Koj2], we might want to use a global argument to prove the boundedness of the solution. And in fact the same idea there can be carried out without major modification. The main steps are just as before. Now we just take the domain V to be the whole manifold X. The background metric has to be kept along the way. But in fact they are not going to bring any essential difference. The closedness of X would easily allow us to patch things up once we've known the local pieces well enough. For our situation, we still need to study the domain V which contains degenerated variety (directions) of ω_∞. The nondegeneracy of directions going out of V would make it easy to globalize whatever we get for V to the whole of X just as discussed in [Koj2].

The good point for global argument is that we don't have to go through the point-picking as mentioned before. Hence it looks more concise. And globally on X, the justification of comparison principle for bounded functions would also be easier since there is no boundary to worry about.

Basically, we have both local and global arguments to prove apriori L^∞ estimate for bounded solution and existence of bounded solution. But for continuity, until now, I can only prove it by local argument since the approximation mentioned before using results in [FoNar] is only local. Anyway, the differences are rather superficial for our argument. The details for the global argument have been carried out in [Zh] [15], and in this work, we'll mainly focus on the local argument which in my opinion, illustrates the idea in a better way.

[15] We include part of its main argument, which contains all the details about the technical differences from the local argument below, in Appendix for readers' convenience.

4.6 Application of Boundedness Result

Before going into the details, let's point out some implication of the boundedness result, which might motivate readers a little.

From Chapter 2, using Kähler-Ricci flow or perturbation methods, we can get a solution for the following degenerate Monge-Ampere equation:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

where $[\omega]$ is nef. and big, Ω is a smooth volume form. The solution would be smooth out of the stable base locus set of $[\omega_\infty]$ with possible singularities along this variety which are described by some degenerate estimates.

Now in the case when the class $[\omega_\infty]$ is also semi-ample, we can further have the boundedness of the solution u. By classic results in pluripotential theory, we can have all kinds of positive currents and Borel measures with the global integrals over X to be the natural (finite) ones, for example

$$\int_X (\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = \int_X {\omega_\infty}^k \wedge \omega^{n-k}, \quad k = 1, \cdots, n$$

where ω is a smooth real closed nonnegative $(1,1)$-form. Of course, the integrals would be the same when restricted to the regular part of u because the stable base locus set is pluripolar. But in fact, those measures would just be the natural smooth ones when restricted to the regular part. So the integrals over that part would also be finite, which are not completely available only from the degenerate estimates got before by maximum principle argument.

Also as we can see in the arguments quoted from [Koj1] and [Koj2], the boundedness is important in applying classic results in pluripotential theory to further study the solution itself (continuity, uniqueness for example). Though there are some results about unbounded plurisubharmonic functions, boundedness is very welcome for the whole business.

Chapter 5

General Adjustment of Original Argument

In this chapter, we adjust Kolodziej's original argument quoted before to our current situation. Most of the modifications are still classic and can be found in [Koj2].

5.1 Obvious Generalization of Classic Results over General Domains

As explained in the previous chapter, in order to go through similar argument as in [Koj1] for our degenerate case, we need to justify those classic results used there, which are originally for plurisubharmonic functions defined over domains in $\mathbb{C}^n$, for functions defined in some general domains and plurisubharmonic with respect to some background (Kähler) metrics with possible degeneration. More precisely, the general domains are (connected) open subsets of the closed manifold, X and the background metrics are $\omega_\epsilon := \omega_\infty + \epsilon\omega > 0$ for $\epsilon \geqslant 0$.

In this section, we are going to see that some of those results are trivially true in this situation by their own locality and the following observation.

The (general) domain can be covered by finitely many coordinate balls and the corresponding local potentials for the background metrics (i.e., background potentials) can be uniformly controlled since everything is on a closed manifold. Though the background metric is changing, the background potentials can be chosen to vary in a mild way such that their own values will cause no trouble for us [1].

[1] Actually we can even manipulate the choice so that the background potentials are increasing or decreasing as we wish.

Now let's list these results. In the following, the potentials u, etc. stand for the functions in $\omega + \sqrt{-1}\partial\bar{\partial}u$ where this current is positive with ω only needs to be a smooth real closed $(1,1)$-form. The item b') below is the correspondent one for b) in the Chapter 3, similar for the others. Notice all the functions appearing are not required to be continuous [2].

b') Relative capacity can be defined in the same way by using positive current $\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u$ instead of $\sqrt{-1}\partial\bar{\partial}u$ as before and also imposing condition on u to be PSH_ω [3] instead of PSH. It's natural to require $\omega_\epsilon \geqslant 0$ in order to make sure such u always exists. Those basic properties as for the classic notion will trivially hold.

Remark 5.1.1. *Obviously we can still extend the definition of relatively extremal function to this case. But then we don't have the relation of these two notions as before. Basically, we don't have the description about the support of the measure correspondent to thus defined relative extremal function since the background potentials will get involved in the pluriharmonic lifting argument used before. But this is not a big deal to us as we have already realized before that the notion of relatively extremal function is actually not that necessary even for the original argument.*

When the background "metric" is ω_∞ which has a global potential for the domain we are considering and so can be taken as a bounded plurisubharmonic function, the treatment can be even more flexible. Actually it can easily be proved that the relative capacities defined with or without ω_∞ will be equivalent up to positive constants. We won't use this fact explicitly, though it is possible to make the argument nicer by taking this into account.

As mentioned before, with only Richiberg's method of approximation at hand, in order to go through the original argument for our case here, we'll have to clarify the difference if we only use continuous plurisubharmonic functions in the definitions. Of course, it turns out to be not so important once we have more advanced methods of getting approximation for plurisubharmonic functions.

c') Weak convergence results of currents from the monotonous convergence of potentials is OK since for this result, everything is local. Here the general case when

[2]The proof of all the results here does not need comparison principle which might need continuity generally speaking as discussed in details later.

[3]Forget the requirement about continuity for now. In fact we can still require u to be PSH as before without changing most of the argument below. The relation between the different definitions is quite obvious.

the convergence of potentials is with respect to capacity can also be considered and is true indeed. The original argument in [Koj2] works perfectly if one includes some fixed background metric.

d') CLN inequalities are basically OK with minor changes, i.e., on the right hand side, we should change $\|u_i\|$ to $\|u_i\|+C_i$ where C_i is coming from the fixed background potentials. The result appearing in the discussion of d) will also have its counterpart in this case.

5.2 Comparison Principle

In this section, we discuss comparison principle for general domains not necessary in $\mathbb{C}^n$. Notice that the statement of comparison principle is rather global. So local argument which has been very successful in the previous section will no longer be enough for justifying it.

In that brief discussion of the proof of classic comparison principle, we need a good approximation of those plurisubharmonic functions which are compared. In case of a domain in $\mathbb{C}^n$, we can use convolution, which is not available for general domains. This is our main concern here.

After all the discussions below, we should be able to apply comparison principle in all the places needed.

5.2.1 Restriction to Continuous Functions

Let's first point out that for our argument, it is only necessary to be able to compare functions like relative extremal functions and smooth plurisubharmonic functions since we basically only consider sets like $\{u < v + s\}$. Moreover, for the definition of relative extremal function, we would like to use functions in $PSH(V) \cap C(\bar{V})$ instead of $PSH(V)$. As described before, for compact sets, we can easily see this definiton would be equivalent to the original definition when considering a hyperconvex domain in $\mathbb{C}^n$. In fact we also want to see similar results for relative capacity. But the situation for it is a little bit more involved as follows.

For K compact in a bounded open set $V \subset \mathbb{C}^n$ which is hyperconvex. Let $u_{_K}$ and $u'_{_K}$ be the original relatively extremal function and the one using only continuous functions respectively. Similarly the two notations, $Cap(K)$ and $Cap_c(K)$, stand for the two relative capacities where the latter one has the lower "c" to indicate the use of $PSH(V) \cap C(\bar{V})$ instead of $PSH(V)$. The (background) domain V is fixed and so we omit it in the expression of capacities for simplicity. Notice that we are considering a domain V in $\mathbb{C}^n$ at this moment.

One direction is trivial by definition,

$$u_{_K} \geqslant u'_{_K}, \quad Cap(K) \geqslant Cap_c(K).$$

The other direction will be proved below. From the hyperconvexity of V, we have that there exists $h \in C(V') \cap PSH(V')$ where V is relatively compact in V', $V = \{h < 0\}$, and $h = 0$ on ∂V. Clearly one only needs to consider any $u \in PSH(V)$ such that $u \leqslant 0$ on V and $u = -1$ on K for $u_{_K}$. We can extend such a function plurisubharmonically to some neighbourhood of V as follows.

Define a function v_ϵ by $Ah + \epsilon$ on $V' \setminus V$ and $max\{u, Ah + \epsilon\}$ over V. For $\epsilon > 0$ sufficiently small and $A > 0$ large enough, it is easy to see that $v_\epsilon = u$ near K and $max\{v, Ah + \epsilon\} = Ah + \epsilon$ near ∂V. Thus v_ϵ is clearly plurisubharmonic in V'. For any small constant $\delta > 0$, by taking V'' to be a sufficiently small neighbourhood of V, we can have $v_\epsilon \in [-1, \delta + \epsilon]$ over V''.

Also as $\epsilon \to 0$, $v_\epsilon \to v$ for $v \in PSH(V')$ and $v \in [-1, \delta]$ over V''. Moreover, $v = u$ near K and $v = 0$ on ∂V [4]. And of course it would still suffice to consider functions like $max\{u, Ah\}$ for $u_{_K}$ since the information near K is completely preserved.

Now by taking convolution of v, we have functions $v_j \in PSH(V) \cap C(\bar{V})$ valued in $[-1, \delta]$ such that $v_j \to v$ decreasingly on $\bar{V}$ (as $j \to \infty$ of course, also for the other convergences in the following). Of course. we have $\frac{v_j - \delta}{1+\delta} \to \frac{v-\delta}{1+\delta}$ where $\frac{v_j-\delta}{1+\delta}$ is valued in $[-1, 0]$ which is good in view of the definition of relative capacity.

Simply by using Dini's Theorem, we can see v_j converges uniformly to v over $\partial V \cup K$ since $v = 0$ on ∂V and $v = -1$ on K. From this, by maximum principle, it is trivial to see $u'_{_K} \geqslant v \geqslant u$ [5]. Thus we can conclude $u_K = u'_K$. [6]

The situation is a little bit more involved for relative capacity as we'll see now. For any function $u \in PSH(V)$ valued in $[-1, 0]$, we can still have the extension of it as v above which is equal to u near K and valued in $[-1, \delta]$ in some small neighbourhood of $\bar{V}$. Using convolution, the decreasing convergence of $\frac{v_j-\delta}{1+\delta}$ to $\frac{v-\delta}{1+\delta}$ clearly gives the weak convergence in the sense of distribution. But we can not say the integral over

[4]In fact v is just the extension of the function $max\{u, Ah\}$.

[5]Here the uniform convergence will take care of the values. There is just a little modification involved anyway.

[6]This has been discussed before. Because the auxiliary functions will also be used below, we give the argument again for convenience.

K also converges. In fact it can't be true if you consider K in lower dimension as in the example before. But we can see that for any open set U containing K, for j large enough,

$$\begin{aligned}\int_U (\sqrt{-1}\partial\bar{\partial}(\frac{v_j-\delta}{1+\delta}))^n &\geqslant \int_V \phi(\sqrt{-1}\partial\bar{\partial}(\frac{v_j-\delta}{1+\delta}))^n \\ &\geqslant \int_V \phi(\sqrt{-1}\partial\bar{\partial}(\frac{v-\delta}{1+\delta}))^n - \epsilon \\ &\geqslant (1+\delta)^{-n}\int_K (\sqrt{-1}\partial\bar{\partial}u)^n - \epsilon,\end{aligned}$$

where ϕ is a properly chosen cut-off function and ϵ is any fixed (small) positive constant.

Thus we have $Cap_c(U) \geqslant (1+\delta)^{-n}Cap(K)$, and so $Cap_c(U) \geqslant Cap(K)$. Hence we can conclude

$$Cap_c^*(K) \geqslant Cap(K)$$

where the upper "$*$" represents the outer capacity using Cap_c. More precisely,

$$Cap_c^*(K) = inf\{Cap_c(U)|K \subset U, U \text{ open }\}.$$

Actually the following chain is obvious

$$Cap^*(K) \geqslant Cap_c^*(K) \geqslant Cap(K) \geqslant Cap_c(K)$$

where "Cap^*" is the outer capacity defined using "Cap", but the first and third $\geqslant$ are the same in our setting (cf. [Koj2]), so the first two "$\geqslant$" are actually "$=$".

Recall that for the discussion of original relative capacity "Cap", there is no need to require the compactness of K, so it is true for any E relatively compact in V. But from our argument above, as in the definition of outer capacity, it seems that we should only take open U such that E is relatively compact in it. Actually it is easy to see that we don't need this for the inequality $Cap_c^*(U) \geqslant Cap(U)$ to be true since:

$$Cap_c(U) \geqslant Cap_c^*(K) \geqslant Cap(K)$$

for any compact set $K \subset U$. Hence $Cap_c(U) \geqslant Cap(U)$ from the property of $Cap(U)$ (i.e., it can be approximated by those of K's), then obviously it should be equal since the other direction is trivial. So we can actually define $Cap_c^*(U) = Cap_c(U)$. [7]

[7] This is consistent with the usual treatment which considers compact sets first and uses them to

The discussion above requires V to be hyperconvex in $\mathbb{C}^n$ except for the previous paragraphy which clearly holds in more general case as used for our case.

We'll see later that the above is enough for us to go through the argument after restricting us to continuous plurisubharmonic functions in the definition of relative capacity. Basically we use "Cap_c^*" instead of "Cap" for generalizing the original argument in [Koj1] [8] and the results above would allow us to use all the classic results about relative capacity after getting into local picture in $\mathbb{C}^n$.

Now we can indeed have a nice approximation of relative extremal functions using continuous plurisubharmonic functions by the (modified) definition of relative capacity. And in order to justify comparison principle for them, the approximation near the boundary should be nice. We'll require the domain to be hyperconvex which will give the nice boundary behavior of the approximation for relatively extremal functions in the following sketch of the argument.

Given the nice continuous approximations for both of the functions being considered (which can just be the function itself if it is already nice enough) which behave well near the boundary, we can use Richberg's smooth approximation for the continuous approximating functions, which is like convolution for general domains [9]. The plurisubharmonicity might be lost a little bit, but we can add some $\epsilon\omega$ to save that with ϵ being as small as we want. Anyway, it is quite routine (just as in [BeTa]) to take all the limits to justify comparison principle in this case.

Remark 5.2.1. *As mentioned before, we only need to consider the continuous plurisubharmonic funtions instead of u'_K provided that the definition of relative capacity is well adjusted as above (i.e., using essentially "Cap_c"). For them, we have the Richberg's approximation just as described above. Then it's easy to see that we actually don't need the hyperconvexity of V here.*

From above, the justification of applying comparison principle for relative extremal functions earlier in this section is quite redundant in some sense. But it makes sure that we are allowed to use comparison principle, which is such an important tool in pluripotential theory, for certain functions in more general domains and this might be helpful for other consideration.

A final remark for the discussion above would be that all these difficulties come

define for (open) sets in general.

[8]This modification clearly won't essnetially do anything for the original case considered there.

[9]Basically the idea is to smooth the function locally in a proper manner and then use a clever way to patch them up (see in [De1] for example).

from the absence of convolution which provides a smooth approximation with value highly related to the original function. So once we are in $\mathbb{C}^n$, it is just the classic case being discussed before. And also if we can have something like convolution for some case, then there should also be no trouble in that case to justify comparison principle.

We still want to discuss a little more modification about classic comparison principle which is about the case when we have some background metric. Basically there should be no major difference when there is a background "metric" [10] involved as clearly suggested in the smooth case where the proof would be exactly the same as in the classic case when we only consider smooth plurisubharmonic functions. The boundary condition is still on the (relative) potentials. It should still be OK when the potentials are continuous since Richberg's approximation will then be enough to reduce the situation to the smooth case. And that is basically what we are going to work with.

We emphasize here the trivial but important fact that for comparing of two functions plurisubharmonic with respect to the same background "metric", it is important that the $(1,1)$-currents are positive after combining the background form with the potential. Clearly we can have some cheap generalization if we have some inequality between the possibly different background "metrics" in the favorable direction. More precisely, in a general bounded domain V which might not be in $\mathbb{C}^n$, suppose $0 \leqslant \omega_1 \leqslant \omega_2$ where both are real smooth closed $(1,1)$-forms. For $u \in PSH_{\omega_2}(V) \cap C^0(\bar{V})$ and $v \in PSH_{\omega_1}(V) \cap C^0(\bar{V})$, if $lim_{x \to \partial V}(u-v)(x) \geqslant 0$, then

$$\int_{\{u<v\}} (\omega_1 + \sqrt{-1}\partial\bar{\partial}v)^n \leqslant \int_{\{u<v\}} (\omega_2 + \sqrt{-1}\partial\bar{\partial}u)^n.$$

The interesting point for this observation is that we can compare two functions with different plurisubharmonicity. The proof is rather easy after realizing for the smooth case, $\omega_2 - \omega_1 \geqslant 0$ is in our favor. For the rest part which is just like the term for classic case, after applying Stokes' Theorem, the sign of boundary contribution is only related to the comparison for the values of u and v. Of course we still use Richberg's approximation. In fact, this can also be trivially seen from the version of comparison principle with the same background form by noticing that we can actually use ω_2 on both sides.

Also see if we are considering a domain in $\mathbb{C}^n$ with a background form which can

[10] It just needs to be a real smooth closed $(1,1)$-form, not necessary to be positive. So sometimes we also call it as background form. But in most cases, it'll be nonnegative.

not be put into a global potential itself on this domain, we still have comparison principle for general (bounded) functions plurisubharmonic with respect to this background form since convolution can still be applied to both the background form and the relative potential by noticing the trivial computation below:

$$\begin{aligned}\sqrt{-1}\partial_w\bar{\partial}_w\big(\int_{\mathbb{C}^n}\rho(|w-z|)u(z)dg_z\big) &= \sqrt{-1}\partial_w\bar{\partial}_w\big(\int_{\mathbb{C}^n}\rho(|z|)u(w-z)dg_z\big)\\ &= \int_{\mathbb{C}^n}\rho(|z|)\sqrt{-1}\partial_w\bar{\partial}_w u(w-z)dg_z.\end{aligned}$$

Here the convergence of the convoluted smooth potentials to the original potential can be seen in more or less the same as for plurisubharmonic functions by considering the local potential for the background form (which is closed). Then we see the classic proof can go through with little modification.

Remark 5.2.2. *There will be more discussion about comparison principle in the next two subsections where we can see it holds for a closed manifold even when the functions are not continuous.*

For a projective manifold X, if the background form is representing a semi-positive class, we need another approximation of general plurisubharmonic functions which makes use of a quite deep extension result.

Finally, a more recent result in [BlKol] allows us to remove the projectivity assumption on X.

5.2.2 Approximation Result for Projective Manifolds

In this subsection, we introduce another approximation of general plurisubharmonic functions which can be used to justify comparison principle for cases other than what we've already discussed before and containing our main interest. For completeness, let's give the details which is essentially quoted from [GuZe].

Proposition 5.2.3. *Let L be a positive holomorphic line bundle over a closed (projective) manifold X. h is a hermitian metric for L with curvature form $\omega > 0$. Then for any $\phi \in PSH_\omega(X)$ [11], there exists a sequence of functions, $\phi_j \in PSH_\omega(X) \cap C^\infty(X)$, which decreases to ϕ.*

Furthermore, considering any hyperconvex domain U in X, for any $\phi \in PSH_\omega(U)$ with upper bound $C < \infty$, there exists a sequence $\phi_j \in PSH_\omega(U) \cap C^\infty(U)$ which decreases to ϕ.

[11] Not identical to $-\infty$ by definition.

Proof. The proof for X or U will be almost the same. Just notice in the case of a closed manifold, we always have $\phi < C$. Let's prove for X below and make the argument general enough to work also for U.

Consider the following Bergman spaces (for each fixed j large enough):

$$H_j := \{s \in \Gamma(X, L^j) | \int_X \|s\|^2 e^{-h_j} \omega^n < \infty\},$$

where $\| \cdot \|$ is norm for the metric h and $h_j = (j - N)\phi$ with N being a big enough integer which will be fixed later. Basically, the curvature form for the hermitian metric $\| \cdot \|^2 e^{-h_j}$ [12] is

$$j\omega + (j - N)\sqrt{-1}\partial\bar{\partial}\phi = (j - N)(\omega + \sqrt{-1}\partial\bar{\partial}\phi) + N\omega \geqslant N\omega.$$

For N large enough, it can dominate all the auxilliary terms coming out and justify the application of the extension result used later.

This space H_j would at least be separable [13]. Taking an orthonormal basis (with respect to the natural norm as appearing in the definition of the space), $\{s_k^j\}$, we set

$$\psi_j := \frac{1}{j} \log(\sum \|s_k^j\|^2) = \frac{1}{j} sup_{s \in B_j} \log \|s\|^2$$

where B_j is the unit ball in the space H_j. The last step actually comes from the fact that for $a_k,\ b_k \geqslant 0$ (Cauchy-Schwarz inequality),

$$(\sum a_k b_k)^2 \leqslant (\sum a_k^2)(\sum b_k^2).$$

There is no need to worry about the convergence of $\sum \|s_k^j\|^2$ since we are considering the holomorphic sections for which the (local) L^2-norm [14] can control all the local C^k-norms. And orthonormalization would reduce the control (for any finite sum) to just one element in B_j. So this summation is actually smooth over X.

It's easy to justify that $\psi_j \in PSH_\omega(X)$ by elementary property of plurisubharmonic functions. For the case of U when it is infinite summation, this is also standard computation for Bergman kernel. In fact, since it is the increasing limit of such functions and also of the form $\log F$ for some nonnegative smooth function F, the plurisubharmonicity should be quite easy to see.

[12] The lack of regularity is not a problem here.

[13] In fact it's of finite dimension for a closed manifold.

[14] Usual local L^2-norm is controlled by the norm for H_j as $\phi < C$ locally. The notation is a little confusing as this L^2 is not the one as in the proof. Fortunately, we do not use this one in the proof.

Now for any $x \in X$ and $s \in H_j$, consider s as a holomorphic function with respect to some local frame. Let the metric h on L be represented by e^{-g} (i.e. $\sqrt{-1}\partial\bar{\partial}g = \omega$ locally). As $|s|^2$ would be plurisubharmonic, and so we have

$$|s(x)|^2 \leqslant \frac{C}{r^{2n}} \int_{B(x,r)} |s|^2 d\lambda \leqslant \frac{C}{r^{2n}} e^{sup_{B(x,r)}\{h_j + jg\}} \int_X \|s\|^2 e^{-h_j} \omega^n$$

for small enough r such that $B(x,r)$ would be in a trivialized open set. All these constants will be uniform. Notice that even in the case of U, the background data is still global over X, and so is controlled uniformly.

Considering all the element s in B_j, by the definition of ψ_j, we arrive at

$$\begin{aligned} \psi_j(x) &\leqslant \frac{1}{j}\big(C - 2n\log r + sup_{B(x,r)}(h_j + jg)\big) - g \\ &\leqslant (1 - \frac{N}{j}) sup_{B(x,r)}\phi + \frac{C - 2n\log r}{j} + sup_{B(x,r)} g - g(x). \end{aligned}$$

For any $\epsilon > 0$, taking proper r, we can see for j large enough,

$$\psi_j(x) \leqslant (1 - \frac{N}{j}) sup_{B(x,r)}\phi + \epsilon \leqslant (1 - \frac{N}{j})(\phi(x) + \epsilon) + \epsilon.$$

Upper semi-continuity (with no uniformality on x) of ϕ is used here, so this is just pointwise argument which is not uniform on x. But for the term $sup_{B(x,r)} g - g(x)$, we have the uniform control. For any point x out of some measure 0 set, $\phi(x) > -\infty$, and so we can get $\psi_j(x) \leqslant \phi(x) + 3\epsilon$ for j sufficiently large.

Anyway, we've already got an upper control of ψ_j by ϕ. Now let's search for a lower one which makes use of a deep extension result of Ohsawa-Takegoshi-Manivel as stated in [De2] which gives the following.

For N large enough, there exists constant C such that for $\forall x \in X$, $\forall j > N$ and any value $s(x) \in L^j_x$, we have an element $s \in \Gamma(X, L^j)$ (with the assigned value at x as suggested by the notation) satisfying

$$\int_X \|s\|^2 e^{-h_j} \omega^n \leqslant C \|s(x)\|^2 e^{-h_j(x)}.$$

This is just a simple application of Theorem 4.1 in [De1]. Notice N is chosen large enough to allow $K_X^{-1} \otimes L^j$ to have a positive enough curvature form where the metric on L^j is twisted by "ϕ" as mentioned before. Moreover those sections of an auxilliary bundle used to characterize the point x can be chosen to have uniform data for all x

which is justified by the projectivity of X.

Now take proper $s(x)$ such that the right hand side of the inequality above is 1, and so the section s would be in B_j. Thus we have

$$\psi_j(x) \geqslant \frac{1}{j}\log\|s(x)\|^2 = \frac{1}{j}\log\frac{1}{Ce^{-(j-N)\phi(x)}} = (1-\frac{N}{j})\phi(x) - \frac{\log C}{j}.$$

As $\phi(x) < \infty$, for any $\epsilon > 0$, we have $\psi_j(x) \geqslant \phi(x) - \epsilon$ for j sufficiently large.

Combining these two estimates, we have the almost everywhere convergence of ψ_j to ϕ, thus also in $L^1(X)$ (or in $L^1_{loc}(U)$). At this moment, the convergence is not yet decreasing and ψ_j may not be smooth everywhere. Also notice that up to now, we don't really need the upper bound of ϕ over X (or U).

First, let's make them decreasing. For below, we'll use lower index N to keep track of the chosen constant N before, for example, $H_{j,N}$ stands for the previous space H_j.

Let $s \in \Gamma(X, L^{j_1+j_2})$ with $\int_X \|s\|^2 e^{-h_{j_1+j_2}}\omega^n < 1$. It can also be viewed as the section over the diagonal Δ of $X \times X$ of the line bundle $\pi_1^* L^{j_1} \otimes \pi_2^* L^{j_2})$ where π_i's for $i = 1, 2$ are the natural projections to each factor.

Still consider the corresponding Bergman spaces

$$H_{j_1,j_2,N} := \{S \in \Gamma(X\times X, \pi_1^*L^{j_1}\otimes\pi_2^*L^{j_2}) | \int_{X\times X} \|S\|^2 e^{-\pi_1^* h_{j_1,\frac{N}{2}} - \pi_2^* h_{j_2,\frac{N}{2}}} \pi_1^*\omega^n \pi_2^*\omega^n < \infty\}$$

where $\frac{N}{2}$ in the lower index gives the constant used in the definition of function h_j's.

Clearly the extension result can also be applied to the case when extending sections from Δ to $X \times X$ with uniform chosen data as before. Thus for the section s chosen before, we have $S \in \Gamma(X \times X, \pi_1^* L^{j_1} \otimes \pi_2^* L^{j_2})$ such that

$$\int_{X\times X} \|S\|^2 e^{-\pi_1^* h_{j_1,\frac{N}{2}} - \pi_2^* h_{j_2,\frac{N}{2}}} \pi_1^*\omega^n \pi_2^*\omega^n \leqslant C \int_X \|s\|^2 e^{-h_{j_1+j_2,N}}\omega^n \leqslant C$$

where C is a uniform constant.

The space $H_{j_1,j_2,N}$ (for $X \times X$) has an orthonormal basis coming from the orthonormal basis of $H_{j_1,\frac{N}{2}}$ and $H_{j_2,\frac{N}{2}}$ (for X). Using this basis and Cauchy-Schwarz inequality as before, it is easy to get:

$$\psi_{j_1+j_2,N} \leqslant \frac{\log C}{j_1+j_2} + \frac{j_1}{j_1+j_2}\psi_{j_1,\frac{N}{2}} + \frac{j_2}{j_1+j_2}\psi_{j_2,\frac{N}{2}}.$$

Now let's use the upper bound of ϕ. In fact we can make it to be 0 without affecting the result. Then we have $\psi_{j,N_1} \leqslant \psi_{j,N_2}$ for $N_1 \leqslant N_2$ which is trivial from the relation between norms for H_{j,N_i} and the expression of ψ_j.

Define $\hat{\psi}_j = \psi_{2^j,N} + 2^{-j}\log C$ and we can see that $\{\hat{\psi}_j\}$ is decreasing.

It remains to make the functions smooth. The only trouble is from the possibly nonempty 0 locus of $\sum_k \|s_k^{2^j}\|^2$. So we can treat just by adding some positive term to it. Indeed, we can use the term $\epsilon_j \sum \|\sigma_l^j\|^2$ where $\{\sigma_l^j\}$ is a basis of $\Gamma(X, L^{2^j})$, where the bundle is positive enough (very ample), and $\epsilon_j > 0$ is properly chosen to make sure this sequence is still decreasing with the limit still being ϕ. The resulting sequence would be the $\{\phi_j\}$ we want.

□

Remark 5.2.4. *As shown above, this new approximation has more global feature then convolution or Richberg's approximation (for continuous functions). So it would be enough to justify comparison principle over a closed manifold (i.e., a bounded domain without boundary). If there is boundary for the domain, then the description near the boundary for the approximating (smooth) functions would not be sufficient to justify the application of comparison principle even for themselves since the boundary values may not have the right relation.*

Of course, since the approximation is also a decreasing one, for the case of a domain with boundary, it might be enough to justify comparison principle for functions which are continuous near the boundary.

Actually, for sets like $\{u < v\}$, suppose that the function v is continuous, and so the approximation for it is actually (locally) uniform. Since the approximation for u is decreasing, so the boundary condition (i.e., "$u > v$") is well preserved for the approximations. [15]

Finally, we want to justify the comparison principle for bounded plurisubharmonic functions with respect to $\omega_\infty \geqslant 0$. The idea is not to directly search for a smooth approximation of functions which are plurisubharmonic with respect to ω_∞, but to get the inequality from comparison principle using background metric $\omega_\infty + \epsilon\omega$ for $\omega > 0$ and let $\epsilon \to 0$. Here we need the facts that $PSH_{\omega_\infty} \subset PSH_{\omega_\infty+\epsilon\omega}$ and $\int_{\{u<v\}}(\omega_\infty+\epsilon\omega+\sqrt{-1}\partial\bar\partial v)^n \to \int_{\{u<v\}}(\omega_\infty+\sqrt{-1}\partial\bar\partial v)^n$ (also for the measure involving u) as $\epsilon \to 0$ [16]. In fact, we don't need the perturbation to be as clean as $\epsilon\omega$.

Let's point out that from the approximation above for a fixed background form $\omega_\infty+\epsilon\omega$, we can easily get a decreasing sequence of smooth plurisubharmonic functions

[15]We'll give another observation in the next subsection which says that approximation like this is enough for our application even in local argument considering a domain with boundary.

[16]For the second fact, you might want to know the function is actually plurisubharmonic in a larger domain in order to guarantee the finiteness of term including ϵ, but that's usually true especially when we can use hyperconvexity of the domain to "extend" the function considered.

with background forms also decreasing to ω_∞ by choosing proper functions from each sequence of approximation before [17]. This is what's useful for the discussion at the end of Chapter 2 for bounded functions. And one might also use this to justify comparison principle for ω_∞ more directly.

For a projective manifold, we can easily justify comparison principle for $\omega_\infty \geqslant 0$ which might not even represent a rational class as follows. By taking a cohomology basis consisting of (finite) rational Kähler classes, $\{[\omega_j]\}$ with ω_j being a Kähler metric, we have $\omega_\infty + \sqrt{-1}\partial\bar{\partial}f = \sum_j a_j\omega_j$ where $a_j \in \mathbb{R}$. Then we can use rational numbers to approximate each a_j from above and the corresponding forms would be the "$\omega_\infty + \epsilon\omega$" above. Clearly the extra smooth function f can be absorbed by modifying the potentials simultaneously and so won't bring us any trouble. Hence comparison principle for ω_∞ is justified.

5.2.3 Approximation Result for Closed Kähler Manifolds

In the recent paper [BlKol] of Blocki and Kolodziej, an improved approximation method has been given which can be applied for a closed complex manifold. More precisely, they proved the following theorem. [18]

Theorem 5.2.5. *For a closed manifold X and a Kähler metric ω over it, suppose $u \in PSH_\omega(X) \cap L^\infty(X)$, then we have a sequence of functions, $u_j \in PSH_\omega(X) \cap C^\infty(X)$, decreasing to u.*

The proof is a generalization of the proof for Richberg's approximation result for continuous case, and so is more elementary than the proof of the previous result for projective manifolds. The construction used is fairly local. So the result can still be used to treat a domain in X. Then the approximation might be for a smaller domain (just as for convolution), but that won't give us any trouble, especially when the domain is hyperconvex and we can extend the function u being approximated.

Using the approximation result above, we can get the following version of comparison principle.

Proposition 5.2.6. *Suppose X is a closed Kähler manifold and ω is a smooth real nonnegative $(1,1)$-form over it. For $u,\ v \in PSH_\omega(X) \cap L^\infty(X)$, we have*

$$\int_{\{v<u\}} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leqslant \int_{\{v<u\}} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n.$$

[17] The decreasing can be made strict by adding proper constants.

[18] The result proved there is more general than this, but the following is all we need.

Notice we only need the semi-positivity of ω. The idea for this is just like what's used in the previous subsection using approximation. So we require X to be Kähler which gives a positive form. The life is slightly easier now.

In the proof, one still needs the boundedness of functions to control the contribution of small capacity sets. That's also why though the approximation they got can be for unbounded functions with Lelong number 0, this comparison principle is still just for bounded functions.

This is a global version over X which will be sufficient for the global argument. We also want a local version for local argument. So we need to take care of the boundary condition for approximation functions. As mentioned before, for set like $\{u < v\}$, by decreasing convergences and Dini's Theorem, we can deal with function v which is continuous near the boundary. We would like to see that this is enough for us.

Basically, the least regular v would appear in the definition of relative capacity. Let's consider the case that continuity of the "test" function is not require there. As we have seen before, one only needs to consider the capacity for a compact set. Let's also notice that the (background) domains we are considering in the definition of relative capacity are hyperconvex. So it's easy to see that by taking the maximum of any bounded function v and a large multiple of the function from hyperconvexity, we can make sure that the resulting function is continuous near the boundary without the same value as u near the compact set being considered. Clearly this function would obviously be valued in the proper range and we only need to consider such functions for the relative capacity of this compact set.

5.3 Presence of Background Form

In this section, we mainly take care of the technical differences when there is a background form involved in the computation of Kolodziej's argument quoted before.

5.3.1 About Parts (1) and (2)

In Part (1) of the quoted argument from [Koj1], we need the relation between relative capacity and relative extremal function of any (compact) set inside the domain we are considering, i.e. $Cap(K, V) = \int_K (\sqrt{-1}\partial\bar{\partial} u_{K})^n$ for K compact set in V. Though we've already seen it's not that crucial, the hyperconvexity of the domain V (in $\mathbb{C}^n$

at that time) behind this fact would still be important for us in a general domain in X.

In view of the discussion about the difficulty coming from the degeneracy of ω_∞ as a metric, the natural idea is to consider a neighbourhood of each component of $\{{\omega_\infty}^n = 0\}$ separately. And from above, we want the neighbourhood chosen to be hyperconvex.

This can be achieved if we assume the set $\{{\omega_\infty}^n = 0\}$ gets mapped to a set of (finite) points in $\mathbb{CP}^N$ because each general domain that we need to consider now is just the preimage of a ball around one of those points in $\mathbb{CP}^N$ (or say $\mathbb{C}^N$ locally) and the exhausting function can be obtained by pulling back the usual one for a ball in $\mathbb{C}^N$. So now let's assume this for simplicity. Of course, in general, the preimage of a ball in $\mathbb{CP}^N$ under the map P would have similar property. So this simplification is not a big deal.

Remark 5.3.1. *This situation we are considering here is quite special but still farily natural to start with as a prototype. Moreover, for complex dimension* 2*, this is the general case. Later on, we'll try to show this is also not needed for generalizing the rest of argument.*

For this crush-to-point case, we have the potential of ω_∞ globally on V. And so it seems that we don't have to treat the case when there is a background form. But since our approximation functions u_ϵ for the function u are from the equations

$$(\omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega,$$

we still need the results with some background form involved in a nice and explicit way. That's what we are going to deal with below. All these are basically quoted from [Koj2].

Now the domain V is a general domain on a closed Kähler manifold X with Kähler metric ω.

For any $u,\ v \in PSH_\omega(\bar{V}) \cap C^0(\bar{V})$ [19] with $U(s) := \{u - s < v\} \neq \varnothing$ and relatively compact in V for $s \in [S, S + D]$. Assume v is valued in $[0, C]$.

[19] For simplicity, we use $PSH_\omega(\bar{V}) \cap C^0(\bar{V})$ which means they are actually defined in a neighbourhood of $\bar{V}$ with these properties, but of course that's not so necessary. Basically, u can be just bounded and v is continuous. At the place where comparison principle will be applied, it's possible that "v" might be a bounded function which are continuous near the boundary depending on the definition chosen for relative capacity. But it's still OK as has been well explained in the discussion about comparison principle before. Let's keep this picture in mind from now on.

Thus for any $w \in PSH_\omega(V) \cap C^0(\bar{V})$ [20] and valued in $[-1,0]$, we have for any $t \geqslant 0$:

$$U(s) \subset V(s) = \{u - s - t - Ct < tw + (1-t)v\} \subset U(s+t+Ct)$$

since $0 \leqslant t + Ct + tw - tv \leqslant t + Ct$. So we have the following computation for $0 < t \leqslant 1$:

$$\begin{aligned}
\int_{U(s)} (\omega + \sqrt{-1}\partial\bar{\partial} w)^n &= t^{-n} \int_{U(s)} \big(t\omega + \sqrt{-1}\partial\bar{\partial}(tw)\big)^n \\
&\leqslant t^{-n} \int_{U(s)} \big(t\omega + \sqrt{-1}\partial\bar{\partial}(tw) + (1-t)\omega + \sqrt{-1}\partial\bar{\partial}((1-t)v)\big)^n \\
&= t^{-n} \int_{U(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v)\big)^n \\
&\leqslant t^{-n} \int_{V(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v)\big)^n \\
&\leqslant t^{-n} \int_{V(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(u - s - t - Ct)\big)^n \\
&\leqslant t^{-n} \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial} u)^n,
\end{aligned}$$

where the next to the last inequality comes from comparison principle. Hence we can conclude

$$t^n \cdot Cap_\omega(U(s), V) \leqslant \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial} u)^n$$

for $t \in (0, min(1, \frac{S+D-s}{1+C})]$. Of course, for our purpose, it is always safe to assume $\frac{S+D-s}{1+C} < 1$. In fact, we'll only apply this result for intervals with the length, D, smaller than 1, so there is no need to worry about the value of t appearing in the computation as in [Koj1].

The definition of this Cap_ω should be clear from above and one uses only continuous functions when taking supremum. [21]

In order for the result to look more like the one in Part (1) in Chapter 3, we

[20] The requirement of continuity may not be that important as discussed before, depending on how one wants to justify comparison principle and the correspondent definition of relative capacity.

[21] This won't bring any essential trouble, and clearly it's also OK if we don't require the continuity as just discussed. Notice that the set $U(s)$ would always be open which makes the outer capacity easier to handle.

rewrite the inequality as:

$$t^n \cdot Cap_\omega(U(s), V) \leqslant (1+C)^n \int_{U(s+t)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n$$

for $t \in (0, S + D - s]$.

It is important to notice that this result will not affected by the choice of ω. In fact it is even not necessary for it to be positive, which is useful if one wants to prove apriori L^∞ bound for bounded solution of the original equation. [22] So this can actually be considered directly as a generalization of the original computation in Part (1) quoted before where $\omega = 0$.

Then provided that we have a similar Condition (A), it is easy to come to a similar conclusion as the one at the end of Part (2). We just have a few more constants involved. In fact for our application of this inequality to prove boundedness result, v can be chosen to be 0, and so there could be no difference between the expression of the results at all.

5.3.2 About Parts (4), (5) and (6)

Part (3) will be the main topic later. Let's take a look at Parts (4), (5) and (6) now. Basically, they are not affected at all.

For Part (4), using the global version of CLN inequality mentioned before with background metric, we can get the (uniform) apriori L^∞ bound in the same way. Notice that though the background metrics are not the same, their local potentials are obviously controlled uniformly and so the CLN inequality has uniform constants. The way to get a bounded solution is by taking the limit of approximation solutions where the convergence is considered locally as in [Koj1] and so can still be applied in our case. In fact, a global version is also discussed in [Koj2] where the background form is fixed, but that's not a big issue as the convergence of background forms for us is very explicit.

For Part (5), as we'll see later, local argument is what we are really going to use to prove continuity. So background form will not appear as before. There will also be an attempt to argue globally later for continuity motivated by the argument in [Koj2]. At that place, the background form would still be the same for each pair of

[22] Of course, the potential still needs to be plurisubharmonic with respect to it.

functions considered in our case, and so will not bring any more trouble than what's treated in [Koj2] which would be quoted there.

For Part (6), there are background forms in the argument quoted in Chapter 3 already, and so that's not a problem for us. Indeed, in this part, we only need to consider the backgound form, ω_∞ and no approximation of it is really involved.

5.4 First try about Part (3)

In this section, we start the discuss about Condition (A) which is the essential part of our generalization. First, Let's make an easy observation which gets rid of all the background metrics. For any $\omega \geqslant 0$, considering the right hand side of Condition (A), it clearly becomes smaller when one uses the correspondent relative capacity without ω. Thus it suffices to justify this condition in the form as in Part (3) there. The only difference would be that now the domain V may no longer be in $\mathbb{C}^n$. The L^p functions have the naturally generalized meaning on a smooth manifold.

Now take a closer look at the argument in Part (3).

From the original discussion, we can see the result needed is still just the "claim" there, i.e., we want to have something like:

$$\lambda(U_s) \leqslant C \cdot e^{-Cs},$$

where $U_s = \{u < -s\}$ and $\int_V (\sqrt{-1}\partial\bar{\partial}u)^n \leqslant 1.$ [23]

At the first sight, it would be helpful to use some kind of correspondence between the general domain V in X and the other picture of it, i.e., the picture of its image $P(X)$ in $\mathbb{CP}^N$ where this domain V really comes from. Here we can use this idea to treat the case when locally the map P will crush a subvariety, i.e., one component of $\{{\omega_\infty}^n = 0\}$, to a point and map a neighbourhood of the subvariety biholomorphically to a neighbourhood of that point in $P(X) \subset \mathbb{CP}^N$ after removing the subvariety and that point respectively. More importantly, we assume that point is also a smooth point on $P(X)$. Basically, one can think of it as a blow-down map from $\hat{B}^{2n} \to B^{2n}$ where $\hat{B}^{2n}$ is the blow-up of B^{2n} at the center. It is quite easy to go through the original argument in this case which just corresponds to using a singular volume form on B^{2n}.

[23] This u is essentially just $(Cap(K,V))^{-\frac{1}{n}} u_K$. Of course here λ is a smooth measure over V corresponding to a nondegenerated smooth volume form.

In this situation, bounded plurisubharmonic functions on these two domains have one-to-one correspondence which is clear from the obvious correspondence for the punctured parts and classic extension result for plurisubharmonic functions. Thus we have convolution for those functions on $\hat{B}^{2n}$. So we can just use the classic definitions of relative capacity, etc. for this domain. And they are actually correspondent to the same things on B^{2n} since it is easy to see the exception divisor will essentially have no contribution.

Also the smooth measure over $\hat{B}^{2n}$ corresponds to a measure with the only singularity like $\frac{1}{r^{2n-2}}$ near the origin of B^{2n} where r is the distance to the origin. Clearly this measure, λ', is L^p on B^{2n} with respect to the standard measure. By studying the extremal case when there is a difference between these two measures on B^{2n} [24], it's easy to see by direct computation that

$$\lambda'(K) \leqslant C \cdot \lambda(K)^{\alpha}$$

for some positive constants $C > 0$, $\alpha \in (0, 1)$ and $K \subset B^{2n}$ where $\lambda(\cdot)$ is the standard measure over B^{2n}. Of course $\lambda'(K)$ can be looked on as the measure of the correspondent set in $\hat{B}^{2n}$ which is defined up to the exceptional divisor (variety) which is of measure 0.

In fact for all the above discussion, we only need to notice that the singularity for the measure λ' with respect to λ is of the form $\frac{1}{r^q}$ and obviously the total integral is finite from the definition of it since we can then see it is actually L^p for some $p > 1$. In other words, once we have the explicit form of singularity like that, the fact that it is L^p for some $p > 1$ comes from it being L^1 by the openness of the condition on q. We have used the same idea before.

Thus for justifying Condition (A), we can see the only modification would be in the proof of the "claim" where $(\star)$ is changed by taking power α and multiplying some constant, which clearly will not affect the argument too much. Hence for this case we can have the boundedness of the approximation solutions and so for the solution.

The situation discussed above might look very similar to what has been treated in Chapter 3 in sight of the smooth image assumption. But the argument now is fairly local in flavor. We actually generalize the original argument a little bit instead of directly applying the original results. In fact we can see that this argument here works for the case when the map P is locally blow-down to a smooth image in $\mathbb{CP}^N$.

[24] This is of course the case when the set considered is a disk centered at origin in B^{2n}.

This should be more general than the situation considered in Chapter 3 where P is required to be birational.

Remark 5.4.1. *There could be similar discussion when the image has an orbifold structure instead* [25]. *More precisely, we can use the convolution on the orbifold coordinate chart to get continuous approximation of bounded plurisubharmonic functions on some general domain in* X [26].

[25] We don't require the map P to be compatible with the orbifold structure now.

[26] Strictly speaking, the construction still needs to go through the image $P(X)$ and the classic extension of plurisubharmonic functions is used. Here we should also need that the convolution of G-invariant plurisubharmonic functions will still be G-invariant where G is the finite group for orbifold structure which is essentially made up of rotations. Basically, we require $G \subset SL(n, \mathbb{C})$ in order to preserve the Euclidean volume form when using this (averaging) convolution.

Chapter 6

Essential Estimate

In this chapter, we justify Condition (A) for our degenerated case in general. The argument we are going to use is different from the original one in [Koj1] as quoted in Part (3). Let's motivate it first.

It might seem natural to push the discussion at the end of last chapter to the case when the image of the domain in $\mathbb{CP}^N$ is singular at the center, i.e., that point onto which a subvariety in X is crushed is no longer a smooth point of $P(X)$. But there seems to be some substantial difficulties when trying to carry out similar argument as explained below.

The difficulty is basically lying in the proof of the "subclaim" where some trivial but important geometric information of $\mathbb{C}^n$ is heavily used. For the singular image case, we still consider plurisubharmonic functions over the original domain V which is smooth. But at the same time we want to push them onto the image to get some help. If we are considering the simple local picture of $P(X)$ as $\{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3$, there are still a lot of problems which make it too difficult to carry through the same argument as described below.

The obvious attempt is to defined similar $\mathcal{L}$-functions on this singular variety since the infinity is still regular (see for example [Ze]). But then notice that in the proof of the subclaim, there is a point "a" which will not be on the origin $(0, 0, 0)$, and so we don't have all the complex lines through it to cover the whole domain. It might seem reasonable to consider general algebraic curves instead of Riemann sphere. But somehow it turns out to be at least too difficult for me at this moment.

6.1 Reduction to Essential Estimate

Let's give up this line by line generalization and search for something different. In fact by taking a closer look at the discussion in Part (3), it is easy to see that before we arrived at the "claim" there, we did something very rough. We can actually do something else.

For our consideration where the measure is L^p for some $p > 1$, just from the original form of Condition (A) and using Hölder inequality for the left hand side, we can see in order to prove Condition (A), it suffices to get for any compact subset $K \subset V$,

$$\lambda(K) \leqslant C \cdot \big(Cap(K,V)(1 + Cap(K,V)^{-\frac{1}{n}})^{-m}\big)^q$$

where λ is the smooth measure over V and q is some positive constant depending on p. Here we have already used the explicit form of function $Q(r)$ there.

Obviously, it is enough to prove that

$$\lambda(K) \leqslant C_l \cdot Cap(K,V)^l \cdots\cdots (\star\star)$$

for l sufficiently large. Of course we have $\lambda(K) < C$, so in fact we can get for every l in between. In the following, we'll consider Condition (A) in this form. Let's observe that this inequality can be easily summed up in the sense that if we have it for different V's, then we can get a similar version for the union of all these V's as long as there are only finitely many of them. Since X is closed, this makes the essential estimate for the global argument available once we get it for all the local pictures.

When $V \subset \mathbb{C}^n$, we've mentioned before that the result for $l = 1$ is trivial from definition. And the general inequality would follow from $(\star)$ in Part (3) before which actually gives the exponential control and is a much stronger result. The estimate $(\star\star)$ we want here is a generalization of it for general domains.

As discussed before about comparison principle used in the proof, we may use "Cap_c" instead of "Cap". [1] Notice in the proof of boundedness result, the function v as in Part (1) is taken to be smooth (constant in fact), so the set $U(s)$ there is open. Thus by the discussion before about "Cap_c", we can in fact use Cap_c^* for the result of generalized Part (1) in the case of general domains. Then for Condition (A), we can also use Cap_c^* on the right hand side, and so we have reduced the proof of

[1] This is how we first came up with the proof, though the discussion before allows more flexibility.

boundedness result to the following inequality:

$$\lambda(K) \leqslant C_l \cdot Cap_c^*(K,V)^l \cdots\cdots (A)$$

uniformly for any K compact in V where l is large enough. It's easy to see that it suffices to prove this when $Cap_c^*(K,V)$ is small.

Remark 6.1.1. *Once we justify the essential estimate above, then we have proved the boundedness result and the careful track-down argument in Chapter 3 would also give us the explicit apriori L^∞ estimate as claimed in the theorem.*

6.2 Proof of Essential Estimate

When $P(X)$ is smooth, the argument at the end of the previous chapter would give us this estimate. Basically, we can directly use the estimate in for domains in $\mathbb{C}^n$ which are in $P(X)$. The singular measure on $P(X)$ can be controlled by power of smooth measure on it. For the rest part, we consider the case when $P(X)$ is singular. Let's start with the easiest case and gradually remove the extra assumptions in order to give a clear picture about what's happening.

6.2.1 Blow-down to Point Case

At the beginning, let's look at the case of crush-to-point which has been illustrated before. Basically, we still require the map P to be biholomorphic away from those points locally, i.e., a blow-down picture locally.

We have $V \subset \bar{V} \subset V'$ where V' is the preimage of a small ball in $\mathbb{CP}^N$ around one of the points being crushed onto. Here "small" means it only contains one singular point in the image and the punctured ball is biholomorphic to V' after removing the corresponding variety being crushed to this point. The domain V, which is our main concern, corresponds to a smaller ball inside. Naturally, V' is referred to whenever we want to extend functions over V to a bigger domain. They are both hyperconvex in the usual sense.

In our proof of the estimate (A), the essential step is to prove the inequality below:

$$\lambda(K) \leqslant C_1 \cdot \epsilon^{N_1} + C_1 \cdot \epsilon^{-N_2}\exp\left(\frac{C_2}{\log\epsilon \cdot Cap_c^*(K,V)^{\frac{1}{n}}}\right) \cdots\cdots (B)$$

for any (sufficiently small) $\frac{1}{2} > \epsilon > 0$ where K is a compact set in V. All the other constants C_i's are positive and NOT depending on ϵ.

After getting this, it is easy to see we can put $\epsilon = Cap_c^*(K,V)^\beta$ for some proper β to justify the inequality for any chosen l we need when $Cap_c^*(K,V) > 0$. Basically, one just needs to use the dominance of exponential growth over polynomial growth (over logarithmic growth). If $Cap_c^*(K,V) = 0$, then it's easy to see $\lambda(K) = 0$ by using the potential of ω_∞, which makes Condition (A) trivially true and won't give us any trouble.

The rest part of this subsection will be devoted to the proof of this inequality in the current situation. The following construction, which will be called "small-piece-cover", is of fundamental importance for this goal.

We can use finite unit coordinate balls on X to cover V'. Essentially, only one needs to cover the crushed variety. Then we take two finite sets of open subsets depending on $\epsilon > 0$ as follows: [2]

$\{U_i\}$ and $\{V_i\}$ with $i \in I$, two finite coverings of $V \backslash W$, where W is ϵ-neighbourhood of the crushed variety, i.e., corresponding to the intersection of a ball of radius ϵ centered at the point being crushed onto with the image, $P(V)$, such that $V_i \subset U_i$ in one of the coordinate balls chosen above. Moreover, when mapped to $\mathbb{CP}^N$, U_i and V_i are in fact the intersections of the image of V with balls of size $\frac{1}{2}\epsilon$ and $\frac{1}{6}\epsilon$. The I is the index set with $|I|$ being controlled by $C \cdot \epsilon^{-N_2}$.

It is easy to justify such a choice by considering the picture in $\mathbb{CP}^N$. Basically in the current case, for the upper picture (i.e., the preimage side of the blow-down map), each coordinate ball on V which covers some part of the crushed variety should contain a positive "cone angle" in the lower picture (i.e., the image side of the blow-down map). That allows us to put in U_i's and V_i's. The numbers of these sets are controlled by the number of balls needed to cover the whole ball in $\mathbb{CP}^n$. One can use the example, $\{x^2 + y^2 + z^2 = 0\}$, to see what's really happening. We know that the tangent cone (lowest order part) of the singular point should be something like that and it'll be basically the situation when we talk about very small scale [3]. Our construction would clearly survive small peturbation from the primitive situation. Furthermore, in the construction above, we can choose the size of the small pieces to be in the scale ϵ^C for any fixed (large) $C > 0$ without affecting the application below. This would be useful for more general cases.

[2]All the ϵ-dependences involved are explicitly stated below.

[3]the situation for small $\epsilon > 0$ is obvious the only essential content of the inequality we are proving.

The essential computation to prove (B) is the following:

$$\begin{aligned}
\lambda(K) &\leqslant \lambda(W) + \sum_{i\in I} \lambda(K\cap \bar{V}_i) \\
&\leqslant C\cdot \epsilon^{N_1} + \sum_{i\in I} C\cdot exp\big(-\frac{C}{Cap(K\cap\bar{V}_i, U_i)^{\frac{1}{n}}}\big) \\
&\leqslant C\cdot \epsilon^{N_1} + \sum_{i\in I} C\cdot exp\big(\frac{C}{\log\epsilon\cdot Cap_c^*(K\cap\bar{V}_i, V)^{\frac{1}{n}}}\big) \\
&\leqslant C\cdot \epsilon^{N_1} + \sum_{i\in I} C\cdot exp\big(\frac{C}{\log\epsilon\cdot Cap_c^*(K, V)^{\frac{1}{n}}}\big) \\
&\leqslant C\cdot \epsilon^{N_1} + C\epsilon^{-N_2}\cdot exp\big(\frac{C}{\log\epsilon\cdot Cap_c^*(K, V)^{\frac{1}{n}}}\big).
\end{aligned} \tag{6.1}$$

That gives just what we want. Here we use C_1 and C_2 because the C's at different places have different affects on the magnitude of the final expression. In the following, we'll justify the computation above.

First, let's notice that the computation above is clearly justified even if some capacity terms are 0. If one doesn't want such terms, then he can take summation over a subset of I such that this won't happen, which would not affect the final control. Anyway, 0 capacity implies 0 measure as pointed out before (using ω_∞).

The only nontrivial steps are the second and third ones. [4] In fact the second one is the direct application of $(\star)$ in Chapter 3, the classic result in $\mathbb{C}^n$ because V_i and U_i are in one of the finitely many unit coordinate balls of V which can of course be taken as the unit Euclidean ball in a uniform way and we are even using a smaller domain U_i as the background domain, which increases the relative capacity. Here we should realize that the only difference is that the measure is not the standard one, but not too different either.

The third step makes use of the following inequality:

$$Cap(K\cap\bar{V}_i, U_i) \leqslant C\cdot(-\log\epsilon)^n\cdot Cap_c^*(K\cap\bar{V}_i, V).$$

This is nontrivial since we are enlarging the background domain from U_i to V, while the coefficient $(-\log\epsilon)^n$ can be really big.

This is also a classic fact in case of a hyperconvex domain $V\subset\mathbb{C}^n$. In the following, we prove it by extending plurisubharmonic function from U_i to V. Argument using the similar idea has appeared before.

[4]The "$\epsilon^{N_1>0}$" control of the measure of W should be clear from the holomorphicity of the map P and the definition of W.

Consider $v \in PSH(U_i)$ valued in $[-1, 0]$. Define the function

$$h = \left(\log(\frac{36|z|^2}{\epsilon^2})\right)^+ - 2,$$

where the upper $+$ means taking maximum with constant 0 which clearly preserves the plurisubharmonicity, on the unit ball in $\mathbb{CP}^N$ but with the coordinate system z centered at the center of V_i [5].

Then it's easy to see the pullback of this function on V, still denoted by h, is plurisubharmonic on V. Of course the function $max\{h, v\}$ on U_i is equal to v near $\bar{V}_i$ and equal to h near ∂U_i. So we can use this function to extend v to the whole domain V while keeping the value near $K \cap \bar{V}_i$ (and so the extended function will correspond to the same measure near $K \cap \bar{V}_i$ as v itself under Monge-Ampere operator.). We call this function H.

We also want to deal with the subtlety that the orginal v may not be continuous but the definition of Cap_c^* requires continuity of the functions that one can use to take supremum. [6] Consider the convolution of $max\{h, v\}$ in U_i [7], H_j. This might be a confusing notation, but notice that it is actually the convolution of H, just not global on V. Since H is in fact equal to h and so continuous near ∂U_i, by Dini's Theorem, H_j converges to H uniformly in some annulus near ∂U_i. Thus the continuous function $max\{h + \delta, H_j\}$ will be equal to $h + \delta$ for some small $\delta > 0$ and j sufficiently large, and so it can be extend to the whole of V continuously by $h + \delta$. Let's call it $H_{\delta,j}$. Notice that near $\bar{V}_i$, this function $H_{\delta,j}$ will be equal to H_j as $H_j \geqslant H$ and h is -2 there. So as $j \to \infty$, it'll still decrease to v near $\bar{V}_i$. Moreover, $H_{\delta,j}$ is valued in $[-1, -2\log\epsilon + C]$.

Now by applying the convergence of $(\sqrt{-1}\partial\bar{\partial}H_{\delta,j})^n$ to $(\sqrt{-1}\partial\bar{\partial}v)^v$ in the sense of distribution over some open neighbourhood of $\bar{V}_i$ as $j \to \infty$, we can draw the conclusion just as before by using cut-off functions and the definitions of those relative capacities. The coefficient $C \cdot (-\log\epsilon)^n$ appears because the values of function $H_{\delta,j}$ range in an interval with length controlled by some multiple of $-\log\epsilon$.

Remark 6.2.1. *The classic result for domains in $\mathbb{C}^n$ can be proved using the relative extremal function as in [AlTa]. It's very likely that we can also do this here if the relation between the relative extremal function and relative capacity is still available. For this consideration, it might be better to work with open sets instead in sight of*

[5]This would only change things in a uniform way. The power of ϵ would be some $2C$ for general cases discussed later, but it clearly won't affect our following discussion.

[6]This part is rather superficial as we can see it now.

[7]This is OK since U_i is in some coordinate ball.

$Cap_c(U) = Cap(U)$ for open sets. Anyway, the idea of extending plurisubharmonic functions by hyperconvexity of the domain is really what's behind the screen.

Also we see it is OK for us to work with open sets only since the sets $U(s) = \{u < v + s\}$ there in the argument, which we'll apply the Condition (A) over, are all open as the v is always chosen to be continuous.

However, for later discussion about stability, in sight of the quoted argument before, we are going to working with sets which might not be open (if the functions are not continuous). In this case, we can come back to the relative capacity "Cap" and we know there should be no worry about comparison priciple.

Thus we conclude our justification of Condition (A) for the case of crush-to-point and get boundedness result.

Finally, let's point out that we get the estimate above with "Cap_c^*" on the right hand side simply because that's what we currently need since continuous functions are used instead of general (bounded) functions. But it is easy to see the argument for this estimate, as sketched below, would actually be simpler if we want "*Cap*" instead, i.e., for any $l \geqslant 0$, there exists a constant $C_l > 0$ such that for any compact set K inside V,

$$\lambda(K) \leqslant C_l \cdot Cap(K, V)^l.$$

Clearly, it still only takes to prove the corresponding inequality (B). In the computation used above for justifying the corresponding (B), we can just change "Cap_C^*" to "*Cap*". In order to prove the third step, the argument is easier since we only have to extend the function on U_i to V basically using the function h and we don't even bother to make it smooth (continuous) now.

Remark 6.2.2. *As mentioned before, there could also be a global version of the argument for boundedness result. It is quite natural to imagine that we'll need a global version of Condition (A). For the classic case as in [Koj2], it is quite easy to get the global version of this condition from the local version by patching things up. For our case here, to carry out the patching argument, the degeneracy of the background metric will cause trouble if one wants to patch thing up in exactly the same way (using coordinate balls). But since we have already treated the neighbourhood of the degenerate part, similar argument can go through now.*

6.2.2 Birational Case

In this section, we remove the assumption about crushing varieties to points. But we still assume $P : X \to P(X)$ to be a birational morphism. In other words, we are considering the case of crushing varieties to varieties [8].

We start by assuming that the map P is locally a blow-down map with (possibly) singular image with the blown-down part to be the set $\{{\omega_\infty}^n = 0\}$. Here the "locally" means we are considering the map P from an open set of X to an open set of $P(X)$ inside some Euclidean ball in $\mathbb{CP}^N$. The one in X is the preimage of the one in $P(X)$. There couldn't be more than one component by the birationality assumption.

At the first sight, this is very different from the previous case because if we want to consider the neighbourhood of $\{{\omega_\infty}^n = 0\}$, it'll no longer correspond to something inside a ball in $\mathbb{CP}^N$ under the map P. Thus we lose all the local functions we can construct on this neighbourhood by pulling back those elementary functions over a (Euclidean) ball inside $\mathbb{CP}^N$. But the following observation would help a lot which can actually be traced back to our original idea of generalizing the argument discussed in Chapter 4.

Basically, we don't have to consider a general domain which contains (one connected component of) the degenerated variety of ω_∞. It's only necessary to make sure that the degenerate directions will not go through the boundary of the domain being considered transitively. Then the semi-positivity of ω_∞ [9] will provide some room D for us which is crucial for the argument. [10]

Now we can easily see that there is actually no difference between the case now and that of the previous subsection. Let's describe the argument a little bit as follows.

Take finite Euclidean balls to cover $P(\{{\omega_\infty}^n = 0\})$ in $\mathbb{CP}^N$. Then the preimages of them are our general domains V. Just as before, we can have a slightly larger ball for each of them. Now it is only needed to consider them one by one.

It is easy to see that we still only need to check Condition (A) in the form as in the previous subsection, i.e., inequality (B) there. The most important part is the construction of small-piece-cover (i.e., the U_i and V_i there). The construction used is pretty local. Basically you just have to avoid the degenerate part by removing the ϵ-neighbourhood of it and then cover the rest by such sets. Clearly it still works here just as before. The justification doesn't even need to be modified except that the

[8]Of course, we expect the dimension to decrease by calling it "crushing".

[9]In fact the convexity of its potential.

[10]We do need ω_∞ to be reduced to potential globally over the general domain considered.

scale of small ball needs to be like ϵ^C with some large but fix $C > 0$.

Hence we can conclude the boundedness result when P is locally blow-down.

Remark 6.2.3. *In fact, as we can see for the discussion above, the only essential part of the proof is just to prove* (B)*. Our proof here is essentially only dependent on the construction of the small-piece-cover before.*

For the construction, we only have to make sure the sets are kind of regular in $\mathbb{CP}^N$ *and can be put in coordinate balls in the neighbourhood* V *in* X*. In fact their preimages don't have to be connected or even in the same coordinate ball respectively. Moreover, the cover we need is not for the whole neighbourhood* V*. In fact we can remove some* ϵ*-neighbourhood of any subvariety in it and the algebraic structure of the (singular) image will generally provide us with the uniformality with respect to all* ϵ *sufficiently small, which is very important for the argument. We only need holomorphic structure on* X *since it's smooth.*

All the observations above provide quite some flexibility for our argument. So it is very natural to guess that we can conclude the boundedness result without assuming the local blow-down picture of P *by finding enough properties for such a holomorphic map* P *with* $dim_{\mathbb{C}}X = dim_{\mathbb{C}}P(X)$*. In fact, combining with the argument in the next subsection, we'll justify that it is indeed the case.*

Now we justify our argument for the case when $P : X \to P(X)$ is a birational morphism. In the following, we are going to make sure that our small-piece-cover construction works in this case which will provide us with the proof for Condition (A) and so the boundedness result.

Let's start with a better description about the map $P : X \to \mathbb{CP}^N$.

First by Proper Mapping Theorem, we know the image $P(X)$ is a subvariety of $\mathbb{CP}^N$. And so we have the local picture of it as a finite-sheet covering of a standard Euclidean ball of the same dimension branched over some subvariety (see in [GrHa] for example).

By the birationality of the map $P : X \to Y$ where $Y = P(X)$, we can find dense open sets $X_o \subset X$ and $Y_o \subset Y$ such that the restriction $P : X_o \to Y_o$ is an isomorphism (i.e., a biholomorphism). We also have several obvious relations as follows.

$Y_o \subset Y \setminus \{singular\ locus\}$, $\{{\omega_\infty}^n = 0\} \subset X \setminus X_o$ where $\{{\omega_\infty}^n = 0\}$ and the image of it are subvarieties of X and $P(X)$ (or $\mathbb{CP}^N$) respectively. Also $Y \setminus Y_o \subset P(X \setminus X_o)$ and the equality will not be true in general. In fact, the equality case is more or less just the crush-down case considered before.

Set $W = P(X \setminus X_o)$. From above, it is a subvariety of Y containing $Y \setminus Y_o$. Finally set $Z = P^{-1}(W)$ which is clearly a subvariety of X containing $X \setminus X_o$ (and $\{{\omega_\infty}^n = 0\}$ of course). We can now summarize the following picture of P:

Z and $W = P(Z)$ are subvarieties of X and $Y = P(X)$ respectively, $\{\omega_\infty^n = 0\} \subset Z$ and the complements are biholomorphic to each other under the map P. Observe that this is essentially what we need from the crush-down picture. The only difference is that we are now "crushing down" more than the degenerate variety of ω_∞.

Now for the construction of small-piece-cover, one can use W and Z instead of the variety being crushed down to $(P(\{{\omega_\infty}^n\} = 0))$ and the variety being crushed $(\{{\omega_\infty}^n = 0\})$ as in the picture of crush-down considered before. More precisely, we consider the local picture of $P(X)$, i.e., the picture of it inside a small Euclidean ball in $\mathbb{CP}^N$, and its preimage in X. Here notice that without loss of generality, we can consider only connected X and so is $P(X)$. In fact $P(X)$ is irreducible, and so the smooth part of it is connected (see in [GrHa] for example) as well as all the open sets appearing above. So we can conclude that $P(X) \cap B$ and its preimage in X are connected open sets in $P(X)$ and X respectively for sufficiently small Euclidean ball B in $\mathbb{CP}^N$.

Moreover, if we want to make sure the pullback of the elementary functions over $P(X) \cap B$ onto the preimage can be used as before as the defining functions of the hyperconvexity of the domain [11], it is important to make sure that the boundary value is correct, in other words, we need

$$\partial\big(P^{-1}(P(X) \cap B)\big) \subset P^{-1}\big(\partial(P(X) \cap B)\big),$$

where for a general open set U, $\partial U := \bar{U} \setminus U$. The relation above is generally true for a continuous function $f : X \to Y$ and an open set U in Y in place of P and $P(X) \cap B$ above. The proof is trivial from definition as follows.

For any $x \in \partial(f^{-1}(U))$, by definition, we know x is not in $f^{-1}(U)$ but there is an element of $f^{-1}(U)$ in any neighbourhood of x. Thus $f(x)$ is not in U by definition and there is an element of U in any neighbourhood of $f(x)$ by the continuity of f, so $f(x) \in \partial U$ and $x \in f^{-1}(\partial U)$. Of course one can also see this by noticing $f^{-1}(\bar{U})$ is a

[11] Actually, hyperconvexity (of the domain in X) is not that necessary for the extension used in our construction. Basically, we only need the pullback function from the image side valued in an interval with controllable length. So the following discussion is only for the sake of a picture more similar to the classic situation. The only place where it's useful might be the hyperconvexity of the domain in a projective manifold which is require in the approximation results discussed in Chapter 5.

closed set in X containing $\overline{f^{-1}(U)}$.

Let's emphasize that the equality may not true in general, but the extra part of the preimage of ∂U is not in our consideration of the domain $f^{-1}(U)$.

Remark 6.2.4. *This result could be a little confusing when one thinks about the map of a close disk in $\mathbb{R}^2$ being projected to an interval. This confusion actually comes from the definition of boundary ∂U. The boundary of the whole space would always be empty using the above definition. But this definition of boundary would correspond to the usual one, which is also what we want, when X and Y are closed (i.e., without boundary). It might get people nervous since $Y = P(X)$ is singular as used before, but for this concern here, the "Y" can be chosen to be $\mathbb{CP}^N$, and so everything is classic.*

$P^{-1}(P(X) \cap B)$ also needs to be able to be enlarged to be a strictly bigger open set in X as part of our requirement of hyperconvexity. But there is also no need to worry about this since it is come from $P(X) \cap B$ in $P(X)$ and so we may take ball B's of different sizes to get all the open sets needed with desirable functions.

Anyway, we've seen from above that the global geometry of the domains (preimages) in X is good enough for us. Now we need to take care of the small pieces. Obviously, it is enough to consider them out of Z and W respectively. Since P is biholomorphic between those parts, we can do just what we did before and there is no need to worry about the geometry which would give the uniformality for the size of the small pieces. Just as mentioned, the point is that in the construction of small-piece-cover, we can avoid any subvariety in the preimage by removing ϵ-neighbourhood of it. The reason for us to introduce all the other varieties above is to make the sure that the neighbourhood in X would correspond to a Euclidean ball in $\mathbb{CP}^N$ in order to use the geometry there.

So in fact we are actually considering more than what we need (domains covering a neighbourhood of $\{{\omega_\infty}^n = 0\}$) in order to justify Condition (A).

The birationality will then make sure that it is good enough to compute the measure of any set in X using the data from $P(X)$ [12]. Again the uniformality with respect to ϵ is from the algebraic picture of $P(X)$ and the complex analytic picture of P described above. Simply speaking, any coordinate neighbourhood which covers part of the variety Z would contains a uniform "cone" pointed at the corresponding part of W.

This ends the proof of boundedness result when P is birational to its image.

[12] In fact it just says the total number of the small pieces are controlled well, and from here we can see the birationality is actually not that crucial.

6.2.3 General Case

Actually when we are considering a semi-ample (and big) class $[L]$ (for example, the canonical class K_X which is nef. and big), for sufficiently large integer m, the map $P : X \to \mathbb{CP}^N$ given by the holomorphic sections of $m[L]$ would be birational to its image. Of course, now the manifold X is algebraic (or projective). So the discussion in the previous subsection would be sufficient to conclude boundedness result with such a (nonnegative) class $[\omega_\infty]$ (rational or ratinal up to a positive real constant) in the degenerated Monge-Ampere equation. The reasoning is simple as follows.

Suppose the class $[L]$ is semi-ample (and big). We know for some (integral) effective divisor E, $[L] - \frac{1}{m}[E]$ would be positive for sufficiently big integer $m > 0$. Thus the holomorphic sections of the holomorphic line bundle $ml[L] - l[E]$ (for large enough integer $l > 0$) would provide an embedding of X to some projective space. Hence the sections of $ml[L]$ would do the same thing (i.e., tell apart the points and tangent vectors) out of E at least, which would give the birationality of the map P thus got.

We can actually do better. The stable base locus set of $[L]$ is the (finite) intersection of of those E's, and so sufficiently large finite multiple of $[L]$ can take care of the tangent vectors at each point out of the stable base locus set since by taking N large enough, we can have $N[L] - N_i[E_i]$ to be the $ml[L] - l[E]$ above for those finite i's and each z out of $\cap_i E_i$ would be out of some E_i. The situation for telling apart points out of the stable base locus set would be a little different since for two points not inside the stable base locus set, they might not be out of the same E_i. So as far as I can see, we only have local isomorphic out of stable base locus set and the image of the complement of the stable base locus set under the map from $N[L]$ might not be smooth. Even for the special case of complex dimension 2, the situation is not much better. Of course, when $[L] = K_X$, the picture is much more clear as we'll see in our application.

Remark 6.2.5. *Ths discussion above actually tells that the map P would be locally isomorphic out of the stable base locue set of $[L]$, telling apart the tangent vectors. So the pullback of Fubini-Study metric on $\mathbb{CP}^N$ to X would be nondegenerated out of the stable base locus set. Thus $\{{\omega_\infty}^n = 0\} \subset \{stable\ base\ locus\}$. We have also mentioned before that Nakayame's result tells that the stable base locus set is just the union of all the subvarities, Z, such that $[L]^{dim_{\mathbb{C}} Z} \cdot Z = 0$. Since $[L] = [\omega_\infty]$, we know $\{stable\ base\ locus\} \subset \{{\omega_\infty}^n = 0\}$. Hence we conclude that for m large enough, $\{{\omega_\infty}^n\} = \{stable\ base\ locus\}$.*

The above discussion tells that we have done enough to get the boundedness result when the class $[L] = [\omega_\infty]$ is semi-ample (and big). This could be the main interest for this problem.

In the following, we'll show that by combining all the previous arguments, the boundedness result can be proved in case of $[\omega_\infty]$ merely coming from a map. More precisely, as stated in the theorem, there exists a holomorphic map

$$P : X \to Y \subset \mathbb{CP}^N$$

with $dim_{\mathbb{C}} Y = dim_{\mathbb{C}} X$, and $\omega_\infty = P^*\omega_Y$ where ω_Y is a Kähler metric defined on a neighbourhood of Y. Here we just need X to be a closed Kähler manifold. This ω_Y is denoted by ω_M before.

As before, it is still only left to justify Condition (A) using the small-piece-cover coming for the map P.

First, it's easy to see the differential of the map P would be invertible at one point as follows.

Use W to denote the singular variety of Y, and $Z = P^{-1}(W)$, a subvariety in X which can't be X itself as $P(X) = Y$. Now for the restriction of the map P from $X \setminus Z$ to $Y \setminus W$ (of the same dimension), by Sard's Theorem, the set of critical points is of measure 0 on Y (and thus a genuine subvariety of Y). So there are actually plenty of points on X at which the differential of P is invertible. Actually, this is also how we know $\int_X P^*(\omega_Y)^n > 0$.

In fact, the injectivity of differential for one point implies also for a dense open subset of X. Thus by removing a big enough variety W in $Y = P(X)$ which contains the singular locus variety of Y and defining $Z = P^{-1}(W)$ which is a subvariety in X just as in the previous subsection, we have the restriction $P : X \setminus Z \to Y \setminus W$ be a covering map (i.e., with injective differential at each point).

For any $y \in Y \setminus W$, $P^{-1}(y)$, which is a subvariety of X consisting of points, should just be a finite set of points. Since $Y \setminus W$ is connected, we can say the map $P : X \setminus Z \to Y \setminus W$ is an γ-sheet covering map for some finite positive integer γ. Then still taking the local picture of $P(X)$, i.e., $P(X) \cap B$ for small Euclidean ball B in $\mathbb{CP}^N$, now the preimage might have several ($\leqslant \gamma$) components and the boundary of each one still gets mappped to the boundary of $P(X) \cap B$ [13]. So we can consider each of these components as the domain V and this would be enough for proving the boundedness result as explained below.

It is easy to confirm that the small pieces can be constructed just as before. There is just a small difference here. Recall that for the construction, the small pieces in $V \subset X$ are the preimages, $V_i \subset U_i$, of the small pieces, $S_i \subset T_i$, in $P(X)$ which are the intersections of the image with small balls in $\mathbb{CP}^N$. Of course, we now only cover the part away from W (or Z). In the cases considered before, each small piece in X would be connected just as in $P(X)$ (from birationality). But for the current situation, the preimage of a small piece in $P(X)$ may have several components even when restricted to V and the numbers of components for preimages of open sets O_1 and O_2 with $O_2 \subset O_1$ may not be the same in V since when one enlarges the open set in the image, the components in preimage might become connected. We are done if it is true that there is one component of T_i for each component of S_i such that both sit in the same coordinate chart in V. But this is still obvious from the consideration of "cone" mentioned before from the analyticity of the map P and the algebraicity of the image $P(X)$. The pullback of the functions from the lower picture in $\mathbb{CP}^N$ would work as before since the boundary values would be preserved by the simple topological result discussed before.

But we have to be careful here since the above statement would be true if we can make sure that the preimage of the small balls in $\mathbb{CP}^N$ will have each component sit in one of those chosen coordinate charts in a relatively compact way. This is not as obvious as in the earlier case when the map is birational at least locally. For the case here, we can't get this no matter how small the neighbourhoods that we choose are since the neighbourhoods of W and Z are considered near these varieties where in general there could be things like branching happening.

To deal with this problem, we still just need to see the size of the "ball" T_i can be chosen as ϵ^C for some fixed $C > 0$ large enough. Again, we need the properties of the map P to justify this.

Actually, there is another way to understand this picture which might give more intuition as follows.

Still choose finitely many coordinate balls on X to cover V. Then let's consider a curve (for a complex direction in $\mathbb{CP}^N$) in $P(X)$ passing through a point $z \in Z$. It's

[13]Classic result for covering spaces and the simple relation proved in the previous subsection are used here.

easy to see that the preimage of it would be the union of several ($\leqslant \gamma$ in fact) curves on X, which intersect W transitively, and a subvariety of W. Now by choosing a complex cone (with some high order as the $C > 0$ in the power of ϵ before) pointed at z around the curve, after removing the part in W from the preimage of it, we can have each component contained relatively compactly in a coordinate ball. Of course we might have to consider in a smaller ball in $\mathbb{CP}^N$ for each curve like that. But we can do this for each curve and the parameter space of all such objects is compact just as all the complex directions from Z. Thus we can definitely choose finitely many such cones which cover $P(X) \cap B'$ for some presumably ball B' smaller than the ball we started with in $\mathbb{CP}^N$ and the preimage of each cone is contained in the coordinate balls in X in a nice way. Now we can have the small pieces in X by pulling back the small balls in those cones and they will work just as before. For the argument, we do not even need to make sure that U_i and V_i are connected.

Also there is another difference coming along which is that the total number of small pieces in this domain V might be a multiple of the number of small pieces in $P(X) \cap B$. Fortunately that's just a uniformly finite ($\leqslant \gamma$) multiple, and so it will not affect our essential computation used before to prove the inequality (B). Thus the inequality (B) can be proved in exactly the same way, and so is Condition (A).

Hence boundedness result has been proved for the general case. Until now, we have finished proving (1) and (2) for Theorem 4.1.2 (or Theorem 1.3.2).

Chapter 7

Continuity of Bounded Solution and Stability

To begin with, let's recall that in the proof for the classic case as in Part (5) of Kolodziej's argument, we pick a special point after assuming the discontinuity of a bounded solution. We can still do the same thing for the degenerated case here. The argument is completely local. If the point lies in the part where $\omega_\infty > 0$, then the original argument would give us the contradiction. So it'll suffice for us to provide the argument when that point thus chosen actually lies on the degenerate set, $\{{\omega_\infty}^n = 0\}$.

As before, we can still get a neighbourhood which has the convexity for the potential of ω_∞. But the problem is that now there is no convolution to provide us with a smooth (decreasing) approximation, which is very involved in the argument, since this domain is no longer in $\mathbb{C}^n$. As pointed out before, we do need the approximation functions to be plurisubharmonic. In other words, for $u \in PSH_{\omega_\infty}(V) \cap L^\infty(V)$, we need $u_j \to u$ decreasingly as $j \to \infty$ with $u_j \in PSH_{\omega_\infty}(V) \cap C^\infty(V)$. This is not available from the approximation results listed before because ω_∞ is not positive. [1]

7.1 Orbifold Image Case

A trivial doable case would be when the map P is locally birational with a smooth image $P(X)$. This is basically the first case considered for the proof of boundedness result in which the original proof in [Koj1] is easily carried through with little modification.

The whole point is that we can use the convolution locally on $P(X)$ for functions

[1]The situation is slightly different from that of comparison principle.

over $V \subset X$. Now V is the preimage of a Euclidean ball B in $P(X)$ by the smoothness. The main observation is that plurisubharmonic functions over V are one-to-one correspondent to the plurisubharmonic functions over B. This can easily be justified by considering the biholomorphism between the dense open parts, which is the reason we require the local birationality, and using the extension property of plurisubharmonic functions to treat the rest part which is analytic (algebraic). [2] Clearly this is enough for us to carry through the original continuity argument for this case.

We can actually generalize the above case a little bit. When $P(X)$ carries an orbifold structure instead, since we also have the continuous approximation of plurisubharmonic functions by using the convolution on the orbifold coordinate chart, the continuity argument still goes through. Let's provide a little detail below.

We can still push a plurisubharmonic function u over V to the image $P(V)$ which has orbifold singularities. Denote the function over $P(V)$ by v. Clearly v is plurisubarmonic on a dense open (smooth) part of $P(V)$, and the values for the rest part would make v satisfy essential upper semi-continuity over $P(V)$. Thus the pullback of v over the orbifold coordinate chart U would be a G-invariant plurisubharmonic function w where G is the local orbifold group and $P(V) = U/G$. The convolution of the oribifold coordinate chart can be descended to the (singular) quotient if it preserves the G-invariance. This would be the case since G usually consists of just some rotations. More precisely, we just need $G \subset SL(n, \mathbb{C})$ which would preserve the Euclidean volume form over U. The functions from this "convolution" would only be continuous in general, but that's is still enough for our application. Here we do not require the map P to be compatible with the orbifold structure.

Remark 7.1.1. *There is also some other thing for the above case. Usually, the functions on X which are pullbacks of orbifold smooth functions on $P(X)$ would actually be Hölder continuous. So we would also get more than the continuity of the solution. This somehow makes us feel the orbifold structure of $P(X)$ should not be that necessary merely for the solution to be continuous.*

One might also want to use the obvious generalization of the results in [Koj1] in orbifold case as mentioned before, i.e., treat a proper equation over the image $P(X)$ instead. But if the map P is not birational, then it'll be impossible to get a proper (singular) volume form corresponding to a general volume form over X.

There might be a small issue about the convolution we get from the image since

[2] More complicated picture will be considered later where more details can be found.

the convolution needs to avoid boundary. But the boundary of the domain where we actually use the convolution might be more than the boundary of the general domain when we consider on X as from the simple topological relation discussed in Chapter 6. But we can deal with this by considering bigger domains. More presisely, the idea is pretty easy as follows. For the application in continuity argument, we have the function in a domain strictly containing V, so in a strictly bigger domain in $P(V)$ and also in the orbifold coordinate chart. Thus we can have the functions from the convolution defined over V. [3]

7.2 Attempt for Global Argument

For a long time, it seems too hard for me to carry through the continuity argument as above in general. So a different route has been tried a little. It's interesting in its own way and provides some useful information about the solution. This section is devoted to this "wrong" way.

Let's provide the global proof mentioned before for the continuity of the solution in the classic case when the background form is actually a (Kähler) metric. We would like to point out that essentially everything is already contained in [Koj2].

Lemma 7.2.1. *Let ω be a Kähler metric over a close manifold X. Suppose we have a sequence of uniformly bounded plurisubharmonic (with respect to ω) functions $\{u_j\}$ such that*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_j)^n = F_j\omega^n$$

with $F_j \geqslant 0$ and uniformly bounded in L^p-norm for some $p > 1$. Then there is a subsequence which uniformly converges to a bounded plurisubharmonic function u.

Remark 7.2.2. *For the original approximation as in [Koj1], u_j's are actually smooth from classic results since the measures, $F_j\omega^n$, are smooth nondegenerate volume forms. So the limit u would also be continuous by the above lemma. In fact, the uniform convergence would tell us that the corresponding measures would converge weakly to some measure which will just be the Monge-Ampere measure of u. Thus this discussion actually provides a slightly different point of view about the results in [Koj1].*

[3] Sometimes we have to extend the original function to achieve this and the hyperconvexity of the domain will be eough for us if we do not worry too much about the values near the boundary which is usually the case.

Proof. Suppose $|u_j| \leqslant C$ for some positive constant $C > \frac{1}{3}$. As usual, the same C with starting value greater than $\frac{1}{3}$ might change during the process. From the earlier discussion about the limit, we can assume u_j converges in L^1-norm to u.

Set $a_{j,k} = Cap_\omega(E_{j,k}(2\delta))$ where δ is some sufficiently small positive constant and $E_{j,k}(2\delta) = \{u_j + 2\delta \leqslant u_k\}$.

For any $v \in PSH_\omega(X) \cap C^0(X)$ valued in $[-1, 0]$, set $V_{j,k} = \{u_j + \delta \leqslant (1 - \frac{\delta}{3C})u_k + \frac{\delta}{3C}v + \frac{\delta}{3}\}$. From the following inequalities

$$u_k - \delta \leqslant (1 - \frac{\delta}{3C})u_k + \frac{\delta}{3C}v + \frac{\delta}{3} \leqslant u_k + \frac{2\delta}{3},$$

we have the following chain of sets:

$$E_{j,k}(2\delta) \subset V_{j,k} \subset E_{j,k}(\frac{\delta}{3}).$$

Comparison principle gives the following:

$$\int_{V_{j,k}} (\omega + \sqrt{-1}\partial\bar{\partial}((1 - \frac{\delta}{3C})u_k + \frac{\delta}{3C}v + \frac{\delta}{3}))^n \leqslant \int_{V_{j,k}} (\omega + \sqrt{-1}\partial\bar{\partial}(u_j + \delta))^n$$

which can be rewritten as

$$\int_{V_{j,k}} \big(\frac{\delta}{3C}(\omega + \sqrt{-1}\partial\bar{\partial}v) + (1 - \frac{\delta}{3C})(\omega + \sqrt{-1}\partial\bar{\partial}u_k)\big)^n \leqslant \int_{V_{j,k}} (\omega + \sqrt{-1}\partial\bar{\partial}u_j)^n.$$

Considering arbitrary such a function v and using the relation between those set, we arrive at

$$a_{j,k}(\delta)\big(\frac{\delta}{3C}\big)^n \leqslant \int_{E_{j,k}(\frac{\delta}{3})} F_j\omega^n.$$

Since $u_k - u_j \geqslant \frac{\delta}{3}$ over $E_{j,k}(\frac{\delta}{3})$, we can have the following computation

$$\begin{aligned} a_{j,k}\frac{\delta^{n+1}}{3^{n+1}C^n} &\leqslant \int_X |u_k - u_j| F_j\omega^n \\ &= \int_{\{F_j > M\}} |u_k - u_j| F_j\omega^n + \int_{\{F_j \leqslant M\}} |u_k - u_j| F_j\omega^n \\ &\leqslant 2C \int_{\{F_j > M\}} F_j\omega^n + M \int_X |u_k - u_j|\omega^n \\ &\leqslant \frac{2C}{M^{p-1}}\|F_j\|_{L^p}^p + M \int_X |u_k - u_j|\omega^n. \end{aligned}$$

From the uniform bound of $\|F_j\|_{L^p}$ and the L^1 convergence of u_j to u, by choosing the constant M large enough, we can have that for any (small) $\epsilon > 0$, if $j,\ k$ are large enough, then the following is true

$$a_{j,k} \leqslant \epsilon\delta^{-n-1}.$$

Let's emphasize that the discussion above is uniform for all $\delta > 0$ sufficiently small. [4]

Suppose $E_{j,k}(3\delta)$ is nonempty. Then the global argument (for Part (1)) quoted before would give for $j,\ k$ sufficiently big,

$$3\delta - 2\delta = \delta \leqslant \kappa(a_{j,k}) \leqslant \kappa(\epsilon\delta^{-n-1}).$$

When ϵ is very close to 0, this would be a contradiction. Hence we conclude that fixing any $\delta > 0$, for large enough $j,\ k$, we have $u_j + 3\delta > u_k$. The indices are symmetric, so we get the uniform convergence of this sequence $\{u_j\}$.

□

Remark 7.2.3. *The proof actually tells that a sequence would be uniformly convergent if it's L^1 convergent provided the measures are uniformly controlled. The background Kähler metric ω is fixed which is important for the symmetry of indices $j,\ k$ mentioned at the end.*

The argument is globally over X and the relative capacity used here is not "Cap_c^*". But it's OK as discussed before.

However, the situation is very different for our current consideration, the approximation solution, $u_\epsilon \in PSH_{\omega_\infty+\epsilon\omega}$, have less plurisubharmonicity than the solution $u \in PSH_{\omega_\infty}$ itself.

More precisely, the plurisubharmonicity is "increasing" as ϵ decreasing to 0. So we'll meet with the same kind of difficulty as in applying the local argument which is discussed before, i.e., not knowing $(\lambda\omega + \omega_\infty + \epsilon\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n$ as an L^p-measure for some $\lambda > 0$ and $p > 1$. Notice now the condition will be on all the approximation solutions (with some uniform bound of the norm) instead of the limiting solution itself. These two conditions are not quite equivalent. In some vague sense, the condition for local argument before looks less restrictive which is sort of natural to accept as the local argument makes use of some fairly delicate construction.

Remark 7.2.4. *In fact, from the way (flow or perturbation method) we get the*

[4] Actually we just need "$1 - \frac{\delta}{3C} \geqslant 0$".

(bounded) solution [5]*, the continuity of the solution out of the stable base locus set is justified. But as we know from results in algebraic geometry, the stable base locus set is contained in* $\{{\omega_\infty}^n = 0\}$. [6] *So we actually do have to consider the degenerate set.*

7.3 General Case

Professor Kolodziej pointed out a classic extension result in [FoNar] which can be used to treat the main difficulty of the argument for continuity, i.e., the lack of smooth (or just continuous) local approximation of plurisubharmonic functions in a general domain which is not in $\mathbb{C}^n$. In this section, we prove (3) in Theorem 4.1.2 (or Theorem 1.3.2).

The idea is again to use the image of the map P. Let's restrict ourselves to the case when the map is birational. This covers our main interest when the map is from a semi-ample (and big) bundle as discussed before. Indeed, it's OK if the map is locally birational which also allow us to push forward the solution to the image locally as explained below.

7.3.1 Weak Plurisubharmonicity

For a birational map $P : X \to P(X)$, we can push forward the solution u for the degenerate Monge-Ampere equation $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega$ over X to the image $P(X)$ as follows.

First, by the local isomorphism from birationality, we get a function v on a dense open part of $P(X)$ (out of a subvariety). For the rest part, we can see the preimage of each point in $P(X)$ would have to be a connected subvariety of X. In order to justify this, we only have to use the connectedness of a neighbourhood in $P(X)$ (which is clearly irreducible) of this point and the birationality of the map P. The background forms ω_{FS} and $\omega_\infty = P^*\omega_{FS}$ [7] have corresponding local potentials, and so we can still get a lawful value of v for this point from the function u. It's quite easy to see the function v thus get will be upper semi-continuous on $P(X)$ just from definition. Of course, P^*v, the pullback of v on X would be u itself from the construction. The function v is clearly bounded.

[5] Ω is a smooth volume form and e^u is on the right hand side of the equation.

[6] In fact, we have seen they are actually the same set using proper choices.

[7] We can use any Kähler metric ω_M as in the statement of the theorems. But ω_{FS} comes with the most geometric interests.

Remark 7.3.1. *It's also natural to think about this pushforward in another way. By the birationality of P, we can get v out of a subvariety of X. Then we can use upper semi-continuization to extend v to the whole $P(X)$, $\tilde{v}$. Clearly, the pullback of this function to X by P would be u itself out of that subvariety, but we can't just say $P^*\tilde{v} = u$ from this since we haven't seen the plurisubharmonicity of $P^*\tilde{v}$.*

But if we use the essential upper semi-continuity (along varieties) discussed below, it'll be easy to see $\tilde{v}$ would exactly correspond to the function over $P(X)$ constructed before. So these two points of views are actually equivalent.

Now the idea is to locally extend v plurisubharmonically (after being combined with the local potential of ω_{FS}) to a neighbourhood of $P(X)$. Everything is inside a Euclidean ball of $\mathbb{CP}^N$. So finally the usual convolution would provide the approximation for v (and also for u by pulling back to X). The following is some discussion about the extension result.

First, we want to make sure that the function v is weakly plurisubharmonic as used in [FoNar]. The definition would be clear from the discussion below.

We have already seen that it is upper semi-continuous. In fact, it's essentially upper semi-continuous just as in classic case for plurisubharmonic functions over a domain in $\mathbb{C}^n$ since a measure 0 set in the image $P(X)$ would have the preimage also with measure 0 in X. [8] Indeed, we can only consider subvarieties on $P(X)$ and X instead of all those measure 0 sets. It might be easier to feel more convinced like this.

The above property is emphasized because it tells that the value at any point of $P(X)$ can be decided from the values for enough points near it. Since we have the plurisubharmonicity of the function v in a big part of $P(X)$ which is isomorphic with its preimage in X, this will make it easier to control the values for the remaining part.

In fact, for the functions in the general domain in X [9], we can go one step further to get essential upper semi-continuity along closed varieties. Classic results tell us (restriction of) any plurisubharmonic function over a variety would have to be a constant. [10]

Combining the observations above, we have the following fact.

Claim: Keep the above notations. Then for any surjective holomorphic map

[8] The measure for $P(X)$ is induced from some metric in $\mathbb{CP}^N$ which is clearly dominated by the pushforward measure from some smooth measure on X.

[9] The following is for any domain in a manifold which contains closed varieties.

[10] The constant can be $-\infty$ apriori, but clearly we don't have to worry about this as the functions are all bounded for our consideration.

$R : Y \to P(X)$ [11], where Y is a smooth manifold with the same dimension as $P(X)$, we have $R^*v \in PSH_{R^*\omega_{FS}}(Y)$.

The dimension assumption guarantees that R would be a covering for a dense open part. This assumption is not very restrictive since R would be a resolution of singularities for $P(X)$ in our application.

The justification of this result is as follows. First, R^*v is plurisubharmonic with respect to $R^*\omega_{FS}$ for a dense open part of Y coming from the regular part of $P(X)$. Over Y, we can extend such a function plurisubharmonically to the whole of Y since it's smooth. This element would be essentially upper semi-continuous along varieties which are preimages of points on $P(X)$. Clearly, this is also true for the original pullback R^*v over Y. Hence they should be the same and we get what we want.

The property left to be justified in order to see that v is weakly plurisubharmonic is the following.

For any holomorphic map $f : \Delta \to P(X)$ where Δ is the unit (open) disk for $\mathbb{C}$, we need to know f^*v is subharmonic over Δ. Clearly, this is a local statement about $P(X)$ and would be the same as requiring v to be plurisubharmonic if $P(X)$ is smooth.

Let's start with a prototype to illustarte the proof first. Suppose $P(X)$ has the local picture of $\{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3$ which has been our favorite choice of singularity. If the image $f(\Delta)$ does not contain $(0, 0, 0)$ which is the only singular point, the subharmonicity of the pullback function is classic. Otherwise, we can assume the following two local cases. One is that $f(\Delta) = \{(0, 0, 0)\}$ which is a trivial case for us. The other one is that only the center of Δ is mapped to $(0, 0, 0)$. In the second case, it's easy to see that the map f can be lifted by the resolution of singularity map for $P(X)$ since one just needs to blow up the point $(0, 0, 0)$ once to make it smooth. [12] As we have seen before, the lift of function v by the resolution of singularity map is indeed plurisubharmonic. So we conclude f^*v is subharmonic.

Using the similar idea, we can treat the general case as follows. In general, by Hironaka's classic result about resolution of singularities, we can make $P(X)$ smooth by blowing up along subvarieties for finitely many times. Thus if we are in the counterpart of the second case considered above, then similar argument would work. Basically, we have a nontrivial tangential direction at $f(0)$ and it can be used to life

[11] The "holomorphic" means that the map $R : Y \to \mathbb{CP}^N$ is holomorphic with the image inside $P(X)$.

[12] Essentially it's the tangential direction that is used to lift f at the center.

the map f, though we might have to do this several times to get a smooth range object as mentioned above. The nontrivial case becomes the counterpart of the first case considered in the above example. Now we have to consider the map $f : \Delta \to W$ where W a subvariety of $P(X)$ with lower dimension which is basically the variety being blown up in the resolution of singularities. Consider the restriction of resolution map $R : Z \to W$, we know for dense open (regular) parts Z_0 and W_0 for Z and W respectively, the map $R : Z_0 \to W_0$ is a smooth holomorphic fibration, and so the map $f : \Delta \to W$ can be lifted locally for this part and if the image is not in the complement of W_0 in W, the local lift can be done for Δ. Then we have the (pluri)subharmonicity of f^*v. Now we only need to further consider the case when $f(X) \subset W \setminus W_0$ and it can be treated similarly using smooth fibration picture as above. The dimension is strictly decreasing and so it must end after finitely many steps.

Hence we conclude the subharmonicity of f^*v in general and get the weak subharmonicity of v. This would allow us to apply a classic extension result which is discussed in the next subsection and provides us with the local plurisubharmonic approximation needed to run through continuity argument.

7.3.2 Extension Result and Application

In [FoNar], a local extension result is proved. Let's state the result below. The proof is not that long, but it's fairly much unrelated to the rest part of this work, and so the details won't be presented here.

Theorem 7.3.2. *Any weakly plurisubharmonic function over a complex space can be locally extended to a plurisubhamonic function over the smooth local ambient space.*

The complex space has local picture of (singular) analytic varieties and so the smooth local ambient space is just a ball in $\mathbb{C}^{\mathbb{N}}$. The locality of the extension depends on all kinds of things including the function itself.

Remark 7.3.3. *The proof actually makes use of the results in another big branch in pluripotential theory. Stein domain, Runge domain, etc. are the main objects there. Rossi's local maximum modulus principle is also used in an essential way. Professor Rossi's help is very important for me to get a hold of the argument used in proving this theorem. A few things will be discussed in Appendix in order to make it possibly easier for people who are not so familiar with these stuffs to understand the proof in [FoNar].*

Now let's see why this result helps us to eventually prove the continuity of any bounded solution of the degenerate Monge-Ampere equation under the assumption that the map P is locally birational.

First, as pointed out before, the original argument (by deriving contradiction), which we want to follow, starts with picking a special point $x \in X$. This point obviously has its own image $P(x) \in P(X)$ which is a variety in $\mathbb{CP}^N$. Moreover, we only need to consider the case when this point is in the general domain V which covers (a part of) the degenerated set $\{{\omega_\infty}^n = 0\}$.

By the assumption for the map P, the solution $\tilde{u}$, (after combining with the potential of the semi-positive background form ω_∞) restricted to the general domain V can be pushed onto $P(V)$. Then the result above allows us to extend this function to a Euclidean ball in $\mathbb{CP}^N$ centered at $P(X)$. We can clearly pick our new domain V from this. Then convolution in this ball gives a smooth decreasing approximation locally for this pushforward function over $P(V)$ and so the pullback to V by P gives a decreasing approximation by smooth plurisubharmonic functions.

In the following, we briefly sketch how this would give the continuity. The punchline of the argument is as follows. Suppose $\{\tilde{u}_j\}$ is the sequence of smooth plurisubarmonic functions constructed above which are defined on a neighbourhood slightly larger than V [13] which decreases to $\tilde{u}$ pointwisely. Then by the construction in [Koj1] quoted before, which is very local and can be easily adjusted to our case, we can prove the sets $\{\tilde{u} + c < \tilde{u}_j\}$ are nonempty and relatively compact [14] inside V for all $c \in (0, a)$ for $a > 0$ and $j > j_0$.

The local argument for L^∞ estimate before gives

$$\frac{a}{2} \leqslant \kappa(Cap(\{\tilde{u} + \frac{a}{2} < \tilde{u}_j\}, V)).$$

We also notice that the relative capacity of the set $\{\tilde{u} + \frac{a}{2} < \tilde{u}_j\}$ would go to 0 as $j \to \infty$. This can be justified by the decreasing convergence and $\frac{a}{2} > 0$. Finally, we can draw the contradiction by letting $j \to \infty$ in the equality above because the right hand side is going to 0.

Remark 7.3.4. *Recently, Professor Kolodziej has proved some Hölder continuity result in the classic case in [Koj3]. Things like convolution and the geometry of*

[13] This is not a problem by the flexibility of our choices.

[14] For the relative compactness of the sets, strictly speaking, we have to use another function (as w there) which is constructed from $\tilde{u}$ linearly instead of $\tilde{u}$ itself in the setting. The quoted argument in Chapter 3 contains details for this.

Euclidean disk seem to be involved in a more essential way. But somehow, this should give us hope for further results in the degenerate case. In fact, when the map P is very nice (but not an embedding), Hölder continuity of solution can be seen from other ways for special equations.

7.4 Stability for Continuous Solution

Let's end this chapter by pointing out that the Kolodziej's argument for stability of continuous solution can be used line by line for our case without any change. So we can have exactly the same conclusion as in Theorem 3.2.12 for the degenerate Monge-Ampere equation. Of course, this gives uniqueness of continuous solution.

Remark 7.4.1. *It's not that satisfying in comparison to Kolodziej's case when the background form is positive. In that case, boundedness implies continuity which means the stability result is also for bounded solution. But now, the assumption on the map P for boundedness result is slightly weaker than that for continuity result. In Appendix, there are some discussions about corresponding results for bounded solution.*

Chapter 8

Applications and Further Problems

In this chapter, we present some applications of the results got from the previous (two kinds of) arguments. Further consideration about this whole program is also discussed.

8.1 Miyaoka-Yau Inequality

In this section, we give an application of the (singular) metric constructed in Chapter 2. As we see below, the result is in a strong flavor of differential geometry, and so unfortunately the results from pluripotential-theoretic argument, which are basically about C^0-norm at least for now, are not so useful. We'll also see one reason why we introduce the perturbation methods after we've already got the metric by flow method.

There is a classic result about the Chern classes for a Kähler-Einstein manifold which could be seen as a standard application of the Kähler-Einstein metric. We want to see that our (singular) K-E metric can also do similar job possibly with proper assumption about the singularities. In the following, we'll use the same convention and notations as in [CheOg] where standard computation is carried out explicitly. The original result in that paper is as follows.

Theorem 8.1.1. *For a closed Kähler-Einstein manifold of complex dimension $n \geqslant 2$, M, we have:*

$$[\omega]^{n-2}c_2 \geqslant \frac{n}{2(n+1)}[\omega]^{n-2}{c_1}^2$$

where ω is Kähler-Einstein [1]*, "$[\ \cdot\ ]$" means taking the corresponding cohomology class*

[1]There is no need for normalization of the constant as is clear from the rescaling invariance of

as usual. c_1 *and* c_2 *are the first and second Chern classes of* X *(i.e., the holomorphic tangent bundle of* X*). The pairing on both sides are the standard topological one. Furthermore, the equality holds only when the manifold* M *is a complex space form.*

The proof basically makes use of the following two classic results (see [CheOg]) which are obtained by pointwise computation.

Lemma 8.1.2. *For any closed Kähler manifold,* M*, with Kähler metric* ω*, we have:*

$$[\omega]^{n-2}c_2 = A \cdot \int_M (\rho^2 - 4\|S\|^2 + \|R\|^2)\omega^n,$$

$$[\omega]^{n-2}{c_1}^2 = 2A \cdot \int_M (\rho^2 - 2\|S\|^2)$$

where ρ*,* S *and* R *are scalar curvature, Ricci tensor and curvature tensor with respect to the metric* ω *respectively and the norm* $\|\cdot\|$ *is also taken with respect to* ω*. "*A*" is some universal positive constant depending only on* n*.*

Remark 8.1.3. *The statement can actually be made pointwise if we use the metric* ω *to compute the Chern forms. In that case we do not have to require the closedness of the manifold. But what we really need is just the cohomological statement above.*

Lemma 8.1.4. *For any Kähler manifold ,*M*, with Kähler metric* ω*, we have pointwisely that*

$$\frac{n(n+1)}{2}\|R\|^2 \geqslant 2n\|S\|^2 \geqslant \rho^2.$$

All the notations have the same meaning as in the previous lemma. The left equality holds when ω *is a complex space form metric. The right equality holds when* ω *is Kähler-Einstein.*

After getting these two results, there is only a little algebra left to conclude the theorem above. In fact we'll do something quite similar for our case with just a little modification. Basically, we'll use Lemma 8.1.2 for the approximation metrics and Lemma 8.1.4 for the limiting metric.

We'll use the perturbation approximation below. So let's recall the setting first

$$\tilde{\omega}_\epsilon^n = (\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}\Omega$$

where $\omega_\epsilon = \omega_\infty + \epsilon\omega$ for a fixed Kähler metric ω, and $[\omega_\infty] = K_X$ nef. and big.

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this inequality.

The following are the main estimates got from before:

$$\tilde{\omega}_\epsilon^n \geqslant C_\alpha |\sigma|^\alpha \omega^n, \qquad \langle \omega, \tilde{\omega}_\epsilon \rangle \leqslant C_\beta |\sigma|^{-\beta}, \tag{8.1}$$

where α, β are positive constants uniform for all $\epsilon \in (0,1]$. Notice that α can be as close to 0 as we want, but that's not the case for β. Actually, we can have the following control of the approximation metrics as metric [2]

$$C|\sigma|^{(n-1)\beta+\alpha}\omega \leqslant \tilde{\omega}_\epsilon \leqslant C|\sigma|^{-\beta}\omega,$$

where the right "$\leqslant$" is obvious from the bound of trace, while the left one can be seen by using contradiction. Notice $\tilde{\omega}_\epsilon$ is a smooth metric for all these ϵ's.

We also have the convergence $\tilde{\omega}_\epsilon \to \tilde{\omega}_\infty = \omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty$ locally smoothly in $X \setminus \{\sigma = 0\}$ as $\epsilon \to 0$. Here the ∞ is used in ω_∞ to indicate it is also the one from flow construction.

The observation below is trivial but useful:

$$\mathrm{Ric}(\tilde{\omega}_\epsilon) = -\sqrt{-1}\partial\bar{\partial}\log(e^{u_\epsilon}\Omega) = -\omega_\infty - \sqrt{-1}\partial\bar{\partial}u_\epsilon.$$

Remark 8.1.5. *We see above that the Ricci form is essentially the metric form for the approximating Kähler metrics up to an explicit term. If we use the flow construction, then $\sqrt{-1}\partial\bar{\partial}(\frac{\partial u}{\partial t})$ comes up. Of course we have the local control of it in $X \setminus \{\sigma = 0\}$ from the bounds for higher derivatives by classic interior estimate, but that is not as explicit as the estimates listed above.*

Now for all $\epsilon > 0$, we suppose

$$[\tilde{\omega}_\epsilon]^{n-2}c_2 = a_\epsilon[\tilde{\omega}_\epsilon]^{n-2}{c_1}^2$$

which is also true if one uses ω_ϵ instead as basically a cohomology result. The limiting situation is

$$[\omega_\infty]^{n-2}c_2 = a_\infty[\omega_\infty]^{n-2}{c_1}^2.$$

Here we have the well-definedness of a_ϵ and $a_\epsilon \to a_\infty$ as $\epsilon \to 0$ because $[\omega_\infty]^{n-2}{c_1}^2 \neq 0$ by noticing the right hand side is some nonzero multiple of $[\omega_\infty]^n > 0$ from nef. and big assumption. The lower index ∞ is still inherited from the flow construction which does not look so natural in this setting.

[2] This has been mentioned and used before as uniform control of the metrics locally out of the set $\{\sigma = 0\}$. Here we just put it into a more global-looking version.

Now we can carry out the following computation:

$$\begin{aligned} 0 &= \int_X \big((1-2a_\epsilon){\rho_\epsilon}^2 - 4(1-a_\epsilon)\|S_\epsilon\|^2 + \|R_\epsilon\|^2\big)\tilde{\omega}_\epsilon^n \\ &\geqslant \int_X \big((1-2a_\epsilon){\rho_\epsilon}^2 - 4(\frac{n}{n+1} - a_\epsilon)\|S_\epsilon\|^2\big)\tilde{\omega}_\epsilon^n \\ &= \int_X \big((1-2a_\epsilon){\rho_\epsilon}^2 - 4(\frac{n}{n+1} - a_\epsilon)\|S_\epsilon\|^2\big)e^{u_\epsilon}\Omega, \end{aligned} \tag{8.2}$$

where ρ_ϵ and $\|S_\epsilon\|^2$ are essentially (up to positive constants) just the following two terms respectively

$$\langle \tilde{\omega}_\epsilon, -\omega_\infty - \sqrt{-1}\partial\bar{\partial}u_\epsilon \rangle,$$

$$(-\omega_\infty - \sqrt{-1}\partial\bar{\partial}u_\epsilon, -\omega_\infty - \sqrt{-1}\partial\bar{\partial}u_\epsilon)_{\tilde{\omega}_\epsilon}.$$

Thus if we can take β in (8.1) to be small enough, then we can apply Dominated Convergence Theorem for the last expression above as $\epsilon \to 0$. In this case, we can of course only consider the integration over $X \setminus \{\sigma = 0\}$ where the limiting metric is Kähler-Einstein. The following is what we'll get:

$$\begin{aligned} 0 &\geqslant \int_{X\setminus E} \big((1-2a_\infty){\rho_\infty}^2 - 4(\frac{n}{n+1} - a_\infty)\|S_\infty\|^2\big)\tilde{\omega}_\infty^n \\ &= 4(n-1)\int_{X\setminus E} (\frac{n}{2(n+1)} - a_\infty)\|S_\infty\|^2\tilde{\omega}_\infty^n. \end{aligned}$$

By noticing S_∞ is essentially just $-\tilde{\omega}_\infty$ over $X \setminus \{\sigma = 0\}$, we get $a_\infty \geqslant \frac{n}{2(n+1)}$.

Hence we arrive at the following inequality:

$$(-1)^n {c_1}^{n-2} c_2 \geqslant (-1)^n \frac{n}{2(n+1)} {c_1}^n$$

because $C \cdot [\omega_\infty] = -c_1$ for some positive constant C due to different convention and $[\omega_\infty]^n > 0$.

Remark 8.1.6. *In the classic case, there is also characterization about when equality would hold. But that'll make use of* (8.2) *for the (singular) Kähler-Einstein metric, which means we have to use Dominated Convergence Theorem there. So we'll need the control for* $\|R_\epsilon\|$. *But just as mentioned before, until now the control we have is not very satisfying for this purpose.*

Hence we conclude the following theorem:

Theorem 8.1.7. *For a smooth projective manifold* X *with* K_X *nef. and big, if there* *is some proper control of the trace of the singular Kähler-Einstein metric constructed before near the singular variety, i.e., the β in* (8.1) *can be taken to be sufficiently small, then we conclude, just as in the case when K_X positive, that:*

$$(-1)^n {c_1}^{n-2} c_2 \geqslant (-1)^n \frac{n}{2(n+1)} {c_1}^n.$$

Remark 8.1.8. *The inequality should hold when X is a projective surface and K_X is only big (see in [Mi]). So it seems reasonable to conjecture that the assumption in the above theorem will always be satisfied in the case of complex dimension* 2.

8.2 Combining Results

In this section, we give an example about how to study interesting geometric objects by combining the results got before from the two kinds of arguments. More specifically, we prove Theorem 1.4.1 stated in Introduction.

From the discussion in Chapter 2, we can already prove statements (1), (2) and (4) of this theorem. Now by the argument from pluripotential theory in Part II of this work, we can get (3) as follows.

Consider the Kähler-Ricci flow equation (2.1) with $S = 0$:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \qquad \tilde{\omega}_0 = \omega_0,$$

where ω_0 is any given Kähler metric.

Let $\omega_\infty = -\mathrm{Ric}(\Omega)$ for a volume form Ω. Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ and $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, we can put (2.1) on the level of potential and more in the Monge-Ampere setting as:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t} + u}\Omega.$$

We have seen in Chapter 2 that the right hand side has a uniform L^p-norm bound for all t with any $1 < p \leqslant \infty$. In fact, we know u and $\frac{\partial u}{\partial t}$ are bounded uniformly from above.

Now we know $[\omega_\infty] = K_X$ is nef. and big by assumption. Hence it would be semi-ample and provide us with a birational map P from X to some $\mathbb{CP}^N$ as mentioned before, and so it falls right into the picture of Theorem 1.3.2.

Though in $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$, we do not assume that $\omega_0 - \omega_\infty > 0$ which would make it more like the perturbation previously used in Part II, by combining with the degenerate lower bound of u from Chapter 2 which works like assuming "$max_X u(t,\cdot)=0$", we can still have the uniform L^∞ estimate for $u(t,\cdot)$ with $t\in[0,\infty)$ simply by using part of ω_∞ in the front to dominate the second term. In other words, let's consider ω_t as $\frac{1}{2}\omega_\infty + e^{-t}\big(\omega_0-(1-\frac{1}{2}e^t)\omega_\infty\big)$ where $\frac{1}{2}\omega_\infty$ can be used as $P^*\omega_M$ in Theorem 1.3.2 (or Theorem 4.1.2) and the rest part is a harmless positive perturbation for our argument for t sufficiently large. [3]

This would give us the boundedness of the potential and so its limit. The continuity follows directly from Theorem 1.3.2 since the map P is birational to its image. Hence (3) in Theorem 1.4.1 is true.

Actually we can also see that the limiting solution is more canonical than what's stated in (4) since it's even unique for the limiting equation by comparison principle.

Remark 8.2.1. *The above is more or less like building connection between different pictures. But we have very good reasons to expect that they can eventually tells us much more about these objects.*

8.3 Surface Case

In complex dimension 2 case, using the rich theory about complex surfaces, we can show some relation between the (singular) Kähler-Einstein metric constructed before with other known metrics as follows.

Now X is a minimal complex surface of general type. This is just another way of saying that K_X is nef. and big. It is well known that a basis of sections of mK_X for some $m>0$ gives rise to a holomorphic map $P: X\to\mathbb{CP}^N$. The map P will contract finitely many rational curves to points (as stated in the appendix of [Za] by Mumford which basically makes use of the results in [Ar]) and the image $\overline{X}=P(X)$ is a Kähler orbifold (see for example in [Du]) with rational double points. We can explain part of this picture using some simple algebraic geometry argument below. It looks simple only because we make use of some results already been mentioned before which might not be easy to prove at all.

Since K_X is nef. and big, we know the map P would be a local isomorphism out of the stable base locus set of K_X for m sufficiently big as discussed before. From Nakamaye's result mentioned and also used before, the stable base locus set should be the union of the varieties over which the class K_X is degenerate as Kähler class.

[3] In fact, the uniqueness result already proved before allows us to only consider the case when $\omega_0>\omega_\infty$. But the treatment here gives some idea about the flexibility of the argument.

In the surface case, it should just be the union of the (irreducible) curves C such that $K_X \cdot C = 0$.

We can use adjunction formula to study these curves.

$0 = K_X \cdot C = K_C \cdot C - C \cdot C = 2g_C - 2 - C \cdot C$. Since $K_X - \epsilon E$ would be Kähler for some $\epsilon > 0$ and curve E from the bigness of K_X, we have $(K_X - \epsilon E) \cdot C > 0$. As $K_X \cdot C = 0$, we have $E \cdot C < 0$ and so $C \cdot C < 0$. By plugging this back into the adjunction formula above, we see $g_C = 0$ and $C \cdot C = -2$. So C is a (rational) (-2)-curve. Clearly, there could only be finitely many of them by cohomological consideration. It's easy to see that the argument above can be reversed. So the stable base locus set of K_X consists of all (-2)-curves. Anyway, we now know that every C is contained in the divisor E used above. And we can see by a simple computation that the intersection between different C's can only be 0 or 1. In the following, C would stand for a connected component of the union of (-2)-curves in sight of the possible intersection 1 between them.

These properties would characterize the singularity of the image and give the picture above. In fact, as $K_X \cdot C = 0$, we know that a holomorphic section of mK_X would have its 0 locus set either not intersect C or contain C. Since mK_X is also base-point-free, both situations should appear. We can easily conclude from this that the map P would contract each C to a point, though we can not see here that the point would be different for each C (a chain of (-2)-curves). [4]

Anyway, we have the picture of the map P described before. The argument in [Ar] tells that $P^* K_{\overline{X}} = K_X$ and now $K_{\overline{X}}$ is an orbifold K-E class. Set $m\omega_\infty = P^*\omega_{FS}$, where ω_{FS} is the standard Fubini-Study metric on $\mathbb{CP}^N$. We know that the (singular) K-E metric (from Kähler-Ricci flow or other perturbation methods) got before is smooth outside those rational curves contracted by P.

In the following, we'll show that the singular metric constructed before actually coincides with the pullback of the unique Kähler-Einstein orbifold metric on $\overline{X}$ with $K_{\overline{X}}$ as its Kähler class.

Let $\overline{\omega} = \frac{1}{m}\omega_{FS}|_{\overline{X}}$. Since it represents $K_{\overline{X}}$, there is a (orbifold) volume form $\overline{\Omega}$ on $\overline{X}$ such that $\mathrm{Ric}(\overline{\Omega}) = -\overline{\omega}$.

Moreover, we can see the pullback of this form to X is a smooth volume form Ω such that $\mathrm{Ric}(\Omega) = -\omega_\infty$ as follows. Basically, since $K_X \cdot C = 0$, we know the

[4] Actually, this situation is called as *A*-*D*-*E* singularity. See for example [BaPetVa] for more systematic discussion.

canonical bundle is actually trivial in a small neighbourhood of C. Thus the orientation bundle is also trivial there. The pushforward of a local volume form $|s|^2$ in that neighbourhood in X, where s is a nowhere 0 holomorphic function (i.e., a section of K_X there), is a volume form away from the rational double point. We can lift it to the orbifold coordinate chart. This volume form is defined only out of a point which is of codimension 2. Applying Hartogs' Theorem to the function s, we know the extension would be a orbifold volume form and that's exactly what we want.

Now write the Kähler-Einstein orbifold metric as $\overline{\omega} + \sqrt{-1}\partial\bar{\partial}v$ [5], where v is a smooth function in the sense of orbifolds. In particular, v is continuous on $\overline{X}$. Furthermore, on $\overline{X}$, it satisfies the following Monge-Ampere equation

$$(\overline{\omega} + \sqrt{-1}\partial\bar{\partial}v)^2 = e^v\overline{\Omega}.$$

This equation can be pulled back to an equation on X:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^2 = e^u\Omega,$$

where $u = P^*v$ clearly belongs to $PSH_{\omega_\infty}(X) \cap C^0(X)$.

By uniqueness result of such a solution for the degenerated Monge-Ampere equation which is rather trivial by applying comparison principle, we know that these two metrics have to be the same.

Remark 8.3.1. *In fact, we don't even need the boundedness of the solution got before (from flow method) to conclude this because we can justify comparison principle for more general class of functions as mentioned at the end of Chapter 2.*

The following corollary is what has been proved.

Corollary 8.3.2. *If X is a minimal complex surface of general type, then the global solution of the Kähler-Ricci flow converges to a positive current $\tilde{\omega}_\infty$ which descends to the Kähler-Einstein orbifold metric on its canonical model. In particular, $\tilde{\omega}_\infty$ is smooth outside finitely many rational curves and has local continuous potential.*

Remark 8.3.3. *Of course the consideration above also works for higher dimensions under proper assumption. But it's of course much more restrictive.*

[5] This can be obtained essentially by applying Yau's argument in [Ya] to the orbifold picture as done in [Koi] for example.

8.4 Direct Generalization

We have kept mentioning the canonical line bundle K_X over X and Kähler-Einstein metric and correspondingly in the equation $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$, we assume $\omega_\infty = -\mathrm{Ric}(\Omega)$ besides the more essential assumption on the class $[\omega_\infty]$, though it is pointed out that this assumption is not necessary for most of our argument. But at that time, the real meaning (or say more geometric meaning) of the equation without this extra assumption is not quite clear. Now we are going to clarify this.

Let's reconsider the equation (2.1) which is the main flow equation being considered on the level of metric:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t + S, \qquad \tilde{\omega}_0 = \omega_0$$

where ω_0 is the initial Kähler metric and S is some smooth real closed $(1,1)$-form.

This is just the $k = -1$ case of (1.1) at the very beginning which is our main interest. Notice that in the general computation in Introduction, S always appears in $S - \mathrm{Ric}(\Omega)$ where Ω is a smooth volume form. So we might want to replace S by $\mathrm{Ric}(\Omega) + L$ where L is another smooth real closed $(1,1)$-form. Now the equation becomes:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t + \mathrm{Ric}(\Omega) + L.$$

In this form, at least cohomologically, it is clear that the flow is a tool for us to deal with the class $[L]$.

If we remove the "$-\mathrm{Ric}(\tilde{\omega}_t) + \mathrm{Ric}(\Omega)$" part on the right hand side of the equation, the remaining part is a rather trivial evolution equation and the limit is trivial L in a strong sense which it is not so interesting. In fact we use the part $-\mathrm{Ric}(\tilde{\omega}_t) + \mathrm{Ric}(\Omega)$ to make sure that the solution (if exists) will be a metric for each time slice while hoping the limit in any sense would preserve the positivity.

Now intuitively we want to get a limit as $t \to \infty$ for the flow which means we are heading for a "metric", $\tilde{\omega}_\infty$, satisfying

$$0 = -\mathrm{Ric}(\tilde{\omega}_\infty) + \mathrm{Ric}(\Omega) - \tilde{\omega}_\infty + L.$$

Formally considering in the level of cohomology, we would have $[\tilde{\omega}_\infty] = [L]$ which means we could find a metric representative for the class L if there is such a limit. Of course we do not expect this to happen for general L, but this will give one way to see how we are going to fail or succeed in finding such a representative. Moreover we might lower our standard of the limit by allowing singularities (essentially allowing weak convergence), which is just what have been done earlier.

Let's explicitly write down what we have to treat. Set $\omega_t = L + e^{-t}(\omega_0 - L)$, and $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. Then formal computation tells us that u will satisfy the following evolution equation:

$$\frac{\partial u}{\partial t} = \log \frac{\tilde{\omega}_t^n}{\Omega} - u, \qquad u(0, \cdot) = 0$$

which should look very familiar up to now. And we'd better focus on this equation instead of the equation on the level of metric as usual. Clearly all the equivalence about these equations can be justified in the same way as before. The limiting equation now becomes

$$(L + \sqrt{-1}\partial\bar{\partial}v)^n = e^v\Omega,$$

and there is no relation between $\mathrm{Ric}(\Omega)$ and L now. In fact there is nothing very fancy here if one notices that in the discussion before, we suppose $S = 0$ and so $L + \mathrm{Ric}(\Omega) = 0$. At this moment we just make it more explicit that it is the class $[L]$ that the flow is heading for and it could in fact be anything.

No difference needs to be made to essentially all the discussions for the flow equation above in dealing with this general case and we briefly sketch the output below.

For $[L]$ positive, for any choice, we always have that the limit exists in C^∞-topology which satisfies

$$(L + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{u_\infty}\Omega$$

for any L and Ω we might choose. This is classic.

For $[L]$ numerically effective, for any choice, we still have the global existence of the solution for the flow equation. In fact the flow exists as long as the class remains to be Kähler in general.

For $[L]$ nef. and big, we still have $[L] - \epsilon E$ positive for some effective divisor E and $\epsilon \in (0, a)$. So there is no change for all the local discussions. We get a limit u_∞, which is smooth out of the stable base locus set of $[L]$ with some estimates near the stable base locus set and globally plurisubharmonic over X, satisfying

$$(L + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{u_\infty}\Omega$$

over the regular part in usual sense [6].

And recall we have seen if the stable base locus set is empty for K_X, then the limit

[6] Over X in the sense of measure since the left hand side is a measure even if u_∞ is unbounded as discussed at the end of Chapter 2.

is a smooth metric which represents $[K_X]$. It is of course also the case here. Hence the following is true just as before:

$$[L] \text{ nef. and big with empty stable base locus set } \iff [L] \text{ positive.}$$

The perturbation method using a positive class is totally unaffected. Now for the other perturbations using $\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$, notice that if $\omega_\infty \geqslant 0$ which is now $L \geqslant 0$, then we can actually take a proper norm $|\cdot|$ which makes all the perturbed background forms actually positive, and this will make the argument looks cleaner. In the case when $[L] = K_X$ (and related ones), by Kawamata's result, nef. and big will imply that $[L]$ is semi-ample, and so it is easy to get such a representative. We don't have this for general class. But clearly our argument is not so affected by this fact.

Remark 8.4.1. *For all the discussion above, we can consider real class, while rational class, which essentially comes from some holomorphic line bundle, is usually considered for notions like nef. or big. But the meaning can clearly be generalized to real case.*

But even when restricted to rational class which is of basic interest for geometric consideration, we yet can not say that there is an $L \geqslant 0$ for such class $[L]$ in general. So it is indeed nice to recognize all the results still hold without requiring $L \geqslant 0$.

In sight of the previous discuss, there would be a tough issue when applying results from our argument using pluripotential theory to general class $[L]$. We need the map P which should come from the semi-ampleness of this class.

8.5 Further Problems

In this section, let's list some directions for further consideration on the degenerate Monge-Ampere equation.

8.5.1 Big Bundle

As pointed out before, the flow method seems to be the most promising one in this situation. It corresponds to the case of $T < \infty$ as in Section 2.4, i.e., finite time singularity.

We've already known if the limiting class $[\omega_\infty]$ is the canonical class K_X which is big, then $[\omega_T]$ would be big and semi-ample for a rational initial Kähler class by

Rationality Theorem and Kawamata's result. Thus all the results before can be applied in this case. The only unsatisfying thing is that the finite time limit of potential u_T, which is continuous, would not satisfy the K-E equation since the term $\frac{\partial u}{\partial t}|_{t=T}$ is not 0.

The problem would become more serious in general since though the class $[\omega_T]$ which would be real nef. and big class for us, it may not even be rational and so the map P is out of reach. Even for the surface case with $[\omega_0]$ to be rational, the semi-ampleness won't follow from the rationality of the class $[\omega_T]$ from results in [Ka2].

In any case, the possible strategy is to continue this flow in some weak sense and get a weak limit. There are two ways of looking at this as we see it now.

The first one is to continue the flow on some other (singular) space. The space might come from the map P from the class $[\omega_T]$ in some cases. The simplest situation for a smooth surface of general type is still of quite some interests. We have had some discussion about it in Chapter 2. At that time, the new space is still a smooth manifold. But we should expect singularities on the new manifold in general, which would obviously bring some substantial difficulties in the analysis.

The second one would be to try keeping the flow over X in some weak sense. As suggested in [Tsh2], Dirichet problem for domains on X might be a proper object to study for this purpose. Here the choice of the domains and the boundary values should be quite subtle.

Somehow, these two points of views should be equivalent and the (weak) limit should be canonical, i.e., independent on all the choices. At this moment, it seems that we need a little bit better estimates for the situation at time T.

This program is suggested in [Ti2] as stated in Introduction.

8.5.2 Nef. Bundles

As discussed before, the equation $(L + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$ can't have a bounded solution when the rational class $[L]$ is nef. but not big. But that doesn't mean this equation is of no interest at all. In fact, as we've seen, the corresponding Kähler-Ricci flow has a global solution for all time by nefness and it would be interesting to study the limiting behavior of the metric along the flow. Basically, no bigness means the volume should be collapsing when $t \to \infty$. It's expected that the volume would degenerate to the dimension equal to the Kodaira dimension which in this case is strictly smaller than the dimension of the manifold X itself.

In Song and Tian's recent work [SoTi], the case when $[L] = K_X$ for X, a minimal surface of Kodaira dimension 1, has been successfully treated with complete description about the collapsing. Quite some arguments there can be applied to general dimensions provided the Kodaira dimension is still 1. Hopefully we can address this problem further by combining the different techniques.

8.5.3 More General Equations

It should also be natural to consider the following version of the degenerate Monge-Ampere equation:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = \Omega \tag{8.3}$$

where $[\omega_\infty]$ is nef. and big and Ω is a smooth volume form. Of course we should require $\int_X \Omega = [\omega_\infty]^n$. In fact this is exactly the main equation considered Part II of this work. But now we focus on the case when Ω is smooth so that the maximum priciple argument used in Chapter 2 can also be applied. Basically, the feature of the degeneration is just like in our main interest before, so we can also try to solve it using the same spirit.

It's easy to see the perturbation methods used before also provide families of approximation equations. We can just rescale the right hand side by positive constants to maintain the equality of integrals over X for both sides.

Now since there is no e^u term on the right hand side, we can not get uniform degenerated C^0 estimates as in Chapter 2. Indeed the classic method as in [Ya] doesn't look feasible to me either at this moment. But once C^0 estimate is available, it is easy to realize that there is no difficulty in carrying out the (degenerate) Laplacian estimate and higher order estimates. Then we can have the same kind of limit as before.

When $[\omega_\infty]$ is big and semi-ample (or just comes from a map P as in Theorem 1.3.2), we can have the uniform C^0 estimate for the (normalized) approximation solutions which even better than what we've used before in Chapter 2. [7]

The solution is actually continuous when $[\omega_\infty]$ is semi-ample and big (or the map P locally birational). The uniqueness of such a solution is clear from the stability result mentioned before. Hence we have proved the following theorem.

Theorem 8.5.1. *For the equation (8.3), if $[\omega_\infty]$ is semi-ample and big, we have a unique bounded solution $u \in PSH_{\omega_\infty}(X)$ which is smooth out of the stable base locus*

[7] This is also the case for equation with e^u on the right hand side.

set of $[\omega_\infty]$ *and continuous over* X*. We also have some controls about the singularities along the stable base locus set of* $[\omega_\infty]$*.*

It would be a different story if one uses flow method to study the equation above. Clearly that type of flow introduced in Introduction is not going to help us. But using the original flow equation in the level of potential, it is easy to cook up the following flow equation that'll have (8.3) as the limiting equation:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega}, \qquad u(0,\cdot) = 0 \tag{8.4}$$

where $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$. Then we can trace back to see what is the equation in the level of metric. Set $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ as before. We can get the following by taking "$\sqrt{-1}\partial\bar{\partial}$" on both sides of the above flow equation:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) + \mathrm{Ric}(\Omega) - e^{-t}(\omega_0 - \omega_\infty), \qquad \tilde{\omega}_0 = \omega_0$$

which is no longer canonical as before, i.e., depending on the choices of ω_∞, ω_0 and Ω.

We still have the long time existence when $[\omega_\infty]$ is numerically effective. In fact, general existence result can be proved using similar argument as in Chapter 2.[8]

Quite mysteriously, in order to get a similar convergence result under proper assumptions on the limiting class $[\omega_\infty]$, the main obstruct, as I see it now, is merely the lack of a proper lower bound for $\frac{\partial u}{\partial t}$. And even for the case when the limiting class is Kähler, such a bound is not yet available in general as far as I know at this moment. Related discussion can be found in Appendix.

[8]One can do a change of the unknown to create a "$-u$" for (8.3). It won't do any harm when searching for finite time estimates.

Chapter 9

Appendix

9.1 Inequalities for Laplacian Estimate

In this section, we clarify the computation for the inequalities used in Chapter 2 to get Laplacian estimate. And there are also some other things we might want to pay attention to when using the inequalities for local estimate. A very brief discussion for higher order derivative estimates will also be provided. It's the flow method that requires the most intention. Essentially all these are a direct application of the result in Yau's original work [Ya]. So let's recall what Yau's classic computation gives at the beginning.

On a (closed) Kähler manifold X with $dim_{\mathbb{C}}X = n > 1$, consider the following equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^F \omega^n,$$

where ω is a Kähler metric and F is a smooth function. u is supposed to be a smooth solution. Set $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}u$.

Then we have the following inequality at any point $p \in X$:

$$\begin{aligned} e^{Cu}\Delta_{\tilde{\omega}}\big(e^{-Cu}\langle\omega,\tilde{\omega}\rangle\big) \geqslant (\Delta_{\omega}F - n^2 inf_{i\neq j}R_{i\bar{i}j\bar{j}}) \\ - Cn\langle\omega,\tilde{\omega}\rangle + (C + inf_{i\neq j}R_{i\bar{i}j\bar{j}})e^{-\frac{F}{n-1}}\langle\omega,\tilde{\omega}\rangle^{\frac{n}{n-1}}, \end{aligned}$$

where C is a positive constant such that $C + inf_{i\neq j,X}R_{i\bar{i}j\bar{j}} > 0$ (i.e., the sum is a positive constant) and $R_{i\bar{i}j\bar{j}}$ is for the metric ω where i and j are for some local orthonormal frame for the holomorphic tangent bundle with respect to the metric ω. Let's emphasize that this inequality is pointwise and the "inf" is taken at the point p. So there is no requirement on the global topology of X which is closed in all our consideration.

Let's consider the flow case first. The equation considered is the following:

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u,$$

where $\omega_t = L + e^{-t}(\omega_0 - L)$ is uniform as metric for the range in which we use the computation and everything is smooth. Here we use L to match the notation in general case. This equation can be rewritten as:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t}+u+\log\frac{\Omega}{{\omega_t}^n}}{\omega_t}^n.$$

Now we can apply Yau's computation to get: ($\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$)

$$\begin{aligned} e^{Cu}\Delta_{\tilde{\omega}_t}\big(e^{-Cu}\langle\omega_t, \tilde{\omega}_t\rangle\big) \geqslant \big(&\Delta_{\omega_t}(\frac{\partial u}{\partial t} + u + \log\frac{\Omega}{{\omega_t}^n}) - n^2 inf_{i\neq j}R_{i\bar{i}j\bar{j},t}\big) \\ &- Cn\langle\omega_t, \tilde{\omega}_t\rangle + (C + inf_{i\neq j}R_{i\bar{i}j\bar{j},t})e^{-\frac{\frac{\partial u}{\partial t}+u+\log\frac{\Omega}{{\omega_t}^n}}{n-1}}\langle\omega_t, \tilde{\omega}_t\rangle^{\frac{n}{n-1}}, \end{aligned}$$

where C is a positive constant such that $C + inf_{i\neq j}R_{i\bar{i}j\bar{j},t} > 0$ and $R_{i\bar{i}j\bar{j},t}$ is for the metric ω_t (i and j are for local frame correspondent to ω_t). This is still pointwise (for (t, x)) and here we just use Yau's computation for each fixed t.

The t-derivative can be treated directly as follows:

$$\begin{aligned} e^{Cu}(-\frac{\partial}{\partial t})(e^{-Cu}\langle\omega_t, \tilde{\omega}_t\rangle) &= C\frac{\partial u}{\partial t}\langle\omega_t, \tilde{\omega}_t\rangle - \frac{\partial}{\partial t}\langle\omega_t, \tilde{\omega}_t\rangle \\ &= C\frac{\partial u}{\partial t}\langle\omega_t, \tilde{\omega}_t\rangle - \langle\omega_t, \frac{\partial\omega_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}\rangle + (\frac{\partial\omega_t}{\partial t}, \tilde{\omega}_t)_{\omega_t} \\ &= C\frac{\partial u}{\partial t}\langle\omega_t, \tilde{\omega}_t\rangle - (\frac{\partial\omega_t}{\partial t}, \omega_t)_{\omega_t} - \langle\omega_t, \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}\rangle + (\frac{\partial\omega_t}{\partial t}, \tilde{\omega}_t)_{\omega_t} \\ &= C\frac{\partial u}{\partial t}\langle\omega_t, \tilde{\omega}_t\rangle + (\frac{\partial\omega_t}{\partial t}, \sqrt{-1}\partial\bar{\partial}u)_{\omega_t} - \Delta_{\omega_t}(\frac{\partial u}{\partial t}). \end{aligned}$$

Here the second "=" can be justified either by local explicit computation using coordinates or can be seen intuitively by using the diagonal form of the metrics [1]. We have also used the obvious relation $\langle\omega_t, \cdot\rangle = (\omega_t, \cdot)_{\omega_t}$ above. This computation is general, i.e., not relying on the special form of metric ω_t.

[1] In fact this intuitive way can be made rigorous by noticing we can use $\omega_{t_0} + \frac{\partial\omega_t}{\partial t}|_{t=t_0} \cdot (t - t_0)$ instead of ω_t for each t_0, and then we can diagonalize them simultaneously.

Noticing $\Delta_{\omega_t} u = -n + \langle \omega_t, \tilde{\omega}_t \rangle$, we can sum up the above two to get

$$
\begin{aligned}
& e^{Cu}(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})\big(e^{-Cu}\langle \omega_t, \tilde{\omega}_t \rangle\big) \\
\geqslant\ & \big(\Delta_{\omega_t}(\log\frac{\Omega}{{\omega_t}^n}) - n^2 inf_{i\neq j} R_{i\bar{i}j\bar{j},t} - n\big) + (-Cn + C\frac{\partial u}{\partial t} + 1)\langle \omega_t, \tilde{\omega}_t \rangle \\
& + (\frac{\partial \omega_t}{\partial t}, \sqrt{-1}\partial\bar{\partial} u)_{\omega_t} + (C + inf_{i\neq j} R_{i\bar{i}j\bar{j},t}) e^{-\frac{\frac{\partial u}{\partial t} + u + \log\frac{\Omega}{{\omega_t}^n}}{n-1}} \langle \omega_t, \tilde{\omega}_t \rangle^{\frac{n}{n-1}}.
\end{aligned}
$$

Now we include the relation $\omega_t = L + e^{-t}(\omega_0 - L)$:

$$
\begin{aligned}
(\frac{\partial \omega_t}{\partial t}, \sqrt{-1}\partial\bar{\partial} u)_{\omega_t} &= \big(-e^{-t}(\omega_0 - L), \tilde{\omega}_t - \omega_t\big)_{\omega_t} \\
&= (L - \omega_t, \tilde{\omega}_t - \omega_t)_{\omega_t} \\
&= (L, \tilde{\omega}_t)_{\omega_t} - (L, \omega_t)_{\omega_t} - (\omega_t, \tilde{\omega}_t)_{\omega_t} + (\omega_t, \omega_t)_{\omega_t} \\
&\geqslant -C\langle \omega_t, \tilde{\omega}_t \rangle - C,
\end{aligned}
$$

where for the last step, we use $-C\omega_t < L < C\omega_t$ from the uniformality of ω_t as metric.

By the uniformality of ω_t as metric, we can get uniform controls for $\log\frac{\Omega}{{\omega_t}^n}$, $\Delta_{\omega_t}(\log\frac{\Omega}{{\omega_t}^n})$ and $inf_{i\neq j} R_{i\bar{i}j\bar{j},t}$. Noticing the uniform upper bounds for u and $\frac{\partial u}{\partial t}$ always got before taking care of Laplacian estimate, the above would be sufficient for the Laplacian estimate before we localize the estimates by including "$\log|\sigma|^2$".

Now we want to justify the inequality when terms like $\log|\sigma|^2$ are involved. The equation is now in the following form: [2]

$$
\big(\omega_{t,\delta} + \sqrt{-1}\partial\bar{\partial}(u - \delta\log|\sigma|^2)\big)^n = e^{\frac{\partial u}{\partial t} + u + \log\frac{\Omega}{{\omega_{t,\delta}}^n}} {\omega_{t,\delta}}^n,
$$

where $\omega_{t,\delta} = L + \delta\sqrt{-1}\log|\sigma|^2 + e^{-t}(\omega_0 - L)$ is uniform as metric [3] for the range we are applying this inequality. Again by Yau's computation, we have

$$
\begin{aligned}
& e^{C(u-\delta\log|\sigma|^2)}\Delta_{\tilde{\omega}_t}\big(e^{-C(u-\delta\log|\sigma|^2)}\langle \omega_{t,\delta}, \tilde{\omega}_t \rangle\big) \\
\geqslant\ & \big(\Delta_{\omega_{t,\delta}}(\frac{\partial u}{\partial t} + u + \log\frac{\Omega}{{\omega_{t,\delta}}^n}) - n^2 inf_{i\neq j} R_{i\bar{i}j\bar{j},t,\delta}\big) - Cn\langle \omega_{t,\delta}, \tilde{\omega}_t \rangle \\
& + (C + inf_{i\neq j} R_{i\bar{i}j\bar{j},t,\delta}) e^{-\frac{\frac{\partial u}{\partial t} + u + \log\frac{\Omega}{{\omega_{t,\delta}}^n}}{n-1}} \langle \omega_{t,\delta}, \tilde{\omega}_t \rangle^{\frac{n}{n-1}},
\end{aligned}
$$

[2]Everything should be understood out of $\{\sigma = 0\}$ now.

[3]It can be considered as over X to feel more comfortable about the uniform bounds.

where $C + inf_{i\neq j} R_{i\bar{i}j\bar{j},t,\delta} > 0$ and $R_{i\bar{i}j\bar{j},t,\delta}$ is of course for the metric $\omega_{t,\delta}$.

One can also compute the t-derivative part as before:

$$
\begin{aligned}
& e^{C(u-\delta\log|\sigma|^2)}(-\frac{\partial}{\partial t})(e^{-C(u-\delta\log|\sigma|^2)}\langle\omega_{t,\delta},\tilde{\omega}_t\rangle) \\
= & C\frac{\partial u}{\partial t}\langle\omega_{t,\delta},\tilde{\omega}_t\rangle - \frac{\partial}{\partial t}\langle\omega_{t,\delta},\tilde{\omega}_t\rangle \\
= & C\frac{\partial u}{\partial t}\langle\omega_{t,\delta},\tilde{\omega}_t\rangle + \langle\omega_{t,\delta}, e^{-t}(\omega_0 - L)\rangle - \langle\omega_{t,\delta}, \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}\rangle - \big(e^{-t}(\omega_0 - L), \tilde{\omega}_t\big)_{\omega_{t,\delta}} \\
= & C\frac{\partial u}{\partial t}\langle\omega_{t,\delta},\tilde{\omega}_t\rangle - \Delta_{\omega_{t,\delta}}(\frac{\partial u}{\partial t}) - \big(e^{-t}(\omega_0 - L), \sqrt{-1}\partial\bar{\partial}(u-\delta\log|\sigma|^2)\big)_{\omega_{t,\delta}}.
\end{aligned}
$$

Sum them up to get: $(\Delta_{\omega_{t,\delta}} u = -\langle\omega_{t,\delta},\omega_t\rangle + \langle\omega_{t,\delta},\tilde{\omega}_t\rangle)$

$$
\begin{aligned}
& e^{C(u-\delta\log|\sigma|^2)}(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})\big(e^{-C(u-\delta\log|\sigma|^2)}\langle\omega_{t,\delta},\tilde{\omega}_t\rangle\big) \\
\geqslant & \big(\Delta_{\omega_{t,\delta}}(\log\frac{\Omega}{{\omega_{t,\delta}}^n}) - n^2 inf_{i\neq j} R_{i\bar{i}j\bar{j},t,\delta} - \langle\omega_{t,\delta},\omega_t\rangle\big) \\
& + (-Cn + C\frac{\partial u}{\partial t} + 1)\langle\omega_{t,\delta},\tilde{\omega}_t\rangle - \big(e^{-t}(\omega_0 - L), \sqrt{-1}\partial\bar{\partial}(u-\delta\log|\sigma|^2)\big)_{\omega_{t,\delta}} \\
& + (C + inf_{1\neq j} R_{i\bar{i}j\bar{j},t,\delta}) e^{-\frac{\frac{\partial u}{\partial t} + u + \log\frac{\Omega}{{\omega_{t,\delta}}^n}}{n-1}} \langle\omega_{t,\delta},\tilde{\omega}_t\rangle^{\frac{n}{n-1}}.
\end{aligned}
$$

We can still get rid of the only term not explicitly containing $\langle\omega_{t,\delta},\tilde{\omega}_t\rangle$ by the following consideration:

$$
\begin{aligned}
& - \big(e^{-t}(\omega_0 - L), \sqrt{-1}\partial\bar{\partial}(u-\delta\log|\sigma|^2)\big)_{\omega_{t,\delta}} \\
= & (L + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 - \omega_{t,\delta}, \tilde{\omega}_t - \omega_{t,\delta})_{\omega_{t,\delta}} \\
= & (L + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2, \tilde{\omega}_t)_{\omega_{t,\delta}} - (L + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2, \omega_{t,\delta})_{\omega_{t,\delta}} \\
& - \langle\omega_{t,\delta},\tilde{\omega}_t\rangle + (\omega_{t,\delta},\omega_{t,\delta})_{\omega_{t,\delta}} \\
\geqslant & -C\langle\omega_{t,\delta},\tilde{\omega}_t\rangle - C,
\end{aligned}
$$

where the last step is justified still because $\omega_{t,\delta}$ is uniform as metric in the range considered. Actually in our application, since $L + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0$, one can just use $(L + \delta\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2, \tilde{\omega}_t)_{\omega_{t,\delta}} > 0$, and so the coefficient for $\langle\omega_{t,\delta},\tilde{\omega}_t\rangle$ can be just -1 in the final expression.

Still from the uniformality of $\omega_{t,\delta}$ as metric and the upper bounds for u and $\frac{\partial u}{\partial t}$, we can get the controls for the coefficients in the inequality above. Notice that for the term $\langle\omega_{t,\delta},\omega_t\rangle$, we have ω_t uniform as form.

This should be enough for the Laplacian estimate for the flow method.

If we want to get the estimates for higher order derivatives directly, we can just use Yau's computation for $G = \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}^{s\bar{r}}u_{i\bar{l}s}u_{\bar{j}k\bar{r}}$. In the case of flow equation, we have to do the computation for t-derivative directly, but it is not hard to see that the terms containing "$\frac{\partial u}{\partial t}$" will still cancel themselves just as before and finally we'll get an inequality with $\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t}$ instead of Δ_{ω} in Yau's work. The idea for getting local estimates is very similar to what's used in [CafNiSp]. Though elliptic equation is considered there, the argument can be used for parabolic case without essential modification. From the (local) estimates of G, we have enough derivative estimates to run the iteration using standard estimates in parabolic case (as standard Schauder estimates in elliptic case) to get all the higher ones.

All these above should be enough for the flow case.

For the perturbation methods, there is no t-derivative, so it should be a more direct application of Yau's computation. In case when the perturbation is only for the background "metric" (as in Subsection 2.5.1), it's rather trivial to see the original Yau's computation is enough.

For the case when perturbation happens on both sides of the equation as in Subsection 2.5.2, our equation becomes, out of $\{\sigma = 0\}$ where everything is smooth,

$$\big(\omega_{\epsilon,\delta} + \sqrt{-1}\partial\bar{\partial}(u_\epsilon - \delta\log\|\sigma\|^2)\big)^n = e^{u_\epsilon + \epsilon\log|\sigma|^2 + \log\frac{\Omega}{\omega_{\epsilon,\delta}{}^n}}\omega_{\epsilon,\delta}{}^n,$$

where $\omega_{\epsilon,\delta} = L + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 + \delta\sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^2$ is uniform as metric for fixed small $\delta > 0$ and sufficiently small $\epsilon > 0$. Notice there is even no need for $\omega_\epsilon = L + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ to be a metric. Once more, Yau's computation gives:

$$\begin{aligned}
&e^{C(u_\epsilon - \delta\log\|\sigma\|^2)}\Delta_{\tilde{\omega}_\epsilon}(e^{-C(u_\epsilon - \delta\log\|\sigma\|^2)}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle) \\
\geqslant &\big(\Delta_{\omega_{\epsilon,\delta}}(u_\epsilon + \epsilon\log|\sigma|^2 + \log\frac{\Omega}{\omega_{\epsilon,\delta}{}^2}) - n^2 inf_{1\neq j}R_{i\bar{i}j\bar{j},\epsilon,\delta}\big) - Cn\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle \\
&+ (C + inf_{i\neq j}R_{i\bar{i}j\bar{j},\epsilon,\delta})e^{-\frac{u_\epsilon + \epsilon\log|\sigma|^2 + \log\frac{\Omega}{\omega_{\epsilon,\delta}{}^n}}{n-1}}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle,
\end{aligned}$$

where $C + inf_{i\neq j}R_{i\bar{i}j\bar{j},\epsilon,\delta} > 0$ and $R_{i\bar{i}j\bar{j}}$ is for $\omega_{\epsilon,\delta}$.

Just as before, using the equation $\Delta_{\omega_{\epsilon,\delta}}u_\epsilon = \langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle - \langle\omega_{\epsilon,\delta}, \omega_\epsilon\rangle$, we can continue the above inequality as follows:

$$\begin{aligned}
\geqslant &\big(\Delta_{\omega_{\epsilon,\delta}}(\epsilon\log|\sigma|^2 + \log\frac{\Omega}{\omega_{\epsilon,\delta}{}^n}) - n^2 inf_{i\neq j}R_{i\bar{i}j\bar{j},\epsilon,\delta} - \langle\omega_{\epsilon,\delta}, \omega_\epsilon\rangle\big) - (Cn-1)\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle \\
&+ (C + inf_{i\neq j}R_{i\bar{i}j\bar{j},\epsilon,\delta})e^{-\frac{u_\epsilon + \epsilon\log|\sigma|^2 + \log\frac{\Omega}{\omega_{\epsilon,\delta}{}^n}}{n-1}}\langle\omega_{\epsilon,\delta}, \tilde{\omega}_\epsilon\rangle^{\frac{n}{n-1}}.
\end{aligned}$$

By noticing that $u_\epsilon + \epsilon \log|\sigma|^2 < C$ got before in Subsection 2.5.2, $\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ being the curvature for the line bundle "E" with respect to the norm $|\cdot|$, and $\omega_{\epsilon,\delta}$ being uniform as metric, we can get the controls for all the coefficients. This is enough for the Laplacian estimate there.

In fact if we want to get the estimates for higher order derivatives directly, we can also use Yau's computation for $G = \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}^{s\bar{t}}u_{i\bar{l}s}u_{\bar{j}k\bar{t}}$. In this case, since there is no "t-derivative", we just have to use the original computation by Yau since we already have local uniformality of $\tilde{\omega}$ as metric [4]. From the (local) estimates of G, we have enough derivative estimates to run the iteration using standard Schauder estimates to get all the higher ones.

9.2 Uniform Estimates

In this part, let's take care of the uniformality of the estimates in [Ya] mentioned in Subsection 2.5.2. We are considering the following family of equations:

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{u_\epsilon}|\sigma|^{2\epsilon}\Omega$$

for $\epsilon \in [0, a]$, where $|\cdot|$ is fixed, and a fixed Kähler metric ω. We want to get some uniform estimates for all u_ϵ's.

Following Yau's argument, we just need to find uniform estimates for the family of equations: [5]

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_{\epsilon,\delta})^n = e^{u_{\epsilon,\delta}}(|\sigma|^2 + \delta)^\epsilon\Omega$$

where $\epsilon \in [0, a]$ and $\delta \in (0, C]$.

We know that a smooth solution $u_{\epsilon,\delta}$ exists and is unique for each of these equations. For simplicity, the lower indices will be omitted below. All the positive constants C in the following should be independent on ϵ and δ above which is the uniformality wanted. The most essential part will be proving the C^0 uniform bound for u's.

Lower bound is easy to get as follows. Considering the minimal value point of (each) u, we get at that point,

$$e^u(|\sigma|^2 + \delta)^\epsilon\Omega \geqslant \omega^n.$$

[4] Since the condition is local, it will give us local estimates for G as in [CafNiSp].

[5] In fact this also gives the solvability of each equation with $\epsilon > 0$ above.

So $u \geqslant \log\frac{\omega^n}{\Omega} - \epsilon\log(|\sigma|^2+\delta) \geqslant -C$ at that point, and so for the whole of X.

The proof for uniform upper bound is more involved as follows. Consider the family of equations

$$(\omega + \sqrt{-1}\partial\bar{\partial}v)^n = C_{\epsilon,\delta}(|\sigma|^2+\delta)^\epsilon\Omega$$

where $C_{\epsilon,\delta}$ is the proper constant such that $\int_X C_{\epsilon,\delta}(|\sigma|^2+\delta)^\epsilon\Omega = \int_X \omega^n$. Clearly $C_{\epsilon,\delta}$ can be uniformly controlled for all the above ϵ and δ. We know that the smooth solution exists for each equation and is unique by requiring $\int_X v\omega^n = 0$ [6].

It's easy to see the uniform upper bound for such v above by using Green's function for ω over X. If we have uniform lower bound for v, then we can take a contant A such that $\tilde{v} = v + A > C_{\epsilon,\delta}$. Then we see

$$(\omega + \sqrt{-1}\partial\bar{\partial}\tilde{v})^n < e^{\tilde{v}}(|\sigma|^2+\delta)^\epsilon\Omega$$

with $\tilde{v}$ still being uniformly bounded. Thus we get by taking quotient:

$$\big(\omega + \sqrt{-1}\partial\bar{\partial}u + \sqrt{-1}\partial\bar{\partial}(\tilde{v}-u)\big)^n < e^{\tilde{v}-u}(\omega + \sqrt{-1}\partial\bar{\partial}u)^n.$$

Considering the minimal value point of $\tilde{v}-u$, one gets at that point, $\tilde{v}-u > 0$. So we have $\tilde{v} > u$ on the whole of X. Hence we get a uniform upper bound of u.

Then the argument will be quite standard after noticing

$$\Delta_\omega(|\sigma|^2+\delta) \geqslant \frac{|\sigma|^2}{|\sigma|^2+\delta}\Delta_\omega\log|\sigma|^2.$$

We can only get local estimates for higher order derivatives since the volume bound is not so good from the equation itself. But we already have enough to get uniform $C^{1,\alpha}$-estimate for all the solutions with $\alpha \in [0,1)$ and that'll be enough for getting a $C^{1,\alpha}$ solution by taking limit.

So now it only remains to show the uniform lower bound for v. Let's recall the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}v)^n = C(|\sigma|^2+\delta)^\epsilon\Omega$$

with proper C controlled uniformly for $\epsilon,\ \delta$ and the normalization $\int v = 0$. One still

[6] X and ω^n will be omitted in the integration expression later for simplicity.

uses Yau's computation to get: ($\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}v$, $e^F\omega^n = \Omega$)

$$\begin{aligned}
&\Delta_{\tilde{\omega}}(e^{-Cv}\langle\omega,\tilde{\omega}\rangle)\\
&\geqslant e^{-Cv}\big(\epsilon\Delta_\omega \log(|\sigma|^2+\delta) + \Delta_\omega F - n^2 inf_{i\neq j}R_{i\bar{i}j\bar{j}}\big) - Cne^{-Cv}\langle\omega,\tilde{\omega}\rangle\\
&\quad + (C + inf_{i\neq j}R_{i\bar{i}j\bar{j}})C(|\sigma|^2+\delta)^{-\frac{\epsilon}{n-1}}e^{-\frac{F}{n-1}}e^{-Cv}\langle\omega,\tilde{\omega}\rangle^{\frac{n}{n-1}}\\
&\geqslant -Ce^{-Cv} - Ce^{-Cv}\langle\omega,\tilde{\omega}\rangle + Ce^{-Cv}\langle\omega,\tilde{\omega}\rangle^{\frac{n}{n-1}},
\end{aligned}$$

where one requires $C + inf_{i\neq j}R_{i\bar{i}j\bar{j}} > 0$ and notices $(|\sigma|^2+\delta)^{-\frac{\epsilon}{n-1}} \geqslant C$ for all ϵ and δ.

Applying the arithmetic-geometric inequality in the form $C_\eta + \eta a^{\frac{n}{n-1}} \geqslant Ca$ for any positive C and η, we arrive at:

$$\Delta_{\tilde{\omega}}(e^{-Cv}(n+\Delta_\omega v)) \geqslant -Ce^{-Cv} + Ce^{-Cv}\Delta_\omega v.$$

Using the volume form $\tilde{\omega}^n = C(|\sigma|^2+\delta)^\epsilon e^F\omega^n$ to integrate both sides of above over X, we get:

$$\begin{aligned}
&C\int(|\sigma|^2+\delta)^\epsilon e^{-Cv}\\
&\geqslant \int(|\sigma|^2+\delta)^\epsilon e^{-Cv}\Delta_\omega v\\
&= -\int(d((|\sigma|^2+\delta)^\epsilon e^{-Cv}), dv)_\omega\\
&= C\int(|\sigma|^2+\delta)^\epsilon e^{-Cv}|dv|^2 - \epsilon\int(|\sigma|^2+\delta)^{\epsilon-1}e^{-Cv}(d|\sigma|^2, dv)_\omega.
\end{aligned}$$

For the case $\epsilon = 0$, of course we have

$$\int(|\sigma|^2+\delta)^\epsilon e^{-Cv}|dv|^2 \leqslant C\int(|\sigma|^2+\delta)^\epsilon e^{-Cv}.$$

For the case $\epsilon \in (0,a]$, we can do the following computation: [7]

$$\begin{aligned}
&\int(|\sigma|^2+\delta)^{\epsilon-1}e^{-Cv}(d|\sigma|^2, dv)_\omega\\
&= \int e^{-Cv}\big(C(|\sigma|^2+\delta)^{\frac{\epsilon-2}{2}}d|\sigma|^2, C(|\sigma|^2+\delta)^{\frac{\epsilon}{2}}dv\big)\\
&\leqslant C\int e^{-Cv}(|\sigma|^2+\delta)^{\epsilon-2}|d|\sigma|^2|^2 + C\int e^{-Cv}(|\sigma|^2+\delta)^\epsilon|dv|^2\\
&\leqslant C\int e^{-Cv}(|\sigma|^2+\delta)^\epsilon + C\int e^{-Cv}(|\sigma|^2+\delta)^\epsilon|dv|^2,
\end{aligned}$$

[7]The case when $\epsilon = 0$ can be included below, but it's not necessary.

where the last step is justified by the fact that $|d|\sigma|^2|^2 \leqslant C|\sigma|^2 \leqslant C(|\sigma|^2+\delta)$.

Thus by combining with the previous inequality, we can get

$$\int e^{-Cv}(|\sigma|^2+\delta)^\epsilon |dv|^2 \leqslant C\int e^{-Cv}(|\sigma|^2+\delta)^\epsilon$$

by choosing C's properly, which corresponds to the trivial one claimed for $\epsilon = 0$ above. Here the C's are positive constants, but may not be the same at different places.

Now we can have from the discussion above that

$$\begin{aligned} &\int |d((|\sigma|^2+\delta)^{\frac{\epsilon}{2}} e^{-\frac{Cv}{2}})|^2 \\ \leqslant\, & C\int (|\sigma|^2+\delta)^{\epsilon-2} e^{-Cv}|d|\sigma|^2|^2 + C\int (|\sigma|^2+\delta)^{\epsilon} e^{-Cv}|dv|^2 \\ \leqslant\, & C\int (|\sigma|^2+\delta)^{\epsilon} e^{-Cv}. \end{aligned}$$

Afterwards, Yau's original argument can be applied to get a uniform lower bound for v without any modification.

9.3 Relation between Different Flows

The following general flow equation is our main interest

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) + k\cdot\tilde{\omega}_t + S, \qquad \tilde{\omega}_0 = \omega_0$$

where $k = -1,\ 0,\ or\ 1$ for simplicity and S is some real smooth closed $(1,1)$-form.

When $k = -1\ or\ 1$, the equation can be reformulated as follows:

$$\frac{\partial(e^{-kt}\tilde{\omega}_t)}{\partial(-\frac{1}{k}e^{-kt})} = -\mathrm{Ric}(e^{-kt}\tilde{\omega}_t) + S.$$

We will call it as the rescaled flow in the following. Notice that the time and metric are getting rescaled simultaneously and Ricci curvature form is invariantly under rescaling.

Different k makes the new time parameter $-\frac{1}{k}e^{-kt}$ behave in a very different way.

•$k = 1$: $-e^{-t} \in [-1, 0)$, so it becomes finite time situation. As before, we can have the changing background form as:

$$\omega_t = \mathrm{Ric}(\Omega) - S + e^t(\omega_0 - \mathrm{Ric}(\Omega) + S) = (e^t - 1)(\omega_0 - \mathrm{Ric}(\Omega) + S) + \omega_0.$$

If $[\mathrm{Ric}(\Omega) - S]$ is positive, then we can take a proper initial metric ω_0 with $[\omega_0] = [\mathrm{Ric}(\Omega) - S]$. The original flow will exist forever ($t \in [0, \infty)$) which is not hard to see and will be discussed later.

The rescaled flow has $[e^{-t}\tilde{\omega}_t] = e^{-t}[\omega_0]$ go to 0 (cohomologically) as $t \to \infty$. So singularities should appear when $-e^{-t}$ approaches 0. The fairly famous recent work of Perelman's ([Per]) on finite time singularity might be applied to study the current situation.

Clearly, as $t \to \infty$, the convergence of $e^{-t}\tilde{\omega}_t$ is a much weaker result than the convergence of $\tilde{\omega}_t$ itself. If we know more information about the possible convergence of $\tilde{\omega}_t$, for example, it converges to a Kähler-Einstein metric or a soliton (which has to be shrinking), then we have $e^{-t}\tilde{\omega}_t \to 0$ as $t \to \infty$ in a very nice way. But clearly, this is not the only situation we can expect from this simple-minded consideration.

We can also consider the case when $[\omega_0 - \mathrm{Ric}(\Omega) + S]$ is nef., then the original flow will still exist forever just as before. Now we have $[e^{-t}\tilde{\omega}_t] \to [\omega_0 - \mathrm{Ric}(\Omega) + S]$ as $t \to \infty$. For the rescaled flow, we should still expect finite time singularity when $-e^{-t}$ approaching 0 in this case.

•$k = -1$: $e^t \in [1, \infty)$, so it is still infinite time situation. Basically we are going to see the interaction between this case and $k = 0$ case which has been considered in [Cao]. [8] The two equations considered here and in [Cao] are:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t + S, \quad \frac{\partial \tilde{\phi}_s}{\partial s} = -\mathrm{Ric}(\tilde{\phi}_s) + S$$

and we can easily find the relation $\tilde{\phi}_s = e^t\tilde{\omega}_t$, $s = e^t$ with $\tilde{\phi}_0 = \tilde{\omega}_0 = \omega_0$. We also have the following background forms:

$$\omega_t = -\mathrm{Ric}(\Omega) + S + e^{-t}(\omega_0 - \mathrm{Ric}(\Omega) + S),$$

$$\phi_s = \omega_0 - \mathrm{Ric}(\Omega) + S + s(-\mathrm{Ric}(\Omega) + S).$$

Thus if we want any kind of convergence for $\tilde{\phi}_s$, it is natural to require $[S - \mathrm{Ric}(\Omega)] = 0$.

[8] This kind of interaction might also be used to prove the global existence result for the modified flow discussed at the end of Chapter 8.

The volume form Ω can be chosen properly so that $S = \mathrm{Ric}(\Omega)$. So actually we have $\omega_t = e^{-t}\omega_0$, $\phi_s = \omega_0$.

The result in [Cao] gives us the convergence of $\tilde{\phi}_s$ (or $e^t\tilde{\omega}_t$) as $s \to \infty$ (or $t \to \infty$). Obviously this implies the nice convergence of $\tilde{\omega}_t \to 0$ as $t \to \infty$.

The equivalence relation above also gives the equivalence of long time existence results.

Actually, combining with the results in [Cao], we can draw more information about the convergence as follows.

In the level of potential, we have the following equations:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t}+u}\Omega, \quad (\omega_0 + \sqrt{-1}\partial\bar{\partial}(e^t u))^n = e^{\frac{\partial u}{\partial t}+u+nt}\Omega.$$

Using the parameter $s = e^t$ and setting $h(t) = ne^t(t-1)$, $U(s) = e^t u + h(t)$, we have $\frac{\partial U}{\partial s} = \frac{\partial u}{\partial t} + u + nt$. Thus the second equation is just

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}U)^n = e^{\frac{\partial U}{\partial s}}\Omega$$

which is exactly in the setting of [Cao]. From the convergence result there, we know $e^{-t}\omega_0 + \sqrt{-1}\partial\bar{\partial}u$ converges to 0 since $\omega_0 + \sqrt{-1}\partial\bar{\partial}U$ converges to the K-E metric as metric. So we should expect that u would converge to a constant after proper normalization. In fact, it is quite easy to see this below.

From [Cao]: $\frac{\partial U}{\partial s} \to C$ as $s \to \infty$. Thus for any small $\epsilon > 0$, we have proper constants C_1 and C_2 such that

$$(C-\epsilon)s + C_1 \leqslant U \leqslant (C+\epsilon) + C_2$$

which gives us that

$$C - \epsilon + C_1 e^{-t} \leqslant u + nt - n \leqslant C + \epsilon + C_2 e^{-t}.$$

So we conclude $u + nt - n \to C$ as $t \to \infty$ and the convergence would be in C^∞-topology by all the uniform estimates.

The above can also be seen in the following way. As proved in [Cao],

$$(sup_{X\times\{s\}} - inf_{X\times\{s\}})U(s) \leqslant C.$$

In our setting, this is just

$$(sup_{X\times\{t\}} - inf_{X\times\{t\}})(u+nt) \leqslant Ce^{-t}.$$

Then if we only use the boundedness of $u+nt$ from the discussion before, combining with the higher order estimates, we still get the convergence of $u+nt$ to some constant.

The limits of $u+nt-n$ and $\frac{\partial U}{\partial s}$ are the same. So we have $\frac{\partial u}{\partial t}$ converges to $-n$ as $t \to \infty$. But here we don't have the convergence of U itself, which will be considered later.

In the following, a careless point in [Cao] is pointed out and treated. In fact, we can give further description of the convergence in the level of potential.

Recall the flow in the level of potential as in [Cao]:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega}, \qquad u(0,\cdot) = 0$$

where $\tilde{\omega}_t = \omega + \sqrt{-1}\partial\bar{\partial}u$ with ω being the initial metric and $\mathrm{Ric}(\Omega) = S$. Using Yau's computation for Laplacian estimate, it can be shown that over $X \times [0,T]$:

$$\langle \omega, \tilde{\omega}_t \rangle \leqslant C \cdot e^{C(u - inf_{X\times[0,T]} u)}$$

where C's do not depend on T.

In [Cao], the function $v = u - \frac{1}{Vol_\omega(X)} \int_X u\omega^n$ is considered. Of course $Vol_\omega(X)$ means the volume of X with respect to the metric ω which is just $\int_X \omega^n$. For simplicity of notations, let's assume the volume is 1. All the integration will always be over X.

It's proved there that $|v| \leqslant C$ on $X \times [0,\infty)$. But notice that $\int u\omega^n$ well depends on t. So though we have

$$v - inf_{X\times[0,T]} v = u - \int u\omega^n - inf_{X\times[0,T]}(u - \int u\omega^n),$$

the following inequality

$$inf_{X\times[0,T]}(u - \int u\omega^n) \geqslant inf_{X\times[0,T]} u + inf_{[0,T]}(-\int u\omega^n)$$

means that we don't have the relation

$$u - inf_{X\times[0,T]} u = v - inf_{X\times[0,T]} v$$

or even just "$\leqslant$" which would be enough.

But it's not so hard to get around this by the following simple modification of the original argument. We have $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + C(t)$ where $C(t) = \int \frac{\partial u}{\partial t}\omega^n$ which only depends on t. Since $|\frac{\partial u}{\partial t}| \leqslant C$ by maximum principle argument, we have $|C(t)| \leqslant C$. Anyway, $|\frac{\partial v}{\partial t}| \leqslant C$. Now let's rewrite the flow equation as follows:

$$(\omega + \sqrt{-1}\partial\bar{\partial}v)^n = e^{\frac{\partial v}{\partial t} + C(t)}\Omega.$$

Then we apply the classic Laplacian estimate for this equation to get:

$$\langle \omega, \omega + \sqrt{-1}\partial\bar{\partial}v\rangle \leqslant C$$

which is what we want. This might make people feel that the classic inequality used for Laplacian estimate is not that optimal, but the flexibility is also great.

In [Cao], the convergence of metric $\tilde{\omega}_t$ is proved. More precisely, the convergence of v as $t \to \infty$ is proved together with the convergence of $\frac{\partial u}{\partial t}$ to a proper constant which only depends on the choice of Ω. In fact, we can get more detailed information about the convergence of u itself. Basically, we just carry out more computation similar to those in [Cao] and the convergence about $\tilde{\omega}_t$ is heavily used.

We consider $\psi = \frac{\partial u}{\partial t} - \int \frac{\partial u}{\partial t}\tilde{\omega}_t^n$ as he did. The metric $\tilde{\omega}_t$ is used instead of ω as in the definition of v. It's proved there that $\|\psi\|_{L^2(X)} \leqslant C \cdot e^{-Ct}$.

In the following, the Laplacian Δ will always be with respect to the metric $\tilde{\omega}_t$. For $l \geqslant 1$, we consider the function $\Delta^l\psi = \Delta^l(\frac{\partial u}{\partial t})$. We'll prove below that $\|\Delta^l\psi\| \leqslant C \cdot e^{-Ct}$ for all l's. The essential part is the deduction of an ODE equation just as in [Cao]. Induction is used, i.e., we'll assume this estimate up to $l-1$ for some $l \geqslant 1$. $l = 1$ (i.e., $l - 1 = 0$) case has been proved there. A lot of $\frac{\partial}{\partial t}$'s below are actually just ordinary differential $\frac{d}{dt}$.

We start with the following computation where $\frac{\partial u}{\partial t}$ can be replaced by ψ since $l \geqslant 1$.

$$\frac{\partial}{\partial t}\Big(\int (\Delta^l\frac{\partial u}{\partial t})^2\tilde{\omega}_t^n\Big) = 2\int \frac{\partial}{\partial t}(\Delta^l\frac{\partial u}{\partial t}) \cdot \Delta^l\frac{\partial u}{\partial t}\tilde{\omega}_t^n + n\int (\Delta^l\frac{\partial u}{\partial t})^2\frac{\partial\tilde{\omega}_t}{\partial t}\tilde{\omega}_t^{n-1}.$$

Notice here that the term $\frac{\partial\tilde{\omega}_t}{\partial t} = \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}$ would go to 0 as $t \to \infty$.

Recall the following equations:

$$(\frac{\partial}{\partial t} - \Delta)\frac{\partial u}{\partial t} = 0, \quad \big(\frac{\partial}{\partial t}\Delta\big)f = -(\frac{\partial\tilde{\omega}_t}{\partial t}, \sqrt{-1}\partial\bar{\partial}f)_{\tilde{\omega}_t}.$$

Thus we have the computation

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta^l \frac{\partial u}{\partial t}) &= \Delta^l \frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) + \sum_{l-1\geqslant\alpha\geqslant 0} \Delta^{l-1-\alpha}(\frac{\partial}{\partial t}\Delta)\Delta^{\alpha}(\frac{\partial u}{\partial t}) \\ &= \Delta^{l+1}(\frac{\partial u}{\partial t}) + \sum_{l-1\geqslant\alpha\geqslant 0} \Delta^{l-\alpha-1}\big(\mathrm{Ric}(\tilde{\omega}_t) - S, \sqrt{-1}\partial\bar{\partial}\Delta^{\alpha}(\frac{\partial u}{\partial t})\big)_{\tilde{\omega}_t}.\end{aligned}$$

So we can have the following:

$$\begin{aligned}&\int \frac{\partial}{\partial t}(\Delta^l \frac{\partial u}{\partial t}) \cdot \Delta^l \frac{\partial u}{\partial t}\tilde{\omega}_t^n \\ =& \int \Delta^{l+1}\frac{\partial u}{\partial t} \cdot \Delta^l \frac{\partial u}{\partial t}\tilde{\omega}_t^n + \int \sum_{l-1\geqslant\alpha\geqslant 0} \Delta^{l-1-\alpha}\big(\mathrm{Ric}(\tilde{\omega}_t) - S, \sqrt{-1}\partial\bar{\partial}\Delta^{\alpha}\frac{\partial u}{\partial t}\big)_{\tilde{\omega}_t} \cdot (\Delta^l\psi)\tilde{\omega}_t^n \\ =& -\int |\nabla(\Delta^l \frac{\partial u}{\partial t})|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + \int \sum_{l-1\geqslant\alpha\geqslant 0} \Delta^{l-\alpha}\big(\mathrm{Ric}(\tilde{\omega}_t) - S, \sqrt{-1}\partial\bar{\partial}\Delta^{\alpha}(\frac{\partial u}{\partial t})\big) \cdot (\Delta^{l-1}\psi)\tilde{\omega}_t^n.\end{aligned}$$

By the convergences of $\tilde{\omega}_t$ and $\frac{\partial u}{\partial t}$ and the induction assumption, applying Hölder's inequality and Poincare's inequality, we can conclude that for large enough t,

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int (\Delta^l \frac{\partial u}{\partial t})^2\tilde{\omega}_t^n\Big) &\leqslant -2\int |\nabla(\Delta^l \frac{\partial u}{\partial t})|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + \epsilon\int (\Delta^l \frac{\partial u}{\partial t})^2\tilde{\omega}_t^n + Ce^{-Ct} \\ &\leqslant -C\int (\Delta^l \frac{\partial u}{\partial t})^2\tilde{\omega}^n + C_0 e^{-Ct}.\end{aligned}$$

where all the constants are positive, ϵ is properly chosen to be small enough and the two C's can be chosen to be the same. The metric $\tilde{\omega}_t$ is uniform as metric, so the constants in those inequalities can be uniform. Hence we have for large t,

$$\frac{\partial}{\partial t}\big(e^{Ct}\|\Delta^l\psi\|^2_{L^2(X)}\big) \leqslant C_0$$

where the $L^2(X)$ is with respect to the metric $\tilde{\omega}_t$. This gives $\|\Delta^l\psi\|_{L^2(X)} \leqslant Ce^{-Ct}$ which would be for any l. Of course now the definition of $L^2(X)$-norm can be more flexible in result. [9]

Classic L^p estimates then give $\|\psi\|_{W^{l,2}} \leqslant Ce^{-Ct}$ for any l. Sobolev inequalities would give $\|\psi\|_{C^l} \leqslant Ce^{-Ct}$ for all l. Of course, the constant C would depend on l.

[9] We need a good choice for the computation above.

Now let's consider $\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n$ which is the difference between ψ and $\frac{\partial u}{\partial t}$.

$$\begin{aligned}\frac{\partial}{\partial t}(\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n) &= \int \frac{\partial}{\partial t}(\frac{\partial u}{\partial t})\tilde{\omega}_t^n + n\int \frac{\partial u}{\partial t}\cdot\frac{\partial \tilde{\omega}_t}{\partial t}\tilde{\omega}_t^{n-1} \\ &= \int \Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n + \int \frac{\partial u}{\partial t}\cdot\Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n \\ &= -\int |\nabla\frac{\partial u}{\partial t}|^2_{\tilde{\omega}_t}\tilde{\omega}_t^2 \geqslant -Ce^{-Ct}.\end{aligned}$$

In fact, we only need $\|\Delta\psi\|_{L^2(X)} \leqslant Ce^{-Ct}$ and we have a more delicate proof only for this ($l = 1$) case as follows. It basically makes use of the following inequality: [10]

$$|\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}|^2_{\tilde{\omega}_t} \geqslant \frac{1}{n}(\Delta\frac{\partial u}{\partial t}).$$

And in this case, we have the special form of the equality considered before as

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta\frac{\partial u}{\partial t}) &= \Delta^2\frac{\partial u}{\partial t} + \big(\mathrm{Ric}(\tilde{\omega}_t) - S, \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}\big)_{\tilde{\omega}_t} \\ &= \Delta^2\frac{\partial u}{\partial t} - |\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}|^2_{\tilde{\omega}_t}.\end{aligned}$$

We can carry out the following computation, using the results above, for t sufficiently large,

$$\begin{aligned}\frac{\partial}{\partial t}(\int(\Delta\frac{\partial u}{\partial t})^2\tilde{\omega}_t^n) &= 2\int\frac{\partial}{\partial t}(\Delta\frac{\partial u}{\partial t})\cdot\Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n + n\int(\Delta\frac{\partial u}{\partial t})^2\cdot\frac{\partial\tilde{\omega}_t}{\partial t}\tilde{\omega}_t^{n-1} \\ &= 2\int\Delta^2\frac{\partial u}{\partial t}\cdot\Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n - 2\int|\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}|^2_{\tilde{\omega}_t}\cdot\Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n + \int(\Delta\frac{\partial u}{\partial t})^2\cdot\Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n \\ &\leqslant -2\int|\nabla(\Delta\frac{\partial u}{\partial t})|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n + \int(\Delta\frac{\partial u}{\partial t})^2(1-\frac{2}{n})\Delta\frac{\partial u}{\partial t}\tilde{\omega}_t^n \\ &\leqslant -C\int(\Delta\frac{\partial u}{\partial t})^2\tilde{\omega}_t^n.\end{aligned}$$

It's the second term in the second to the last expression that we use sufficiently large t to control. If $n = 2$, that term is 0 automatically. This final inequality gives the exponential control as before.

Anyway, now we have $0 \geqslant \frac{\partial}{\partial t}(\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n) \geqslant -Ce^{-Ct}$. Let's consider $T_1 \geqslant T_2$,

$$0 \geqslant (\int\frac{\partial u}{\partial t}\tilde{\omega}_t^n)(T_1) - (\int\frac{\partial u}{\partial t}\tilde{\omega}_t^n)(T_2) \geqslant C(e^{-CT_1} - e^{-CT_2}) \geqslant -Ce^{-CT_2}.$$

[10] Recall that the Laplacian Δ is with respect to $\tilde{\omega}_t$.

Hence we have

$$
\begin{aligned}
&\|\frac{\partial u}{\partial t}(T_1) - \frac{\partial u}{\partial t}(T_2)\|_{C^0} \\
&= \|\psi(T_1) - \psi(T_2) + (\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n)(T_1) - (\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n)(T_2)\|_{C^0} \\
&\leqslant \|\psi(T_1)\|_{C^0} + \|\psi(T_2)\|_{C^0} + |(\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n)(T_1) - (\int \frac{\partial u}{\partial t}\tilde{\omega}_t^n)(T_2)| \\
&\leqslant Ce^{-CT_2}.
\end{aligned}
$$

We already know $\frac{\partial u}{\partial t}$ converges to a constant as $t \to \infty$. By choosing Ω properly, the constant would be 0 and indeed it's equivalent to considering $u + Ct$ instead. Anyway, by taking $T_1 \to \infty$, we arrive at

$$
\|\frac{\partial u}{\partial t}(t)\|_{C^0} \leqslant Ce^{-Ct}.
$$

Thus we have $|u| \leqslant C$ and more precisely, the exponential convergence of u in C^0-norm. Together with all the higher space derivative estimates of $\frac{\partial u}{\partial t}$ (just as for ψ), we can see the exponential convergence is for all norms.

Let's try to say something about the limit. It should be different from the one for v which has integral over X with respect to ω be 0. Clearly this is a canonical construction for a metric ω which will decide the choice of Ω as $\mathrm{Ric}(\Omega) = S$ and $\int_X \Omega = \int_X \omega^n$. Of course, if we choose some general constant instead of 0 as the initial value, then the limit would change accordingly. But it seems hard to say more, for example, about the relation between limits with respect to different initial metrics. Even in the same class which means the limiting metrics are the same but the potentials might be up to some constant as I see it.

9.4 Other Flows

For this part, we give brief discussions about some modified flows mentioned in Chapter 8.

9.4.1 Class-Changing Flow for $k = 0$ Case

Let's consider the following flow

$$
\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega}, \qquad u(0, \cdot) = 0
$$

where $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$. ω_0 is any initial Kähler metric and $[\omega_\infty]$ is the limiting class as $t \to \infty$. The goal is to study the equation $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = \Omega$. Ω is a smooth volume form with proper total integral over X. As we have shown before in Chapter 8, the corresponding metric flow is no longer canonical, i.e., the equation itself depends on ω_0 and ω_∞ generally.

We can still have the existence of the solution as long as ω_t remains Kähler. One way is to set $e^t v = u$. Then we can get C^0-norm bound for v similarly as in Chapter 2 and run through the similar argument. So we can still have $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ as the metrics along the flow. [11]

The discussion for convergence will not be so easy. When $[\omega_\infty]$ is nef. and big, the local convergence would be expected for $t \to \infty$. Basically, we just need to get the C^0 estimates for u and $\frac{\partial u}{\partial t}$ to run through the argument used before. The situation is very different and the trick above is not going to work. We can still have the following.

Take t-derivative once to get:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle \tilde{\omega}_t, \omega_0 - \omega_\infty\rangle. \tag{9.1}$$

Another t-derivative gives:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t}\langle \tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \left(\frac{\partial \tilde{\omega}_t}{\partial t}, \frac{\partial \tilde{\omega}_t}{\partial t}\right)_{\tilde{\omega}_t}. \tag{9.2}$$

Sum up the above two equations to arrive at:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left(\frac{\partial \tilde{\omega}_t}{\partial t}, \frac{\partial \tilde{\omega}_t}{\partial t}\right)_{\tilde{\omega}_t}.$$

Maximum principle then tells us that $\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} < C$. Thus $\frac{\partial}{\partial t}(e^t \frac{\partial u}{\partial t}) < Ce^t$ which gives $\frac{\partial u}{\partial t} < C$. If we know $[\omega_\infty]$ is also "semi-ample", then this would give C^0 bound for normalized u by Theorem 1.3.2 proved as one of our main results. Say the normalized u is $v = u - \int_X u\Omega$ if we assume $\int_X \Omega = 1$. But the uniform (degenerated) lower bound for $\frac{\partial u}{\partial t}$ remains to be a big problem.

If we consider the classic situation when ω_∞ is a Kähler metric, it should be true that the convergence is in C^∞-topology over X (to the smooth metric got in [Cao]).

In fact, in this case, once we have the uniform lower bound for $\frac{\partial u}{\partial t}$, then we can run

[11] For these estimates, t is always in a finite time interval, and so the term e^t is not going to give us any trouble. But it causes big trouble in search of the estimates for $t \in [0, \infty)$.

through the argument in [Cao] and get the convergence. Clearly, if $\omega_0 \leqslant \omega_\infty$, then (9.1) gives this bound and hence we have the convergence. It would be an interesting question to get this in general which might help to understand the degenerated case because we can't have $\omega_0 \leqslant \omega_\infty$ when ω_∞ can not be positive.

A final remark would be that for the case when $[\omega_\infty]$ is merely big, i.e., the local convergence should be for $t \to T < \infty$. There are some differences since now the time is always finite, so the term e^t used in proving global existence is also acceptable. So the trick used before might be of some help.

9.4.2 $k = 1$ Case

Now we consider the following evolution equation

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} + u, \qquad u(0, \cdot) = 0$$

where $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$. ω_0 is any initial Kähler metric and $[\omega_\infty]$ is the limiting class as $t \to \infty$. The goal now is to study the equation $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{-u_\infty}\Omega$. As in the previous subsection, the corresponding metric flow is no longer canonical.

This equation would be the classic one if we have $\omega_0 = \omega_\infty$. The Kähler class would then be fixed along the flow, i.e., $\omega_t = \omega_0$. As mentioned in Introduction, we can easily have global existence of the solution in this case. In fact, let's take derivative with respect to t for this equation to get:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial u}{\partial t}.$$

By maximum principle, we can easily get a bound as $|\frac{\partial u}{\partial t}| < Ce^t$ which also gives similar bound for u. These finite time C^0 estimates would be enough for us to carry out finite time higher order estimates through Laplacian estimate as usual. Hence we get the global existence of the solution.

Here we also consider the flow when the class is changing. The limiting equation is still the classic and interesting one. Clearly the way in which ω_t changes, though is quite artificial, is inevitable if the class $[\omega_\infty]$ is no longer Kähler.

Likewise, the global existence (as long as $[\omega_t]$ remains Kähler) is not too hard a problem by direct justification as before. One still needs to notice that t would be in a finite time interval for this question and the form of ω_t can be changed according to the potential u without changing the flow. Meanwhile, in certain cases, we might also use the relation discussed before between the equations with different k's to translate the existence result from one to the other. More precisely, one can set $v = e^{kt}u$ for $k = 1, or\ 1$ and this would change the equation to a form quite similar to the one in the main text. Then we can prove the global existence similarly.

The convergence, as expected in sight of people's usual consideration of this problem, should be a completely different story even in the classic case when the limiting class $[\omega_\infty]$ is Kähler. The degenerate case should even be harder in a sense.

9.5 More Facts in Pluripotential Theory

In this section, we discuss some more results in classic pluripotential theory which are used or briefly described before. For greater details, we refer to the classic works in this field.

• Borel Measure

We explain why $(\sqrt{-1}\partial\bar{\partial}u)^n$ is a (Borel) measure for $u \in PSH(V) \cap L^\infty(V)$. This is clearly just a local statement, and so it won't hurt to assume V is an open subset in $\mathbb{C}^n$.

Classic results in measure theory allow us to reduce this result to prove that it is a positive distribution. As mentioned before, we only need to prove $\sqrt{-1}\partial\bar{\partial}(uT)$ is a positive distribution when u is as above and T is a closed positive (k, k)-current.

We can assume $u > 0$ for simplicity by boundedness. Then uT is a positive distribution. Convolution can be used to locally construct smooth plurisubharmonic functions u_j decreasing converges to u pointwisely (as $j \to \infty$). Results in measure theory would give the convergence of $u_jT \to uT$ in the sense of distribution (i.e., weakly). Thus $\sqrt{-1}\partial\bar{\partial}(u_jT) \to \sqrt{-1}\partial\bar{\partial}(uT)$ weakly by the definition. As $\sqrt{-1}\partial\bar{\partial}(u_jT) = \sqrt{-1}\partial\bar{\partial}u_j \wedge T \geqslant 0$ as T is closed, so the limit is also positive.

• Positive $(1, 1)$-Current

We explain why the $(1, 1)$-current $\sqrt{-1}\partial u \wedge \bar{\partial}u$ is positive for $u \in PSH(V) \cap L^\infty(V)$. This is still local. Basically, it's a definition as follows. Let's again assume

$u > 0$ for simplicity.

$$\sqrt{-1}\partial u \wedge \bar{\partial} u := \frac{1}{2}\sqrt{-1}\partial\bar{\partial}(u^2) - u\sqrt{-1}\partial\bar{\partial}u.$$

The meaning of right hand side is clear as a distribution. To see it's positive, let's use the same u_j as above. Clearly, the equality is classic when we have u_j in the place of u above. [12] We have the convergence of $u_j \to u$ and so the weak convergence for the first term on the right hand side. The weak convergence results in [BeTa], whose special version is introduced before, give the weak convergence for the second one. Hence we conclude the positivity of this current.

Remark 9.5.1. *Starting from this, it's easy to consider the case when the u above is replaced by linear combination of bounded plurisubharmonic functions. Basically, we can do all the classic computation, while using convolution and weak convergence results to justify them.*

• Semi-continuity of Plurisubharmonic Functions

The logic order for this part might be horrible. We just want to illustrate the picture in a simple way.

The upper semi-continuity of plurisubharmonic fucntions can be seen by the decreasing convergence of functions from convolution. But it's not the case for the essential upper semi-continuity which is most useful for us. The rigorous way to describe this fact is discussed (in the best way as I see it) in Lelong's book [Le] by giving several equivalent definitions of plurisubharmonic function. The monotonicity of convergence from convolution plays an important role there.

It's important to keep in mind that plurisubharmonic functions we are talking about are functions with a fixed value at each point. They are not classes of functions as elements in L^1.

The rest of this section is aiming to help beginners understand the proof of the extension result in [FoNar] which is used in our proof for the continuity of bounded solution. The result in [Si] is also important in the proof. The proof for Siu's result is very technical, but the result itself is quite natural and easy to catch (even just by the title of the paper).

• Stein and Runge Property

[12] In fact, the distribution would be positive as long as u has enough regularity.

The definition of Stein space used there is classic as in [Le]. But the definition of Runge space seems to be a little different from the one in [Le]. We would like to know that they are actually the same. Actually this is a very classic observation as showed in [AnNar] which is also where I got to know it.

Suppose Y is an open set in a Stein space X. We claim Y is Runge in X iff for any compact set K, $\hat{K}_X \subset Y$. The definition in [Le] only requires $\hat{K}_X \cap Y$ is compact in Y. $\hat{K}_X$ is the holomorphically convex hull of K in X, i.e., the largest set where the C^0-norm of any holomorphic function over X over K controls that over this set. [13].

To prove the equivalence, We need to see $\hat{K}_X$ is actually in Y since it's compact because X is Stein. It'll be done once we know the fact that each component of $\hat{K}_X$ would intersect K since we have $\hat{K}_X \cap Y$ compact.

Using Oka-Weil's approximation result, this fact can be proved once we know that there is a Runge neighbourhood basis for $\hat{K}_X$. The idea is the following. By contradiction, we can have a Runge neighbourhood of $\hat{K}_X$ which has at least two components with K in only one of them but both containing part of $\hat{K}_X$. Then the characteristic function for the other part can be approximated by holomorphic functions using Oka-Weil's result. This would contradicts the definition of holomorphically convex hull.

The construction of a Runge neighbourhood basis for $\hat{K}_X$ (any holomorphically convex set) is classic by using sublevel sets of the norms of holomorphic functions as shown in [Nar].

• Local Maximum Modulus Principle

We now explain the application of Rossi's local maximum modulus principle in [FoNar] which is crucial for that argument. The picture there is as follows. Suppose X is an affine variety in the disk in $\mathbb{C}^N$ (which is of course Stein) and K is a compact set in it. If we have a point $p \in \hat{K}_X \setminus K$, then set $E = \partial B_p(r) \cap X$ with r small enough so that $E \cap K$ is empty. The claim is that $p \in \hat{E}_X$.

This result was pointed out and explained to me by Professor Rossi. The proof makes use of local maximum modulus principle.

Suppose it's not true. Then there exists a holomorphic function over X, f such that $|f(p)| > |f(q)|$ for $q \in \partial B_p(r)$ which might apriori be empty [14]. Hence the (obviously nonempty) set $\{|f| \geqslant |f(p)|\}$ has a component inside $B_p(r)$ because it

[13] In our application, it would be the same as polynomially convex hull and even plurisubharmonically convex hull as we are basically considering over a subvariety in the unit disk.

[14] It's indeed impossible as the proof indicates.

doesn't intersect the boundary from above. However, by (a later version of) local maximum modulus principle in [Ro], we know this component would intersect the Silov boundary of $\hat{K}_X$. Classic results tells us that this Silov boundary has to be contained in K which brings us a contradiction. In fact, the argument in [Ro] tells how to construct a holomorphic function which takes its global maximum of modulus over X only in $B_p(r)$ and this would automatically contradicts the definition of $\hat{K}_X$ without using the knowledge about Silov boundary.

9.6 Stability for Bounded Solution

In this section, we basically consider uniqueness of bounded solution for the degenerate Monge-Ampere being considered (with or without e^u on the right hand side). No continuity is assumed and so Part (6) in Kolodziej's argument can not be applied directly. This consideration makes sense when one does not require the map P to be locally birational.

9.6.1 Integration by Part

For two bounded plurisubharmonic solutions of the equation

$$(\omega_\infty + +\sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

with $\Omega > 0$ almost everywhere and $\omega_\infty \geqslant 0$. The manifold X is at least a closed Kähler in our problem and so comparison principle is justified. By comparison principle, we have the set where these two equations are different would have measure 0, and so they have to be the same by plurisubharmonicity.

There is actually another way of proving uniqueness of bounded solution for this equation which is by justifying the classic argument for the smooth and nondegenerated case as follows. Suppose $u,\ v \in L^\infty(X) \cap PSH_{\omega_\infty}(X)$ satisfying

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega, \quad (\omega_\infty + \sqrt{-1}\partial\bar{\partial}v)^n = e^v\Omega.$$

where $[\omega_\infty]$ is semi-ample (and big) and Ω is a smooth volume form. Let's use ω_u and ω_v to denote the two positive $(1,1)$-currents (singular metric). Then we have

$$0 \leqslant \int_X (u-v)(e^u - e^v)\Omega = \int_X (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)({\omega_u}^{n-1} + \cdots + {\omega_u}^{n-1}).$$

In the classic case when everything is smooth, integration by part can give that the integral on the very right is nonpositive which tells $u = v$. Now we want to justify this with less regularity of the functions.

Basically, we use a fact introduced in the previous section. For $u \in Ł^{\infty}_{loc}(U) \cap PSH(U)$ where U is a domain in $\mathbb{C}^n$, we can define the following positive $(1,1)$-current

$$\sqrt{-1}\partial u \wedge \bar{\partial} u := \frac{1}{2}\partial\bar{\partial}(u^2) - \sqrt{-1}u\partial\bar{\partial}u.$$

We've mentioned that this fact can be generalized to the case when the function u are linear combination of bounded plurisubharmonic functions and in fact, we can consider the function $u - v$ with $u,\ v \in L^{\infty}(X) \cap PSH_{\omega}(X)$ when ω has a local potential representation. Thus we have the following computation

$$\begin{aligned}
&\int_X (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)({\omega_u}^{n-1} + \cdots + {\omega_v}^{n-1}) \\
=&\int_X \big(\frac{1}{2}\sqrt{-1}\partial\bar{\partial}(u-v)^2 - \sqrt{-1}\partial(u-v) \wedge \bar{\partial}(u-v)\big)({\omega_u}^{n-1} + \cdots + {\omega_v}^{n-1}) \\
=&-\int_X \sqrt{-1}\partial(u-v) \wedge \bar{\partial}(u-v)({\omega_u}^{n-1} + \cdots + {\omega_v}^{n-1}) \leqslant 0
\end{aligned}$$

which gives the uniqueness result.

Remark 9.6.1. *Apparently, this argument looks better than the previous one using comparison principle as there is no assumption on ω_∞ (and even the Kählerity of X). But as we have seen before, those assumptions are needed for the existence of such solutions.*

Recall that in Chapter 2, we have repeatedly used maximum principle to prove that the solutions got by all kinds of methods (flow, perturbations) are actually the same and also not dependent on all the possible choices. But if we have the boundedness of the solutions, it's easy to see that in order to prove all these solutions are the same, we only need to use the fact that the solutions got are smooth out of some subvariety of X, which is the same for all solutions, namely, the stable base locus set of $[\omega_\infty]$ [15], using the following argument.

[15] We do not actually need this fact here.

$$
\begin{aligned}
0 &\leqslant \int_X (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)({\omega_u}^{n-1}+\cdots+{\omega_v}^{n-1}) \\
&= -\int_X \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)({\omega_u}^{n-1}+\cdots+{\omega_v}^{n-1}) \\
&\leqslant -\int_{X\setminus E} \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)({\omega_u}^{n-1}+\cdots+{\omega_v}^{n-1}) \leqslant 0,
\end{aligned}
$$

where E is the subvariety out of which u and v are both smooth with ω_u and ω_v being smooth metrics. The positivity of the currents justifies the second $\leqslant$ above which is actually $=$ since the set E is pluripolar. Clearly, this tells that $\partial(u-v)=0$ on $X\setminus E$ which means u and v are the same up to a constant out of E, and so over the whole of X which gives $u=v$ from this equation. Here we need the regularity of both u and v.

A good point about this discussion is that it works, to some extent, for the case when there is no e^u on the right hand side of the equation. So now let's discuss the following equation:

$$
(\omega_\infty+\sqrt{-1}\partial\bar{\partial}u)^n=\Omega
$$

where Ω is a smooth volume form with proper total integral over X. We've seen in the argument for Theorem 8.5.1, if we can have boundedness of the approximation solutions (with certain normalization of course), then the Laplacian estimate and higher derivative estimates can be obtained just as for the equation $(\omega_\infty+\sqrt{-1}\partial\bar{\partial}u)^n=e^u\Omega$, and so we can obtain a bounded solution with similar kind of regularity. But now uniqueness result for bounded plurisubharmonic solution of this equation can't be proved by merely applying comparison principle since there is no e^u term on the right hand side of the equation. There are two ideas to treat this.

The first one would be trying to generalize the original argument in [Koj2] quoted before. Unfortunately, there is a little fact involved which seems to rely on the continuity of the functions a lot as I see it right now. But this method should work for more general Ω as discussed in the next subsection.

The other idea would be the justification of the argument for the classic case. But the argument earlier would not work, which is not surprising since the argument should be different even in the classic case for these two kinds of equations. We can do it in a slightly different way as follows.

For two (bounded plurisubharmonic) solutions u and v for this equation above. We want to see that they are the same. Obviously it's enough to prove for the case when one of them (say u) is the solution got before by perturbation methods (and so with some more regularity). We still have the following computation.

$$\begin{aligned}
0 &= \int_X (u-v)({\omega_u}^n - {\omega_v}^n) \\
&= \int_X (u-v)\sqrt{-1}\partial\bar{\partial}(u-v)({\omega_u}^{n-1} + \cdots + {\omega_v}^{n-1}) \\
&= -\int_X \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)({\omega_u}^{n-1} + \cdots + {\omega_v}^{n-1}) \\
&\leqslant -\int_{X\setminus E} \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v){\omega_u}^{n-1},
\end{aligned}$$

where the subvariety E is for u from before. Since ω_u is a metric over $X\setminus E$, we have the current $\sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v) = 0$ over $X\setminus E$. This actually tells that $u-v$ is a constant over $X\setminus E$. The proof is as follows which is local argument.

By convolution, we can have smooth w_j's converges weakly to $u-v$ (of course as $j\to\infty$). Also from the discussion about the current $\sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v)$, we know $\sqrt{-1}\partial w_j\wedge\bar{\partial}w_j \to \sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v) = 0$ weakly. This would imply that $Dw_j\to 0$ weakly (or even locally in L^2) essentially from the definition. But we also have $Dw_j \to D(u-v)$ weakly. Thus we see $D(u-v)=0$ which gives the desired result (as an exercise in [Ev]).

For the last part of discussion above, we can also try to argue on the manifold X instead of $X\setminus E$. The advantage is that on X, Poincare inequality, which directly gives the constant difference between the two solutions, is at least easier to be justified. The idea is quite natural since on a closed manifold, (local) interior result is enough for the global result and the spectrum of Laplacian is very clear. The detail is as follows. Let's keep the setting above.

We have already seen $\sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v) = 0$ as (positive) current over $X\setminus E$. This current is actually defined over X and positive globally. Notice that the set E is a subvariety and so is pluripolar. Hence we actually have $\sqrt{-1}\partial(u-v)\wedge\bar{\partial}(u-v) = 0$ over X by splitting the integration into two parts and taking a limit. By the same kind of local argument as above, we know $u-v\in W^{1,2}(X)$ and $D(u-v)=0$ on X. Now we can finish by using the following Poincare inequality over X

$$\int_X |D(u-v)|^2 d\mu \geqslant C\int_X |(u-v)-C_0|^2 d\mu$$

which can be justified by the obvious one for smooth case from the spectrum of Laplacian and using approximation argument.

Remark 9.6.2. *Most of the argument before only uses pluripotential theory, so there is no need to require Ω to be a smooth volume form. In fact let Ω be an L^p measure, which will be enough for the boundedness result, and we can argue similarly. But there are also several places where we do need more regularity.*

9.6.2 Generalizing Kolodziej's Stability Argument

In this part, we can see that for the the equation without e^u on the right hand side considered before with general $L^{p>1}$ measure on the right hand side which we have existence of bounded solution, all the solutions that we can get (by approximation) are the same.

We basically generalize Kolodziej's original argument for stability result to our case. But as mentioned before, there is a fact which needs continuity and we further discuss below.

V is an open set in $\mathbb{C}^n$. Suppose $u,\ v \in PSH(V) \cap C^0(V)$ with $(\sqrt{-1}\partial\bar{\partial}u)^n \geqslant g \cdot d\lambda$ and $(\sqrt{-1}\partial\bar{\partial}v)^n \geqslant g \cdot d\lambda$ for nonnegative $g \in L^1$ where $d\lambda$ is standard Euclidean measure, then $(\sqrt{-1}\partial\bar{\partial}u)^k \wedge (\sqrt{-1}\partial\bar{\partial}v)^{n-k} \geqslant g \cdot d\lambda$.

When everything is smooth, this is just an application of algebraic-geometric mean value inequality. Approximation is used to get in this case. Be careful that we have to make sure the approximation is nice for the potential ("$\to u$") and the measure ("$\geqslant g$"), and so convolution will not do the job well. In the above case, one has to set up proper Dirichlet problems to get the approximation needed.

If one can justify this result for bounded instead of continuous functions, then Kolodziej's stability argument quoted before can be applied in our case to get stability for bounded solution.

In the following, we try to get around with this. Indeed, it'll be OK if the solutions we considered are from approximation (i.e., continuity method). It's more or less like we are using this in the place of the Dirichlet problem used above. The "g" above is now well chosen and in $L^{p>1}$. And we only need to make sure that the convergence of potentials (approximation solutions) is good enough (to gaurantee the weak convergence of distribution).

Actually, we can construct a decreasing sequence of functions converging to the solution which comes from approximation solutions. Here it's OK that the plurisubharmonicity of those functions is weaker than that of the solution. More precisely, the construction makes use of the global argument for continuity quoted before in Chapter 7.

Recall the way we get a solution for $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^{ku}\Omega$ where $k = 0$ *or* 1 and Ω is a nonnegative volume form with L^p-norm bounded for some $p > 1$. Of course we need $\int_X e^{ku}\Omega = \int_X {\omega_\infty}^n$ to be possible. [16].

Consider the perturbation $(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = C_\epsilon e^{ku_\epsilon}\Omega$ where $\omega_\epsilon = \omega_\infty + \epsilon\omega$ for some Kähler metric ω and C_ϵ is some positive constant to make this equation cohomologically possible. For all $\epsilon > 0$, we have a continuous solution u_ϵ uniformly bounded (after proper normalization for $k = 0$ case). Then we have a sequence $\{u_{\epsilon_i}\}$ for ϵ_i decreasing to 0 (as $i \to \infty$) converges to a bounded solution u of the original solution in L^1-norm and pointwisely almost everywhere.

Now let's construct a decreasing sequence of functions converging to u from the sequence above. For simplicity, we use u_i to denote u_{ϵ_i} above.

From the global argument for continuous quoted from [Koj2] before, for any $\delta > 0$, there exists N_δ such that $u_j - u_i \leqslant \delta$ if $j \geqslant i > N_\delta$. Take subsequence $\{u_{k_\alpha}\}$ such that $u_{k_\alpha} - u_{k_\beta} \leqslant \frac{\delta}{2^\beta}$ for $\alpha \geqslant \beta$. Then set $v_\alpha = u_{k_\alpha} + \frac{\delta}{2^{\alpha-1}}$.

We have $v_\alpha - v_\beta = u_{k_\alpha} - u_{k_\beta} + \frac{\delta}{2^{\alpha-1}} - \frac{\delta}{2^{\beta-1}} \leqslant \frac{\delta}{2^\beta} + \frac{\delta}{2^{\alpha-1}} - \frac{\delta}{2^{\beta-1}} \leqslant 0$ for $\alpha > \beta$. So $\{v_\alpha\}$ is a uniform bounded decreasing sequence of continuous solutions of the perturbed equations which clearly still converges to u. This is what we need for the weak convergence of distribution with mixed terms.

Remark 9.6.3. *Actually, this tells us that the obstruction for proving the continuity of the bounded solution from approximation will be the same with or without e^u on the right hand side and for general volume form Ω if we want to use the global argument in [Koj2].*

Hence, we can get uniqueness of solution from approximation for this equation. In fact, we can have similar stability result as Theorem 3.2.11. But this is not that satisfying.

9.7 Global Argument for Boundedness Result

In this section, we present the global argument for boundedness result as in Theorem 1.3.2. It has already appeared in [Zh] and so we only include the main part here for readers' convenience. We also provide a little more explanation along the way.

[16] This is known as the cohomological condition which just means that Ω is not almost everywhere 0 when $k = 1$.

9.7.1 Preparation

Many classic results in pluripotential theory are quite local, for example, weak convergence results, and so can be used in our situation automatically. Many definitions also have their natural versions for the domain V with background metric which can't be reduced to potential level globally in V, for example, relative capacity. Since it is of the most importance for us in this work, let's give the definition below in the case when $V = X$, which is the version that we are going to use in the following discussion.

Definition 9.7.1. *Suppose ω is a (smooth) nonnegative $(1,1)$-form. For any (Borel) subset K of X, we define the relative capacity of K with respect to ω as follows:*

$$Cap_\omega(K) = sup\{\int_K (\omega + \sqrt{-1}\partial\bar{\partial}v)^n | v \in PSH_\omega(X), \quad -1 \leqslant v \leqslant 0\}.$$

We require ω to be nonnegative so that $PSH_\omega(X)$ is not empty. We also point out that usually, it only takes to consider any compact set K in order to study any set by approximation.

In fact, if X can be covered by finitely many domains which are hyperconvex with respect to the local potentials of ω, then the global definition above would be equivalent to the summation of locally defined relative capacity. The case that we are interested in here is clearly in this situation. Basically, one just needs to give a method to extend any bounded local plurisubharmonic function to an element in $PSH_\omega(X)$ valued in an interval with a proper length. Let me illustrate the idea which is from [Koj2]. Suppose we have $V_1 \subset\subset V_2$ where both of them are hyperconvex domain mentioned above [17]. Let's consider the relative capacity for any compact set $K \in V_1$. For $v \in PSH(V_2)$ valued in $[-1, 0]$, the uniform multiple of a proper chosen local potential of ω, ψ, can have $max\{v, \psi\}$ equal to v near K and equal to ψ near ∂V_2. Then the extension to the rest part of X would be very clear (by ψ). The multiple used is uniform controlled and won't bring us any trouble. In fact this argument above also works if the background form used in tha global definition is no smaller than ω.

The only thing which is not so trivially adjusted to our situation might be comparison principle which is so important and has a global feature. Now let's state the version of comparison principle we are going to use later.

Proposition 9.7.2. *For X as above, suppose $u,\ v \in PSH_\omega(X) \cap L^\infty(X)$ where ω is*

[17] This is our usual setting of a local picture.

a smooth nonnegative closed $(1,1)$*-form, then*

$$\int_{\{v<u\}}(\omega+\sqrt{-1}\partial\bar{\partial}u)^n \leqslant \int_{\{v<u\}}(\omega+\sqrt{-1}\partial\bar{\partial}v)^n.$$

This version is slightly different from other more classic versions because X may not be projective, ω may not be positive and the functions may not be continuous. The brief description of justification is as follows.

Basically we still just need a decreasing approximation for any bounded plurisubharmonic function by smooth plurisubharmonic functions according to the argument in [BeTa]. This is not as easy as in Euclidean space where convolution is available. And the possible loss of projectivity of X makes it difficult to use some other classic results.

But according to the recent result of Blocki and Kolodziej in [BlKol], we can have a decreasing smooth approximation for plurisubharmonic function over X. The approximation result need the background form to be positive (i.e., a Kähler metric), but clearly nonnegative form (as ω_∞ for us) is acceptable when it comes down to comparison principle by simple approximation argument by a simple approximation used before. [18] This is also why we can now have X to be just Kähler instead of projective as stated in [TiZh].

The next few subsections will be devoted to prove each of the statements about boundedness result in Theorem 1.3.2.

9.7.2 Apriori L^∞ Estimate

• Bound Relative Capacity by Measure

In the following, ω is a (smooth) nonnegative closed $(1,1)$-form. Keep in mind that ω stands for $\omega_\infty+\epsilon\phi$ [19] for any $\epsilon\in[0,1]$ for our application and all the constants do not depend on ϵ.

For $u,\ v\in PSH_\omega(X)\cap L^\infty(X)$ with $U(s):=\{u-s<v\}\neq\varnothing$ for $s\in[S,S+D]$. Also sssume v is valued in $[0,C]$.

Then $\forall w\in PSH_\omega(X)$ valued in $[-1,0]$, for any $t\geqslant 0$, since

$$0\leqslant t+Ct+tw-tv\leqslant t+Ct,$$

[18] Of course, we do need X to be Kähler to guarantee such an approximation.
[19] This ϕ is a Kähler metric.

we have the following chian of sets:

$$U(s) \subset V(s) = \{u - s - t - Ct < tw + (1-t)v\} \subset U(s+t+Ct).$$

So we have for $0 < t \leqslant 1$:

$$\begin{aligned}
\int_{U(s)} (\omega + \sqrt{-1}\partial\bar{\partial} w)^n &= t^{-n} \int_{U(s)} \big(t\omega + \sqrt{-1}\partial\bar{\partial}(tw)\big)^n \\
&\leqslant t^{-n} \int_{U(s)} \big(t\omega + \sqrt{-1}\partial\bar{\partial}(tw) + (1-t)\omega + \sqrt{-1}\partial\bar{\partial}((1-t)v)\big)^n \\
&= t^{-n} \int_{U(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v)\big)^n \\
&\leqslant t^{-n} \int_{V(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v)\big)^n \\
&\leqslant t^{-n} \int_{V(s)} \big(\omega + \sqrt{-1}\partial\bar{\partial}(u - s - t - Ct)\big)^n \\
&\leqslant t^{-n} \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial} u)^n.
\end{aligned}$$

Comparison principle is applied to get the second to the last $\leqslant$. All the other steps are rather trivial from the setting. Thus from the definition of Cap_ω, we conclude

$$t^n \cdot Cap_\omega(U(s)) \leqslant \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial} u)^n.$$

for $t \in (0, min(1, \frac{S+D-s}{1+C})]$. Of course, for our purpose, it is always safe to assume $\frac{S+D-s}{1+C} < 1$. In fact we can just choose $D < 1$. Now let's rewrite this inequality as:

$$t^n \cdot Cap_\omega(U(s)) \leqslant (1+C)^n \int_{U(s+t)} (\omega + \sqrt{-1}\partial\bar{\partial} u)^n$$

for $t \in (0, S + D - s]$ by rescaling the t before.

Intuitively, the constant D can be seen as the gap where the values of u can stretch over.

- Bound Gap D by Capacity

We are still in the previous setting. Now assume that for any (Borel or compact) subset E of X, we have:

$$\int_E (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leqslant A \cdot \frac{Cap_\omega(E)}{Q\big(Cap_\omega(E)^{-\frac{1}{n}}\big)}$$

for some constant $A > 0$, where $Q(r)$ is an increasing function for positive r with positive value. From now on, this condition will be denoted by Condition (A).

The result to be proved in this subsection is as follows:

$$D \leqslant \kappa(Cap_\omega(U(S + D)))$$

for the following function

$$\kappa(r) = (1 + C) \cdot C_n A^{\frac{1}{n}} \Big(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1}(Q(y))^{-\frac{1}{n}} dy + \big(Q(r^{-\frac{1}{n}})\big)^{-\frac{1}{n}} \Big),$$

where C_n is a positive constant only depending on n and $1 + C$ comes from the resacling at the end of the previous step.

The proof is a little technical but quite elementary in spirit. We will briefly describe the idea below.

The previous part gives us an inequality as "$Cap \leqslant measure$".

Condition (A) gives the other direction "$measure \leqslant Cap$".

We can then combine them to get some information about the length of the interval which comes from t in the inequality proved before. The assumption of nonemptiness of the sets is needed because we have to divide $Cap_\omega(U(\cdot))$ from both sides in order to get something purely for t.

Finally, we can sum all these small t's up to control for D. [20]

Of course we'd better use a delicate way to carry out all these just in sight of the rather complicated final expression of the function κ. It has been done beautifully in [Koj1]. In that paper we do not have the extra $1 + C$ for κ, but it won't bring any essential difference here as we can just add it into the computation there. We just need to rescale the interval a little. In fact, for boundedness consideration, we can choose v to be 0 (so $C = 0$) and there would be no difference at all. But we do need general v when apply this result to prove continuity.

Let's emphasize that in the argument, we do not have a positive lower bound for the t's to be summed up, so it is important that the inequality proved in the previous

[20] We use the trivial fact that nonemptiness, nonzero (Lebesgue) measure and nonzero capacity are equivalent for such sets $U(s)$ from the fundamental properties of plurisubharmonic functions.

part holds (uniformly) for all small enough $t > 0$.

• Bound Capacity

For $u \in PSH_\omega(X) \cap L^\infty(X)$ and $u \leqslant 0$, suppose K is a compact set in X which can well be X itself, then there exists a positive constant C such that:

$$Cap_\omega(K \cap \{u < -j\}) \leqslant \frac{C\|u\|_{L^1(V)} + C}{j}.$$

This is aiming for a uniform upper bound for the relative capacity appearing on the right hand side of the inequality proved in the previous subsection.

Proof. For any $v \in PSH_\omega(X)$ and valued in $[-1, 0]$, consider any compact set $K' \subset K \cap \{u < -j\}$, using CLN inequality [21] in [Koj2]:

$$\begin{aligned}\int_{K'} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n &\leqslant \frac{1}{j}\int_K |u|(\omega + \sqrt{-1}\partial\bar{\partial}v)^n \\ &\leqslant \frac{C\|u\|_{L^1(V)} + C}{j}.\end{aligned}$$

From the definition of relative capacity, this would give the inequality above.

□

Now we consider the L^1-norm for those approximation solutions u_ϵ (and also the solution u if it exists by assumption). The following is just the standard Green's function argument. Strictly speaking, the computation needs the function to be smooth, but we can achieve the final estimate by using approximation sequence given by the result in [BlKol] for our situation. So let's pretend that we have the needed regularity in the following.

For fixed $\epsilon \in [0, 1]$, suppose $u_\epsilon(x) = 0$ and $C > G$ where G is the Green function for the metric $\omega_1 = \omega_\infty + \phi$. Also since $\omega_\infty + \epsilon\phi + \sqrt{-1}\partial\bar{\partial}u_\epsilon \geqslant 0$, we have

$$\Delta_{\omega_1} u_\epsilon = \langle \omega_1, \sqrt{-1}\partial\bar{\partial}u_\epsilon \rangle \geqslant -\langle \omega_1, -\omega_\infty - \epsilon\phi \rangle \geqslant -C$$

where C is uniform for $\epsilon \in [0, 1]$. Basically, this tells that there should be no worry

[21] The global version of this inequality over X is quite easy to justify in sight of the locality of the result.

for the changing background metric. Then we have:

$$
\begin{aligned}
0 = u_\epsilon(x) &= \int_X u_\epsilon {\omega_1}^n + \int_{y\in X} G(x,y)\Delta_{\omega_1} u_\epsilon \cdot {\omega_1}^n \\
&= \int_X u_\epsilon {\omega_1}^n + \int_{y\in X} \big(G(x,y) - C\big)\Delta_{\omega_1} u_\epsilon \cdot {\omega_1}^n \\
&\leqslant \int_X u_\epsilon {\omega_1}^n - C\int_{y\in X} \big(G(x,y) - C\big){\omega_1}^n \\
&\leqslant \int_X u_\epsilon {\omega_1}^n + C.
\end{aligned}
\tag{9.3}
$$

This gives the uniform L^1 bound for u_ϵ's by noticing that they are all nonpositive.

Hence we know *the set where u_ϵ has very negative value should have (uniformly) small relative capacity.*

• Conclusion

Combining all the results above, if we assume Condition (A) for some function $Q(r)$ and set the function v at the beginning to be 0, we have:

$$D \leqslant \kappa(\frac{C}{D})$$

if $U(s) = \{u < -s\}$ nonempty for $s \in [-2D, -D]$ where C is a positive constant.

Furthermore, if we can choose the function $Q(r)$ to be $(1+r)^m$ for some $m > 0$ so that Condition (A) holds, this would imply that the function u only take values in a bounded interval since D can not be too large [22]. This D can well be larger than 1 and so can't be used directly as the gap D before. But since it's clear that the existence of a big gap would imply the existence of small ones (with length smaller than 1), one can still get contradiction from above. Of course, one can also use argue for an interval like $[S, S+\frac{1}{2}]$ and see S can't be too negative. There is no essential difference between these ways to draw the conclusion.

That's enough for the lower bound in sight of the normalization $sup_X u = 0$. The more explicit bound claimed in the theorem is not hard to get by carefully tracking down the relation just as in the main text.

• Condition (A)

In this part, we justify Condition (A) under the measure assumption in the main theorem. This part is the essential generalization of Kolodziej's original argument.

[22] As D goes to ∞, κ goes to 0.

In our case, $f \in L^p$ for some $p > 1$, which is the measure on the left hand side of Condition (A) from the equation we want to solve. For the approximation equations, the measures are different, but clearly we can bound the L^p-norm uniformly.

Applying Hölder inequality, we know that it suffices to prove the following inequality:

$$\lambda(K) \leqslant C \cdot \big(Cap_\omega(K)(1 + Cap_\omega(K)^{-\frac{1}{n}})^{-m}\big)^q,$$

where λ is the smooth measure over X and q i s some positive constant depending on $p > 1$ [23]. Obviously, it would be enough to prove:

$$\lambda(K) \leqslant C_l \cdot Cap_\omega(K)^l \cdots\cdots (A)$$

for l sufficiently large.

Of course we have $\lambda(K) < C$, and so in fact we can get for any nonnegative l if the above is true. And the case when measures or capacities of some sets are 0 trivially brings no harm just as in the main text. In the following, we'll consider Condition (A) in this form.

For ω (uniformly) positive, this can be easily reduced to a Euclidean ball. As by the argument quoted in Chapter 2 from [Koj2], using a classic measure theoretic result in [Tsm], we have:

$$\lambda(K) \leqslant C \cdot exp\big(-\frac{C}{Cap_\omega(K)^{\frac{1}{n}}}\big) \cdots\cdots (\star).$$

This is actually stronger than the version above after noticing small capacity situation is of the main interest.

In the following proof of Condition (A), the essential step is to prove the following inequality:

$$\lambda(K) \leqslant C_1 \cdot \epsilon^{N_1} + C_1 \cdot \epsilon^{-N_2} exp\big(\frac{C_2}{\log\epsilon \cdot Cap_\omega(K)^{\frac{1}{n}}}\big) \cdots\cdots (B)$$

for sufficiently small $0 < \epsilon < 1$. All positive constants C_i's do NOT depend on ϵ. This ϵ has nothing to do with the ϵ appearing before in $\omega_\infty + \epsilon\omega$. After proving this, by putting $\epsilon = Cap_\omega(K)^\beta$ for properly chosen $\beta > 0$, we can justify Condition (A) for any chosen l by noticing the dominance of exponential growth over polynomial growth.

It is easy to notice that we can have uniform constants for all ω's related once

[23] $\frac{1}{q} + \frac{1}{p} = 1$ and $q = 1$ when $p = \infty$.

we get for ω_∞ from the favorable direction of the control we want. And we also only need to prove Condition (A) for sets close to the subvariety $\{{\omega_\infty}^n = 0\}$ in sight of the results in [Koj2] by localizing the problem.

The rest part of this section will be devoted to the proof of inequality (B). The following construction is of fundamental importance for this goal.

Let's start with a better description of the map $P : X \to P(X) \subset \mathbb{CP}^N$. For simplicity, we'll assume here that P provides a birational morphism between X and $P(X)$. This assumption will be removed at the end.

Using this assumption, we have subvarieties $Y \subset X$ and $Z \subset P(X)$ such that $X \setminus Y$ and $P(X) \setminus Z$ are isomorphic under F and $P(Y) = Z$. Clearly Z should contain the singular subvariety of $P(X)$. It's the situation near Y (or Z) that is of the main interest to us.

Now we use finitely many unit coordinate balls on X to cover Y. The union of the half-unit balls will be called V. Then we take two finite sets of open subsets depending on $\epsilon > 0$ as follows.

$\{U_i\}$, $\{V_i\}$, with $i \in I$, finite coverings of $V \setminus W$, where W is the intersection of ϵ-neighbourhood of Y [24] with $P(X)$, such that each pair $V_i \subset U_i$ is in one of the chosen unit coordinate balls. Moreover, $P(U_i)$ and $P(V_i)$ are the intersections of $P(X)$ with balls of sizes $\frac{1}{2}\epsilon^C$ and $\frac{1}{6}\epsilon^C$ where some fixed $C > 0$ are chosen to be big enough in order to justify the above construction.

Clearly $|I|$ is controlled by $C \cdot \epsilon^{-N_2}$.

For any compact set K in V, we have the following computation:

$$\begin{aligned}
\lambda(K) &\leqslant \lambda(W) + \sum_{i\in I} \lambda(K \cap \bar{V}_i) \\
&\leqslant C \cdot \epsilon^{N_1} + \sum_{i\in I} C \cdot exp(-\frac{C}{Cap(K \cap \bar{V}_i, U_i)^{\frac{1}{n}}}) \\
&\leqslant C \cdot \epsilon^{N_1} + \sum_{i\in I} C \cdot exp(\frac{C}{\log\epsilon \cdot Cap_{\omega_\infty}(K \cap \bar{V}_i)^{\frac{1}{n}}}) \\
&\leqslant C \cdot \epsilon^{N_1} + \sum_{i\in I} C \cdot exp(\frac{C}{\log\epsilon \cdot Cap_{\omega_\infty}(K)^{\frac{1}{n}}}) \\
&\leqslant C \cdot \epsilon^{N_1} + C\epsilon^{-N_2} \cdot exp(\frac{C}{\log\epsilon \cdot Cap_{\omega_\infty}(K)^{\frac{1}{n}}}).
\end{aligned}$$

That's just what we want. C_1 and C_2 are used in the original statement of (B)

[24] That's a neighbourhood of Y correspondent to the intersection of balls of radius ϵ in $\mathbb{CP}^N$ covering Z.

since the C's at different places have different affects on the magnitude of the final expression. Of course, the same C for each term in the big sum have to be really the same constant. In the following, we justify the computation above. The only nontrivial steps are the second and third ones.

The second one is the direct application of $(\star)$, the classic result in $\mathbb{C}^n$, as V_i and U_i are in one of the finitely many unit coordinate balls which clearly can be taken as the unit Euclidean ball in a uniform way, and we are using U_i instead of the big ball here.

The third step uses the following inequality:

$$Cap(K \cap \bar{V_i}, U_i) \leqslant C \cdot (-\log\epsilon)^n \cdot Cap_{\omega_\infty}(K \cap \bar{V_i}).$$

This result also has its primitive version in classic pluripotential theory for domains in $\mathbb{C}^n$. Extension of plurisubharmonic function is all what we need to prove it as described below.

For any $v \in PSH(U_i)$ valued in $[-1, 0]$. If we can "extend" this function to an element $-C\log\epsilon \cdot \tilde{v}$ where $\tilde{v}$ is plurisubharmonic with respect to ω_∞ valued in $[-1, 0]$ over X, and also make sure that the measures $(\sqrt{-1}\partial\bar{\partial}v)^n$ and $(\omega_\infty + \sqrt{-1}\partial\bar{\partial}\tilde{v})^n$ are the same over $\bar{V_i}$, then this would clearly imply the inequality above from the definition of relative capacity.

The construction will be done mostly on $P(X)$. The function v can be considered over $P(U_i)$. We'll "extend" it to a neighbourhood $P(X) \setminus O_i$ in $\mathbb{CP}^N$ where O_i is a neighbourhood of $\bar{V_i}$ in U_i.

Let's first extend it locally in $\mathbb{CP}^N$. We can safely assume that the construction happens in (finite) half-unit Euclidean balls in $\mathbb{CP}^N$ which cover the variety Z and have ω_M defined on the correspondent unit balls. ω_M can be expressed in the level of potential, and so the construction is merely about functions.

Consider the plurisubharmonic function function

$$h = \left(\log(\frac{36|z|^2}{\epsilon^{2C}})\right)^+ - 2,$$

where the upper $+$ means taking maximum with 0, on the unit ball in $\mathbb{CP}^N$ but with the coordinate system z centered at the center of $P(V_i)$. It's easy to see that the pullback of this function, still denoted by h, is plurisubharmonic and $max(h, v)$ on U_i is equal to v near $\bar{V_i}$ and equal to h near ∂U_i. So this function extends v to the preimage of the unit ball in $\mathbb{CP}^N$ while keeping the values near $\bar{V_i}$.

Now we want to extend further to the whole of X. We still work on $F(X) \subset \mathbb{CP}^N$. And it's only left to extend the function h for the remaining part where the value is less restrictive.

$|h|$ is bounded by $-C \cdot \log\epsilon$ in the unit ball. So we can have

$$\sqrt{-1}\partial\bar{\partial}h = -C \cdot \log\epsilon \cdot (\omega_M + \sqrt{-1}\partial\bar{\partial}H)$$

for H plurisubharmonic with respect to ω_M valued in $[-1, 0]$ in the unit ball [25]. Then using the same argument as in [Koj2], which has been illustratd before to give the equivalence of globally and locally defined relative capacities, we can extend H to (uniformly bounded) $\tilde{H} \in PSH_{\omega_M}(O)$, where O is a neighbourhood of $P(X)$, using the positivity of ω_M. Finally we just take $\tilde{v} = P^*\tilde{H}$.

This ends the argument for the case when $P : X \to P(X)$ is a birational map.

Now we want to remove the birationality condition. In fact, after removing proper subvarieties Y and $Z = P(Y)$ of X and $P(X)$ respectively, we can have $P : X \setminus Y \to P(X) \setminus Z$ is a finitely-sheeted covering map, since the map is clearly of full rank there and the finiteness of sheets can be seen by realizing the preimage of any point in $P(X \setminus Z)$ should be a finite set of points.

Then it's easy to see that the argument before would still work in this situation. Basically, we can still have the construction used before, and now the only difference is that the numbers of small pieces U_i and V_i need to be multiplied by (at most) the number of sheets, which clearly won't affect the previous argument too much.

Hence we get the apriori L^∞ bound in general.

Existence of bounded solution follows after getting this apriori bound.

[25]We can clearly achieve this by requiring the potential of ω_M is valued in a short interval. Actually, it's OK if we have H valued in a interval with length $C > 1$ as long as C is uniformly controlled which is of course the case here.

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